An Introduction to the A Posteriori Error Analysis of Elliptic Optimal Control Problems

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Adaptive Finite Element Methods for Optimal Control Problems


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Recent Books on Optimal Control of PDEs


Contents

- The optimality systems for unconstrained, control, state, and gradient-state constrained elliptic optimal control problems
- Review of the a posteriori error analysis of adaptive finite element methods
- A posteriori error analysis of unconstrained elliptic optimal control problems
- Residual-type a posteriori error analysis of control constrained problems
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Modeling Our Complex World ...
Application I: Optimal Control of Induction Hardening

Optimal hardening of a steel workpiece by electromagnetic induction and quenching: The creation of martensitic structures increases the durability of the steel.

\[
\inf_{u \in K, y} \frac{1}{2} \| \theta(\cdot, T) - \theta^d \|^2_{\Omega_2} + \frac{\alpha}{2} \int_0^T \| u \|^2_{\Gamma_S} dt,
\]

where the control \( u \in K := \{ v \in L^2((0, T), L^2(\Gamma_2)) \mid u_{\text{min}} \leq u \leq u_{\text{max}} \text{ a.e.} \} \) is a current density applied at \( \Gamma_S \subset \Omega_1 \) and the state \( y = (\varphi, A, \theta, z) \) consists of the electric potential \( \varphi \), the magnetic vector potential \( A \), the temperature \( \theta \), and the phase function \( z \).
The state \( y = (\varphi, A, \theta, z) \) satisfies the state equations

\[
\begin{align*}
\sigma \frac{\partial A}{\partial t} + \text{curl}(\mu^{-1}\text{curl} A) + \sigma \nabla \varphi &= 0 \quad \text{in } Q := D \times (0, T), \\
A \wedge n_{\partial D} &= 0 \quad \text{on } \Sigma := \partial D \times (0, T), \quad A(\cdot, 0) = A_0 \quad \text{in } D, \\
-\sigma \Delta \varphi &= 0 \quad \text{in } \Omega_1 \times (0, T), \quad n_{\partial \Omega_1} \cdot \nabla \varphi = \begin{cases} u & \text{on } \Gamma_s \times (0, T) \\ 0 & \text{on } (\partial \Omega_1 \setminus \Gamma_s) \times (0, T) \end{cases}, \\
\rho c \frac{\partial \theta}{\partial t} - \nabla \cdot (\kappa \nabla \theta) &= -\rho L \frac{\partial z}{\partial t} + \sigma |\frac{\partial A}{\partial t}|^2 \quad \text{in } Q_2 := \Omega_2 \times (0, T), \\
 n_{\partial \Omega_2} \cdot \kappa \nabla \theta &= 0 \quad \text{on } \Sigma_2 := \partial \Omega_2 \times (0, T), \quad \theta(\cdot, 0) = \theta_0 \quad \text{in } \Omega_2, \\
\tau \frac{dz}{dt} = g(\theta, z) \quad \text{in } (0, T), \quad z(0) = z_0.
\end{align*}
\]
Application II: Optimal Control of AF$^4$

AF$^4$ (Asymmetric Flow Field Flow Fractionation) is a process for the efficient separation of particles of different size ($\mu$m - nm) in microfluidic flows. AF$^4$ is used in chemical analytics, hematology, pharmacology, proteomics, and cytometry. The principle of AF$^4$ relies on the separation in a microchannel due to a force induced by a cross flow through a porous membrane permeable for the carrier fluid, but impermeable for the particles.
where the control $u$ is the inflow velocity at the inlet and the state $y = (v, p, c)$ satisfies the Navier-Stokes Brinkman equations for $(v, p)$ and advection-diffusion equations for the analytes $c = (c_1, \cdots, c_M)^T$. 
The Adaptive Cycle
The Loop in Adaptive Finite Element Methods (AFEM)

Adaptive Finite Element Methods (AFEM) consist of successive loops of the cycle

\[
\text{SOLVE} \implies \text{ESTIMATE} \implies \text{MARK} \implies \text{REFINE}
\]

**SOLVE:** Numerical solution of the FE discretized problem

**ESTIMATE:** Residual and hierarchical a posteriori error estimators
- Error estimators based on local averaging
- Goal oriented weighted dual approach
- Functional type a posteriori error bounds

**MARK:** Strategies based on the max. error or the averaged error
- Bulk criterion for AFEMs

**REFINE:** Bisection or ‘red/green’ refinement or combinations thereof
Elliptic Optimal Control Problems
Unconstrained Case
Optimize first, then discretize
Elliptic Optimal Control Problems: Unconstrained Case

Given $y^d \in L^2(\Omega)$ and $\alpha > 0$, find $(y, u) \in H^1_0(\Omega) \times L^2(\Omega)$ such that

$$\inf_{(y, u)} J(y, u) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx,$$

subject to $- \Delta y = u$ in $\Omega$,

$y = 0$ on $\Gamma$.

Reduced formulation: Denoting by $G : H^{-1}(\Omega) \to H^1_0(\Omega)$ the control-to-state map, which assigns to a control $u \in L^2(\Omega)$ the solution $y = G(u) \in H^1_0(\Omega)$ of the state equation, the reduced formulation reads:

$$\inf_u J_{\text{red}}(u) := \frac{1}{2} \int_{\Omega} |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx.$$
Unconstrained Minimization in Function Space

**Theorem.** Let $V$ be a reflexive Banach space and assume that $J : V \to (-\infty, +\infty]$ is a proper convex, lower semicontinuous (lsc), and coercive functional. Then, the unconstrained minimization problem

$$\inf_{v \in V} J(v)$$

has a solution $u \in V$.

If $J$ is strictly convex, the solution is unique.
**Elliptic Optimal Control Problems: Unconstrained Case**

**Theorem.** The unconstrained optimal control problems admits a unique solution.

**Proof.** Minimizing sequence argument.

**Theorem.** If \((y, u)\) is the optimal solution, then there exists \(p \in H^1_0(\Omega)\) such that

\[
-\Delta p = y^d - y \quad \text{in } \Omega,
\]

\[
p = 0 \quad \text{on } \Gamma,
\]

and

\[
p = \alpha u \quad \text{in } \Omega.
\]
Proof. Let $u \in L^2(\Omega)$ be the unique solution of the optimal control problem. The necessary (and here also sufficient) optimality condition for

$$\inf_u J_{\text{red}}(u) := \frac{1}{2} \int_\Omega |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx.$$

reads

$$(J'_{\text{red}}(u), v)_{0, \Omega} = (G(u) - y^d, G(v))_{0, \Omega} + \alpha(u, v)_{0, \Omega} = 0, \quad v \in L^2(\Omega),$$

where $J'_{\text{red}}(u)$ is the Gâteaux derivative of $J_{\text{red}}$ at $u$. Straightforward computation yields

$$(J'_{\text{red}}(u), v)_{0, \Omega} = (G^*(G(u) - y^d) + \alpha u, v)_{0, \Omega} = 0, \quad v \in L^2(\Omega),$$

and hence, $p = G^*(y^d - y)$ and $p - \alpha u = 0$. 
Optimality Conditions: Lagrange Multiplier Approach

Let $A : H^1_0(\Omega) \to H^{-1}(\Omega)$ be the operator associated with the bilinear form $a(y, v) := (\nabla y, \nabla v)_{0, \Omega}$. Couple the PDE constraint $Ay = u$ by a Lagrange multiplier $p \in H^1_0(\Omega)$:

$$\inf_{y, u} \sup_{p} \mathcal{L}(y, u, p), \quad \mathcal{L}(y, u, p) := J(y, u) + \langle Ay - u, p \rangle_{H^{-1}, H^1_0}.$$ 

Optimality Conditions:

$$\mathcal{L}_p(y, u, p) = Ay - u = 0 \quad \Rightarrow \quad Ay = 0,$$

$$\mathcal{L}_y(y, u, p) = y - y^d + A^* p = 0 \quad \Rightarrow \quad A^* p = y^d - y,$$

$$\mathcal{L}_u(y, u, p) = \alpha u - p = 0 \quad \Rightarrow \quad p = \alpha u.$$
Finite Element Approximation of the Distributed Control Problem

Let $\mathcal{T}_h(\Omega)$ be a shape regular, simplicial triangulation of $\Omega$ and let

$$V_h := \{ v_h \in C(\Omega) \mid v_h|_T \in P_1(T), \ T \in \mathcal{T}_h(\Omega), \ v_h|_{\partial\Omega} = 0 \}$$

be the FE space of continuous, piecewise linear finite elements.

Consider the following FE Approximation of the distributed control problem

Minimize $J(y_h, u_h) := \frac{1}{2} \| y_h - y^d \|_{L^2(\Omega)}^2 + \frac{\alpha}{2} \| u_h \|_{L^2(\Omega)}^2,$ over $(y_h, u_h) \in V_h \times V_h,$ subject to $a(y_h, v_h) = (u_h, v_h)_{L^2(\Omega)}, \ v_h \in V_h.$
Optimality Conditions for the FE Discretized Control Problem

There exists an adjoint state $p_h \in V_h$ such that the triple $(y_h, p_h, u_h)$ satisfies

\[ a(y_h, v_h) = (u_h, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h, \]
\[ a(p_h, v_h) = -(y_h - y^d, v_h)_{L^2(\Omega)} \quad , \quad v_h \in V_h, \]
\[ p_h - \alpha u_h = 0. \]

Algebraic Formulation:

\[
\begin{pmatrix}
A_h & 0 \\
M_h & A_h
\end{pmatrix}
\begin{pmatrix}
y_h \\
p_h
\end{pmatrix}
= 
\begin{pmatrix}
M_h u_h \\
y^d
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
A_h & -\alpha^{-1} M_h \\
M_h & A_h
\end{pmatrix}
\begin{pmatrix}
y_h \\
p_h
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
y^d
\end{pmatrix}.
\]

Solver: Multigrid with preconditioned Uzawa as a smoother
Multigrid Solvers for Elliptic Optimal Control Problems

A. Borzi, K. Kunisch, and D. Y. Kwak; Accuracy and convergence properties of the finite difference multigrid solution of an optimal control optimality system.

A. Borzi and V. Schulz; Multigrid methods for PDE optimization.

Elliptic Optimal Control Problems
Control Constraints
Elliptic Optimal Control Problems: Control Constrained Case

Given \( y^d \in L^2(\Omega), \alpha > 0 \), and the closed convex set

\[
K := \{ v \in L^2(\Omega) \mid v \leq \psi \text{ a.e. in } \Omega \},
\]

where \( \psi \) is an affine function, find \((y, u) \in H^1_0(\Omega) \times K\) such that

\[
\inf_{(y, u) \in H^1_0(\Omega) \times K} J(y, u) := \frac{1}{2} \int_\Omega |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_\Omega |u|^2 \, dx,
\]

subject to

\[
- \Delta y = u \quad \text{in } \Omega,
\]

\[
y = 0 \quad \text{on } \Gamma.
\]
Elliptic Optimal Control Problems: Control Constrained Case

**Theorem.** The control constrained optimal control problem has a unique solution.

**Proof.** Minimizing sequence argument.

**Theorem.** If \((y, u) \in H^1_0(\Omega) \times K\) is the optimal solution, then there exists an adjoint state \(p \in H^1_0(\Omega)\) and an adjoint control \(\lambda \in L^2(\Omega)\) such that

\[
-\Delta p = y^d - y \quad \text{in } \Omega,
\]
\[
p = 0 \quad \text{on } \Gamma,
\]
\[
p = \alpha u + \lambda \quad \text{in } \Omega,
\]
\[
\lambda \in L^2_+(\Omega), \quad \psi - u \geq 0, \quad (\lambda, \psi - u)_{0, \Omega} = 0.
\]
Elliptic Optimal Control Problems: Control Constrained Case

Reduced formulation: Denoting by $G : H^{-1}(\Omega) \to H^1_0(\Omega)$ the control-to-state map, which assigns to $u \in H^{-1}(\Omega)$ the solution $y = G(u) \in H^1_0(\Omega)$ of the state equation, the reduced formulation reads:

$$\inf_{u \in K} J_{\text{red}}(u) := \frac{1}{2} \int_{\Omega} |G(u) - y|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx.$$  

Unconstrained formulation: Let $I_K$ be the indicator function of the constraint set $K$. Then, the unconstrained formulation of the control constrained optimal control problem is given by

$$\inf_{u \in L^2(\Omega)} \hat{J}(u) := J_{\text{red}}(u) + I_K(u).$$
Proof. The necessary and sufficient optimality condition is given by
\[ 0 \in \partial \hat{J}(u) = J'_\text{red}(u) + \partial I_K(u), \]
where \( \partial I_K(u) \) is the subdifferential of \( I_K \) at \( u \). Hence, there exists \( \lambda \in \partial I_K(u) \) such that
\[ G^*(G(u) - y^d) + \alpha u + \lambda = 0 \quad \implies \quad p = \alpha u + \lambda. \]

Since \( \partial I_K(u) = \{ \mu \in L^2(\Omega) \mid (\mu, u - v)_{0,\Omega} \geq 0, \ v \in I_K(u) \} \), choosing \( v = u - w_+ \), \( w_+ \in L^2_+(\Omega) \), it follows that \( \lambda \in L^2_+(\Omega) \). On the other hand, choosing \( v = \psi \) allows to deduce
\[ (\lambda, \psi - u)_{0,\Omega} = 0. \]
Moreau-Yosida Approximation of Multivalued Maps I

Weighted Duality Mapping: Assume that $V$ is a Banach space with dual $V^*$ and let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous and non-decreasing function such that $h(0) = 0$ and $h(t) \to \infty$ as $t \to \infty$. Then the mapping $J_h : V \to 2^{V^*}$

$$J_h(u) := \{u^* \in V^* \mid \langle u^*, u \rangle = \|u\|\|u^*\|, \|u^*\| = h(\|u\|)\}$$

is called the duality mapping with weight $h$.

Example: For $V = L^p(\Omega), V^* = L^q(\Omega), 1 < p, q < \infty, 1/p + 1/q = 1,$ and $h(t) = t^{p-1}$ we have

$$J_h(u)(x) = \begin{cases} |u(x)|^{p-1} \text{sgn}(u(x)) , & u(x) \neq 0 \\ 0 , & u(x) = 0 \end{cases}.$$
Moreau-Yosida Approximation of Multivalued Maps II

Moreau-Yosida proximal map: Let $f : V \to \bar{\mathbb{R}}$ be a lower semi-continuous proper convex function with subdifferential $\partial f$. For $c > 0$, the Moreau-Yosida proximal map $P_c^{\partial f} : V \to 2^V$ is defined such that $P_c^{\partial f}(w), w \in V$, is the set of minimizers of

$$\inf_{v \in V} f(v) + c j_h\left(\frac{v - w}{c}\right),$$

where $\partial j_h = J_h$.

Moreau-Yosida approximation: If $J_h$ is single-valued, then for $c > 0$ the Moreau-Yosida approximation $(\partial f)_c$ of $\partial f$ is given by

$$(\partial f)_c(w) := J_h(c^{-1}w - c^{-1}P_c^{\partial f}(w)).$$
Moreau-Yosida Approximation of $\partial I_{K_C}$

Idea: Approximate $\partial I_{K_C}$ by its Moreau-Yosida approximation $(\partial I_{K_C})_c$.

Theorem. For any $c > 0$, we have

$$\lambda \in (\partial I_{K_C})_c,$$

if and only if there holds

$$\lambda = c \left( u + c^{-1} \lambda - \Pi_{K_C} (u + c^{-1} \lambda) \right) = c \max(0, u + c^{-1} \lambda - \psi),$$

and this is equivalent to

$$u = \Pi_{K_C} (u + c^{-1} \lambda),$$

where $\Pi_{K_C}$ denotes the $L^2$-projection onto $K_C$. 

Elliptic Optimal Control Problems: Control Constrained Case

**Problem:** The subdifferential $\partial I_K(u)$ is a multivalued function.

\[
\partial I_K(u) = \begin{cases} 
0 & , u < \psi \\
[0, +\infty) & , u = \psi .
\end{cases}
\]

**Remedy:** Moreau-Yosida approximation of multivalued functions.

\[
(\partial I_K)_c(u) = \begin{cases} 
0 & , u < \psi \\
c(u - \psi) & , u \geq \psi .
\end{cases}
\]

\[
\lambda \in (\partial I_K)_c(u) \iff \lambda = \max(0, \lambda + c(u - \psi)).
\]
Primal-Dual Active Set Strategy I

Step 1 (Initialization):
Choose $c > 0$, start-iterates $y_h^{(0)}, u_h^{(0)}, \lambda_h^{(0)}$ and set $n = 1$.

Step 2 (Specification of active/inactive sets):
Compute the active/inactive sets $\mathcal{A}_n$ and $\mathcal{I}_n$ according to

$$\mathcal{A}_n := \{1 \leq i \leq N \mid (u_h^{(n-1)} + c^{-1}\lambda_h^{(n-1)})_i > (\psi_h)_i\}, \quad \mathcal{I}_n := \{1, \ldots, N\} \setminus \mathcal{A}_n.$$

Step 3 (Termination criterion):
If $n \geq 2$ and $\mathcal{A}_n = \mathcal{A}_{n-1}$, stop the algorithm. Otherwise, go to Step 4.
Primal-Dual Active Set Strategy II

Step 4 (Update of the state, adjoint state, and control):

Compute $y_h^{(n)}, p_h^{(n)}$ as the solution of

$$ (A_h y_h^{(n)})_i = \begin{cases} (\psi_h)_i, & \text{if } i \in A_n \\ \alpha^{-1} (M_h^{-1} p_h^{(n)})_i, & \text{if } i \in I_n \end{cases}, \quad A_h p_h^{(n)} = -M_h y_h^{(n)} + y_h, $$

and set

$$ (u_h^{(n)})_i := \begin{cases} (\psi_h)_i, & \text{if } i \in A_n \\ \alpha^{-1} (M_h^{-1} p_h^{(n)})_i, & \text{if } i \in I_n \end{cases}. $$

Step 5 (Update of the multiplier):

Set $\lambda_h^{(n)} := p_h^{(n)} - \alpha M_h u_h^{(n)}$, $n := n + 1$, and go to Step 2.
Elliptic Optimal Control Problems
State Constraints
Elliptic Optimal Control Problems: State Constrained Case

Given $y^d \in L^2(\Omega)$, $\alpha > 0$, and the closed convex set

$$K := \{ v \in W^{1,r}(\Omega) \cap H^1_0(\Omega), \ r > d \mid v \leq \psi \text{ in } \Omega \},$$

where $\psi \in W^{1,\infty}(\Omega)$, $\psi|_{\Gamma} > 0$, find $(y, u) \in K \times L^2(\Omega)$ such that

$$\inf_{(y, u) \in K \times L^2(\Omega)} J(y, u) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx,$$

subject to

$$- \Delta y = u \quad \text{in } \Omega,$$
$$y = 0 \quad \text{on } \Gamma.$$
Elliptic Optimal Control Problems: State Constrained Case

Reduced formulation: Denoting by \( G : W^{-1,s}(\Omega) \to W^{1,r}_0(\Omega) \) the control-to-state map, which assigns to \( u \in W^{-1,s}(\Omega) \) the solution \( y = G(u) \in W^{1,r}_0(\Omega) \) of the state equation, the reduced formulation reads:

\[
\inf_{G(u) \in K} J_{\text{red}}(u) := \frac{1}{2} \int_{\Omega} |G(u) - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx.
\]

Unconstrained formulation: Let \( I_K \) be the indicator function of the constraint set \( K \). Then, the unconstrained formulation of the control constrained optimal control problem is given by

\[
\inf_{u \in L^2(\Omega)} \hat{J}(u) := J_{\text{red}}(u) + (I_K \circ G)(u).
\]
Elliptic Optimal Control Problems: State Constrained Case

**Theorem.** The state constrained optimal control problem has a unique solution.

**Proof.** Minimizing sequence argument.

**Theorem.** Assume that the following Slater condition holds true:

There exists $u_0 \in L^2(\Omega)$ such that the associated solution $y_0 = G(u_0) \in W^{1,r}_0(\Omega)$ satisfies $y_0 \in \text{int}(K)$. If $(y, u) \in K \cap L^2(\Omega)$ is the unique solution of the state constrained optimal control problem, there exist

$$p \in W^{1,s}_0(\Omega), \quad \frac{1}{r} + \frac{1}{s} = 1 \quad \text{and} \quad \lambda \in M(\bar{\Omega}) \quad \text{s.th.}$$

$$\langle \nabla p, \nabla v \rangle_{L^s, L^r} = (y^d - y, v)_{0, \Omega} - \langle \lambda, v \rangle_{M(\bar{\Omega}), C(\bar{\Omega})}, \quad v \in W^{1,r}_0(\Omega),$$

$$p = \alpha u,$$

$$\lambda \in M_+(\bar{\Omega}), \quad \psi - y \geq 0, \quad \langle \lambda, \psi - y \rangle_{M(\bar{\Omega}), C(\bar{\Omega})} = 0.$$
Proof. The necessary and sufficient optimality condition reads

\[ 0 \in \partial \hat{J}(u) = J'_{\text{red}}(u) + \partial(I_K \circ G)(u). \]

What do we know about \( \partial(I_K \circ G)(u) \)?
Subdifferential Calculus: Subdifferential of Composite Maps

Theorem. Let $X, Y$ be Banach spaces with duals $X^*, Y^*$. Let $f : X \to (-\infty, +\infty]$ be proper convex and lsc, and let $A : Y \to X$ be a bounded linear operator. Assume that there exists $\tilde{u} \in Y$ such that $f$ is continuous and finite at $A\tilde{u}$. Then there holds

$$\partial (f \circ A)(u) = A^* \partial f(A(u)).$$
Proof. The necessary and sufficient optimality condition reads
\[ 0 \in \partial \hat{J}(u) = J'_\text{red}(u) + \partial (I_K \circ G)(u). \]

Due to the Slater condition there holds
\[ \partial (I_K \circ G)(u) = G^* (\partial I_K(G(u))). \]

Hence, there exists \( \lambda \in \partial I_K(y) \) such that
\[
G^*(G(u) + \lambda) + \alpha u = 0 \quad \Rightarrow \quad p = \alpha u \quad \text{and} \quad p = G^*(y^d - y - \lambda).
\]

PDE theory tells us that \( p \in W^{1,s}_0(\Omega) \).
Elliptic Optimal Control Problems
Constraints on the Gradient of the State
Elliptic Optimal Control with Pointwise Gradient-State Constraints

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded polygonal domain with boundary \( \Gamma \), \( y^d \in L^2(\Omega) \) a desired state, \( f \) a forcing term, \( \psi \in L^2(\Omega) \) s.th. \( \psi \geq \psi_{\text{min}} > 0 \) a.e. in \( \Omega \), and \( \alpha > 0 \), find \( (y, u) \in H^1_0(\Omega) \times L^2(\Omega) \) such that

\[
\inf_{(y, u)} J(y, u) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx,
\]

subject to

\[
Ly := -\nabla \cdot a \nabla y + cy = f + u \quad \text{in} \ \Omega,
\]
\[
y = 0 \quad \text{on} \ \Gamma,
\]
\[
\nabla y \in K := \{ v \in L^2(\Omega)^2 \mid |v| \leq \psi \ \text{a.e. in} \ \Omega \}.
\]
Pointwise Gradient-State Constraints: State-Reduced Formulation

Let $\hat{V} \subset H^1_0(\Omega)$ be a reflexive Banach space and let $\hat{G} : L^2(\Omega) \to \hat{V}$ be the map that assigns to the rhs $f + u$ the solution $y = \hat{G}(f + u)$ of the state equation. Assume that $\hat{G}$ is a bounded linear operator which is invertible such that $u = \hat{G}^{-1} y - f$. This leads to the state-reduced formulation:

Find $y \in \hat{K} := \{ v \in \hat{V} \mid |\nabla v| \leq \psi \text{ bf a.e. in } \Omega \}$ such that

$$\inf_{y \in \hat{K}} J_{\text{red}}(y) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |\hat{G}^{-1} y - f|^2 \, dx.$$

Unconstrained formulation:

$$\inf_{y \in \hat{V}} J_{\text{red}}(y) + I_{\hat{K}}(y)$$

where $I_{\hat{K}}$ stands for the indicator function of the set $\hat{K}$. 
State-Reduced Formulation: Optimality Conditions

Theorem. The gradient-state constrained optimal control problem admits a unique solution \((y, u) \in \hat{K} \times L^2(\Omega)\) which is characterized by the existence of a unique pair \((p, w) \in L^2(\Omega) \times \hat{V}^*\) satisfying

\[
Lp = -\nabla \cdot (a\nabla p) + cp = y^d - y - w \quad \text{in} \ \hat{V}^*,
\]

\[
p = \alpha u \quad \text{in} \ L^2(\Omega),
\]

\[
w \in N_{\hat{K}}(y) := \{\xi \in \hat{V}^* \mid \langle \xi, z - y \rangle_{\hat{V}^*, \hat{V}} \leq 0, \ z \in \hat{K}\}.
\]

Remark. If \(\hat{V} = W^{2,r}(\Omega) \cap H^1_0(\Omega), r > 2\), there exists a Slater point, i.e., \(y_0 \in \text{int} \ \hat{K}\) and \(|\nabla (y_0 + v)| \leq \psi\) in \(\Omega\) for all \(v \in C^1(\overline{\Omega})\) s.th. \(\|v\|_{C^1(\overline{\Omega})} \leq \delta\) for sufficiently small \(\delta > 0\).

\[
0 \in J'_{\text{red}}(y) + \partial(I_{\hat{K}} \circ \nabla)(y) = J'_{\text{red}}(y) - \nabla \cdot \partial I_{\hat{K}}(\nabla y),
\]

i.e., there exists \(\mu \in \partial I_{\hat{K}}(\nabla y) \subset M(\overline{\Omega})^2\) such that \(w = -\nabla \cdot \mu\).
Control-Reduced Formulation and Dual Problem

Denoting by \( G : H^{-1}(\Omega) \to H^1_0(\Omega) \) the solution operator associated with the state equation, the optimal control problem can be written according to

\[
\inf_{u \in L^2(\Omega)} \mathcal{F}(u) + G(\Lambda u)
\]

where

\[
\mathcal{F}(u) := J(G(f + u), u), \quad G(q) := I_K(q), \quad \Lambda := \nabla G.
\]

Denoting by \( \mathcal{F}^* \) and \( \mathcal{G}^* \) the Fenchel conjugates of \( \mathcal{F} \) and \( \mathcal{G} \)

\[
\mathcal{F}^*(u^*) = \frac{1}{2} \| u^* + G^* y^d + \alpha f \|^2_{M^{-1}}, \quad \mathcal{G}^*(q^*) = \int_{\Omega} \psi |q^*| dx,
\]

where \( M := G^*G + \alpha I \) and \( \| \cdot \|^2_{M^{-1}} := (M^{-1} \cdot, \cdot)_{0,\Omega} \), the dual problem reads as follows:

\[
(D) \quad \sup_{q^* \in L^2(\Omega)} - \mathcal{F}^*(\Lambda^* q^*) - \mathcal{G}^*(-q^*) \iff \inf_{\mu \in L^2(\Omega)} \frac{1}{2} \| G^*(\nabla^* \mu + y^d) + \alpha f \|^2_{M^{-1}} + \int_{\Omega} \psi |\mu| dx.
\]
The Fenchel Conjugate (Polar Function)

Let \( f : V \to (-\infty, +\infty] \) be a proper convex function. The Fenchel conjugate \( J^* : V^* \to (-\infty, +\infty] \) is defined by means of

\[
J^*(u^*) := \sup_{u \in V} \left( \langle u^*, u \rangle - J(u) \right).
\]

Example. Let \( K \subset V \) be a closed convex set with indicator function \( I_K \). The Fenchel conjugate \( I_K^* \) is given by

\[
I_K^*(u^*) = \sup_{u \in K} \langle u^*, u \rangle.
\]
The Fenchel Conjugate of $\mathcal{G} : L^2(\Omega)^2 \to \mathbb{R}$, $\mathcal{G}(q) := I_K(q)$

We claim $\mathcal{G}^*(q^*) = \int_\Omega \psi |q^*| dx$

Proof. We have

$$\mathcal{G}^*(q^*) = \sup_{q \in K} (q^*, q)_{0,\Omega}.$$ 

Since $|q| \leq \psi$, there obviously holds

$$(q^*, q)_{0,\Omega} \leq \int_\Omega \psi |q^*| dx.$$ 

On the other hand, the special choice $q := \psi q^* |q^*|^{-1}$ implies

$$(q^*, q)_{0,\Omega} = (q^*, \psi q^* |q^*|^{-1})_{0,\Omega} = \int_\Omega \psi |q^*| dx.$$
Tightened Formulation of the Primal Problem

Consider the following tightened formulation of the primal problem

\[ \hat{P} \quad \inf_{(y, u) \in \hat{V} \times L^2(\Omega)} J(y, u) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx, \]

subject to

\[ Ly = f + u \quad \text{in} \ \Omega, \quad y = 0 \quad \text{on} \ \Gamma, \quad |\nabla y| \leq \psi \quad \text{a.e. in} \ \Omega. \]

**Theorem.** Let \( \{\mu_n\}_N \subset L^2(\Omega)^2 \) be a minimizing sequence for the dual \( \hat{D} \) to \( \hat{P} \).

Then, there exist a subsequence \( \{\mu_n\}' \) and \( \mu \in M(\bar{\Omega})^2 \) such that

\[ w^* - \lim \mu_n = \mu \quad \text{in} \ M(\bar{\Omega})^2 \quad \text{and} \quad w - \lim \nabla \cdot \mu_n = -w \quad \text{in} \ \hat{V}^*. \]

Moreover, the limit \( w \in \hat{V}^* \) satisfies

\[ (**) \quad Ly = f + u \quad \text{in} \ L^2(\Omega), \quad Lp = y^d - y - w \quad \text{in} \ \hat{V}^*, \quad p = \alpha u \quad \text{in} \ L^2(\Omega). \]

**Remark.** A quadruple \( (y, u, p, w) \in V \times L^2(\Omega) \times L^2(\Omega) \times \hat{V}^* \) such that \( (**) \) holds true and \( \nabla y \in (M(\bar{\Omega})^2)^* \setminus C(\bar{\Omega})^2 \), is called a **weak solution** of \( (P) \).
Basic Concepts of Adaptive Finite Element Methods for Elliptic Boundary Value Problems

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Foundations of AFEM I

For a closed subspace $V \subset H^1(\Omega)$ we assume

$$a(\cdot, \cdot) : V \times V \to \mathbb{R}$$

to be a bounded, $V$-elliptic bilinear form, i.e.,

$$|a(v, w)| \leq C \|v\|_{k,\Omega} \|w\|_{k,\Omega}, \quad v, w \in V, \quad a(v, v) \geq \gamma \|v\|_{k,\Omega}^2, \quad v \in V,$$

for some constants $C > 0$ and $\gamma > 0$. We further assume $\ell \in V^*$ where $V^*$ denotes the algebraic and topological dual of $V$ and consider the variational equation: Find $u \in V$ such that

$$a(u, v) = \ell(v), \quad v \in V.$$

It is well-known by the Lax-Milgram Lemma that under the above assumptions the variational problem admits a unique solution.
Foundations of AFEM II

Finite element approximations are based on the Ritz-Galerkin approach: Given a finite dimensional subspace $V_h \subset V$ of test/trial functions, find $u_h \in V_h$ such that

$$ a(u_h, v_h) = \ell(v_h), \quad v_h \in V_h. $$

Since $V_h \subset V$, the existence and uniqueness of a discrete solution $u_h \in V_h$ follows readily from the Lax-Milgram Lemma. Moreover, we deduce that the error $e_u := u - u_h$ satisfies the Galerkin orthogonality

$$ a(u - u_h, v_h) = 0, \quad v_h \in V_h, $$

i.e., the approximate solution $u_h \in V_h$ is the projection of the solution $u \in V$ onto $V_h$ with respect to the inner product $a(\cdot, \cdot)$ on $V$ (elliptic projection). Using the Galerkin orthogonality, it is easy to derive the a priori error estimate

$$ \|u - u_h\|_{1,\Omega} \leq M \inf_{v_h \in V_h} \|u - v_h\|_{1,\Omega}, $$

where $M := C/\gamma$. This result tells us that the error is of the same order as the best approximation of the solution $u \in V$ by functions from the finite dimensional subspace $V_h$. It is known as Céa’s Lemma.
Foundations of AFEM III

The Ritz-Galerkin method also gives rise to an a posteriori error estimate in terms of the residual \( r : V \to \mathbb{R} \)

\[
r(v) := \ell(v) - a(u_h, v), \quad v \in V.
\]

In fact, it follows that for any \( v \in V \)

\[
\gamma \| u - u_h \|_{1,\Omega}^2 \leq a(u - u_h, u - u_h) = r(u - u_h) \leq \| r \|_{-1,\Omega} \| u - u_h \|_{1,\Omega},
\]

whence

\[
\| u - u_h \|_{1,\Omega} \leq \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\| v \|_{1,\Omega}}.
\]
Foundations of AFEM IV

**Definition.** An error estimator $\eta_h$ is called **reliable**, if it provides an upper bound for the error up to data oscillations $\text{osc}_h^{\text{rel}}$, i.e., if there exists a constant $C_{\text{rel}} > 0$, independent of the mesh size $h$ of the underlying triangulation, such that

$$\|e_u\|_a \leq C_{\text{rel}} \eta_h + \text{osc}_h^{\text{rel}}.$$ 

On the other hand, an estimator $\eta_h$ is said to be **efficient**, if up to data oscillations $\text{osc}_h^{\text{eff}}$ it gives rise to a lower bound for the error, i.e., if there exists a constant $C_{\text{eff}} > 0$, independent of the mesh size $h$ of the underlying triangulation, such that

$$\eta_h \leq C_{\text{eff}} \|e_u\|_a + \text{osc}_h^{\text{eff}}.$$ 

Finally, an estimator $\eta_h$ is called **asymptotically exact**, if it is both reliable and efficient with $C_{\text{rel}} = C_{\text{eff}}^{-1}$. 
Reliability and Efficiency of Error Estimators II

Remark. The notion ‘reliability’ is motivated by the use of the error estimator in error control. Given a tolerance $\text{tol}$, an idealized termination criterion would be

$$ \|e_u\|_a \leq \text{tol}. $$

Since the error $\|e_u\|_a$ is unknown, we replace it with the upper bound, i.e.,

$$ C_{\text{rel}} \eta_h + \text{osc}^{\text{rel}}_h \leq \text{tol}. $$

We note that the termination criterion both requires the knowledge of $C_{\text{rel}}$ and the incorporation of the data oscillation term $\text{osc}^{\text{rel}}_h$. In the special case $C_{\text{rel}} = 1$ and $\text{osc}^{\text{rel}}_h \equiv 0$, it reduces to

$$ \eta_h \leq \text{tol}. $$

An alternative, but less used termination criterion is based on the lower bound, i.e., we require

$$ \frac{1}{C_{\text{eff}}} \left( \eta_h - \text{osc}^{\text{eff}}_h \right) \leq \text{tol}. $$

Typically, this criterion leads to less refinement and thus requires less computational time which motivates to call the estimator efficient.
The Role of the Residual

The error estimate
\[ \|u - u_h\|_{\Omega} \leq \frac{1}{\gamma} \sup_{v \in V} \frac{|r(v)|}{\|v\|_{\Omega}} \]
shows that in order to assess the error \( \|e_u\|_a \) we are supposed to evaluate the norm of the residual with respect to the dual space \( V^* \), i.e.,
\[ \|r\|_{V^*} := \sup_{v \in V \setminus \{0\}} \frac{|r(v)|}{\|v\|_a}. \]
In particular, we have the equality
\[ \|r\|_{V^*} = \|e_u\|_a, \]
whereas for the relative error of \( r(v), v \in V \), as an approximation of \( \|e_u\|_a \) we obtain
\[ \frac{(\|e_u\|_a - r(v))}{\|e_u\|_a} = \frac{1}{2} \|v - \frac{e_u}{\|e_u\|_a}\|^2, \quad v \in V \text{ with } \|v\|_a = 1. \]
The goal is to obtain lower and upper bounds for \( \|r\|_{V^*} \) at relatively low computational expense.
Model problem: Let $\Omega$ be a bounded simply-connected polygonal domain in Euclidean space $\mathbb{R}^2$ with boundary $\Gamma = \Gamma_D \cup \Gamma_N$, $\Gamma_D \cap \Gamma_N = \emptyset$ and consider the elliptic boundary value problem

$$Lu := - \nabla \cdot (a \nabla u) = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \Gamma_D, \quad n \cdot a \nabla u = g \quad \text{on } \Gamma_N,$$

where $f \in L^2(\Omega)$, $g \in L^2(\Gamma_N)$ and $a = (a_{ij})_{i,j=1}^2$ is supposed to be a matrix-valued function with entries $a_{ij} \in L^\infty(\Omega)$, that is symmetric and uniformly positive definite. The vector $n$ denotes the exterior unit normal vector on $\Gamma_N$. Setting

$$H^1_{0,\Gamma_D}(\Omega) := \{ v \in H^1(\Omega) \mid v|_{\Gamma_D} = 0 \},$$

the weak formulation is as follows: Find $u \in H^1_{0,\Gamma_D}(\Omega)$ such that

$$a(u, v) = \ell(v), \quad v \in H^1_{0,\Gamma_D}(\Omega),$$

where

$$a(v, w) := \int_\Omega a \nabla v \cdot \nabla w \, dx, \quad \ell(v) := \int_\Omega f v \, dx + \int_{\Gamma_N} g v \, d\sigma, \quad v \in H^1_{0,\Gamma_D}(\Omega).$$
FE Approximation: Given a geometrically conforming simplicial triangulation $\mathcal{T}_h$ of $\Omega$, we denote by

$$S_{1,\Gamma_D}(\Omega; \mathcal{T}_h) := \{ v_h \in H^1_{0,\Gamma_D}(\Omega) \mid v_h\big|_T \in P_1(K), \ T \in \mathcal{T}_h \}$$

the trial space of continuous, piecewise linear finite elements with respect to $\mathcal{T}_h$. Note that $P_k(T)$, $k \geq 0$, denotes the linear space of polynomials of degree $\leq k$ on $T$.

In the sequel we will refer to $\mathcal{N}_h(D)$ and $\mathcal{E}_h(D)$, $D \subseteq \bar{\Omega}$ as the sets of vertices and edges of $\mathcal{T}_h$ on $D$. We further denote by $|T|$ the area, by $h_T$ the diameter of an element $T \in \mathcal{T}_h$, and by $h_E = |E|$ the length of an edge $E \in \mathcal{E}_h(\Omega \cup \Gamma_N)$. We refer to $f_T := |T|^{-1} \int_T f dx$ the integral mean of $f$ with respect to an element $T \in \mathcal{T}_h$ and to $g_E := |E|^{-1} \int_E g ds$ the mean of $g$ with respect to the edge $E \in \mathcal{E}_h(\Gamma_N)$.

The conforming $P_1$ approximation reads as follows: Find $u_h \in S_{1,\Gamma_D}(\Omega; \mathcal{T}_h)$ such that

$$a(u_h, v_h) = \ell(v_h), \quad v_h \in S_{1,\Gamma_D}(\Omega; \mathcal{T}_h).$$
Representation of the Residual I

The residual $r$ is given by

$$r(v) := \int_{\Omega} f \, v \, dx + \int_{\Gamma_N} g \, v \, ds - a(u_h, v), \quad v \in V.$$ 

Applying Green’s formula elementwise yields

$$a(u_h, v) = \sum_{T \in \mathcal{T}_h} \int_{T} a \nabla u_h \cdot \nabla v \, dx = \sum_{E \in \mathcal{E}_h(\Omega)} \int_{E} [n \cdot a \nabla u_h] \, v \, ds + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \int_{E} n \cdot a \nabla u_h \, v \, ds,$$

where $[n \cdot a \nabla u_h]$ denotes the jump of the normal derivative of $u_h$ across $E \in \mathcal{E}_h(\Omega)$ and where we have used that $\Delta u_h \equiv 0$ on $T \in \mathcal{T}_h$, since $u_h|_T \in P_1(T)$. We thus obtain

$$r(v) := \sum_{T \in \mathcal{T}_h} r_T(v) + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_N)} r_E(v).$$
Representation of the Residual II

Here, the local residuals $r_T(v), T \in \mathcal{T}_h$, are given by

$$r_T(v) := \int_T (f - Lu_h)v \, dx,$$

whereas for $r_E(v)$ we have

$$r_E(v) := -\int_E [n \cdot a \nabla u_h] v \, ds, \quad E \in \mathcal{E}_h(\Omega),$$

$$r_E(v) := \int_E (g - n \cdot a \nabla u_h)v \, ds, \quad E \in \mathcal{E}_h(\Gamma_N).$$
A Posteriori Error Estimator and Data Oscillations

The error estimator $\eta_h$ consists of element residuals $\eta_T, T \in \mathcal{T}_h$, and edge residuals $\eta_E, E \in \mathcal{E}_h(\Omega \cup \Gamma_N)$, according to

$$
\eta_h := \left( \sum_{T \in \mathcal{T}_h} \eta^2_T + \sum_{E \in \mathcal{E}_h(\Omega \cup \Gamma_N)} \eta^2_E \right)^{1/2},
$$

where $\eta_T$ and $\eta_E$ are given by

$$
\eta_T := h_T \| f_T - L u_h \|_{0,T}, \quad T \in \mathcal{T}_h,
$$

$$
\eta_E := \begin{cases} 
  h_E^{1/2} \|[n \cdot a \nabla u_h]_{0,E}, & E \in \mathcal{E}_h(\Omega), \\
  h_E^{1/2} \| g_E - n \cdot a \nabla u_h \|_{0,E}, & E \in \mathcal{E}_h(\Gamma_N). 
\end{cases}
$$

The a posteriori error analysis further invokes the data oscillations

$$
\text{osc}_h := \left( \sum_{T \in \mathcal{T}_h} \text{osc}_{T}^2(f) + \sum_{E \in \mathcal{E}_h(\Gamma_N)} \text{osc}_{E}^2(g) \right)^{1/2},
$$

where $\text{osc}_T(f)$ and $\text{osc}_E(g)$ are given by

$$
\text{osc}_T(f) := h_T \| f - f_T \|_{0,T}, \quad \text{osc}_E(g) := h_E^{1/2} \| g - g_E \|_{0,E}.
$$
Clément’s Quasi-Interpolation Operator I

For \( p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N) \) we denote by \( \varphi_p \) the basis function in \( S_{1,\Gamma_D}(\Omega; T_h) \) with supporting point \( p \), and we refer to \( D_p \) as the set

\[
D_p := \bigcup \{ T \in T_h \mid p \in \mathcal{N}_h(T) \}.
\]

We refer to \( \pi_p \) as the \( L^2 \)-projection onto \( P_1(D_p) \), i.e.,

\[
(\pi_p(v), w)_{0,D_p} = (v, w)_{0,D_p}, \quad w \in P_1(D_p),
\]

where \( (\cdot, \cdot)_{0,D_p} \) stands for the \( L^2 \)-inner product on \( L^2(D_p) \times L^2(D_p) \). Then, Clément’s interpolation operator \( P_C \) is defined as follows

\[
P_C : L^2(\Omega) \rightarrow S_{1,\Gamma_D}(\Omega; T_h), \quad P_Cv := \sum_{p \in \mathcal{N}_h(\Omega) \cup \mathcal{N}_h(\Gamma_N)} \pi_p(v) \varphi_p.
\]
Clément’s Quasi-Interpolation Operator II

**Theorem.** Let \( v \in H^1_{0, \Gamma_D}(\Omega) \). Then, for Clément’s interpolation operator there holds

\[
\| P_C v \|_{0,T} \leq C \| v \|_{0, D_T^{(1)}}, \quad \| P_C v \|_{0,E} \leq C \| v \|_{0, D_E^{(1)}}, \quad \| \nabla P_C v \|_{0,T} \leq C \| \nabla v \|_{0, D_T^{(1)}},
\]

\[
\| v - P_C v \|_{0,T} \leq C h_T \| v \|_{1, D_T^{(1)}}, \quad \| v - P_C v \|_{0,E} \leq C h_E^{1/2} \| v \|_{1, D_E^{(1)}}.
\]

Further, we have

\[
\left( \sum_{T \in T_h} \| v \|_{\mu, D_T^{(1)}}^2 \right)^{1/2} \leq C \| v \|_{\mu, \Omega}, \quad 0 \leq \mu \leq 1,
\]

\[
\left( \sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} \| v \|_{\mu, D_E^{(1)}}^2 \right)^{1/2} \leq C \| v \|_{\mu, \Omega}, \quad 0 \leq \mu \leq 1.
\]

where \( D_T^{(1)} := \bigcup \{ T' \in T_h \mid \mathcal{N}_h(T') \cap \mathcal{N}_h(T) \neq \emptyset \} \), \( D_E^{(1)} := \bigcup \{ T' \in T_h \mid \mathcal{N}_h(E) \cap \mathcal{N}_h(T') \neq \emptyset \} \).
**Element and Edge Bubble Functions I**

The element bubble function $\psi_T$ is defined by means of the barycentric coordinates $\lambda_i^T, 1 \leq i \leq 3$, according to

$$
\psi_T := 27 \lambda_1^T \lambda_2^T \lambda_3^T.
$$

Note that $\text{supp } \psi_T = T_{\text{int}}$, i.e., $\psi_T \mid_{\partial T} = 0$, $T \in \mathcal{T}_h$. On the other hand, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ and $T \in \mathcal{T}_h$ such that $E \subset \partial T$ and $p_i^E \in \mathcal{N}_h(E)$, $1 \leq i \leq 2$, we introduce the edge-bubble functions $\psi_E$

$$
\psi_E := 4 \lambda_1^T \lambda_2^T.
$$

Note that $\psi_E \mid_{E'} = 0$ for $E' \in \mathcal{E}_h(T), E' \neq E$. 

Element and Edge Bubble Functions II

The bubble functions $\psi_T$ and $\psi_E$ have the following important properties that can be easily verified taking advantage of the affine equivalence of the finite elements:

**Lemma.** There holds

\[
\|p_h\|_{0,T}^2 \leq C \int_T p_h^2 \psi_T \, dx, \quad p_h \in P_1(T),
\]

\[
\|p_h\|_{2,0,T} \leq C h_T^{-1} \|p_h\|_{0,T}, \quad p_h \in P_1(T),
\]

\[
\|p_h\|_{0,E}^2 \leq C \int_E p_h^2 \psi_E \, d\sigma, \quad p_h \in P_1(E),
\]

\[
|p_h \psi_T|_{1,T} \leq C h_T^{-1} \|p_h\|_{0,T}, \quad p_h \in P_1(T),
\]

\[
\|p_h \psi_T\|_{0,T} \leq C \|p_h\|_{0,T}, \quad p_h \in P_1(T),
\]

\[
\|p_h \psi_E\|_{0,E} \leq C \|p_h\|_{0,E}, \quad p_h \in P_1(E).
\]
Element and Edge Bubble Functions III

For functions $p_h \in P_1(E)$, $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we further need an extension $p_h^E \in L^2(T)$ where $T \in \mathcal{T}_h$ such that $E \subset \partial T$. For this purpose we fix some $E' \subset \partial T$, $E' \neq E$, and for $x \in T$ denote by $x_E$ that point on $E$ such that $(x - x_E) \parallel E'$. For $p_h \in P_1(E)$ we then set

$$p_h^E := p_h(x_E).$$

Further, for $E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)$ we define $D_E^{(2)}$ as the union of elements $T \in \mathcal{T}_h$ containing $E$ as a common edge

$$D_E^{(2)} := \bigcup \{ K \in \mathcal{T}_h \mid E \in \mathcal{E}_h(T) \}.$$
Element and Edge Bubble Functions IV

Lemma. There holds

\[ |p_h^E \psi_E|_{1,D_E^{(2)}} \leq C h_E^{-1/2} \|p_h\|_{0,e}, \quad p_h \in P_1(E), \]

\[ \|p_h^E \psi_E\|_{0,D_E^{(2)}} \leq C h_E^{1/2} \|p_h\|_{0,E}, \quad p_h \in P_1(E). \]

Further, for all \( v \in V \) and \( \mu = 0, 1 \) there holds

\[ \left( \sum_{E \in \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_N)} h_E^{1-\mu} \|v\|_{\mu,D_E^{(2)}}^{2} \right)^{1/2} \leq C \left( \sum_{T \in \mathcal{T}_h} h_T^{1-\mu} \|v\|_{\mu,T}^{2} \right)^{1/2}. \]
Step MARK of the Adaptive Cycle: Bulk Criterion

Given a universal constant $0 < \Theta < 1$, specify a set $\mathcal{M}_T$ of elements and a set $\mathcal{M}_E$ of edges such that (bulk criterion, Dörfler marking)

$$\Theta \left( \sum_{T \in \mathcal{T}_h(\Omega)} \eta_T^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_E^2 \right) \leq \sum_{T \in \mathcal{M}_T} \eta_T^2 + \sum_{E \in \mathcal{M}_E} \eta_E^2.$$

Step REFINE of the Adaptive Cycle: Refinement Rules

- Any $T \in \mathcal{M}_T, E \in \mathcal{M}_E$ is refined by bisection.
- Further bisection is used to create a geometrically conforming triangulation $\mathcal{T}_h(\Omega)$. 
Adaptive Finite Element Methods for Unconstrained Optimal Elliptic Control Problems

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Elliptic Optimal Control Problems: Unconstrained Case

Let $\Omega$ be a bounded polygonal domain with boundary $\Gamma = \partial \Omega$. Given a desired state $y^d \in L^2(\Omega)$, $f \in L^2(\Omega)$, and $\alpha > 0$, find $(y, u) \in H^1_0(\Omega) \times L^2(\Omega)$ such that

$$
\inf_{(y, u)} J(y, u) := \frac{1}{2} \int_{\Omega} |y - y^d|^2 \, dx + \frac{\alpha}{2} \int_{\Omega} |u|^2 \, dx,
$$

subject to $-\Delta y = u$ in $\Omega$,

$$
y = 0 \quad \text{on} \quad \Gamma.
$$
Reduced Optimality Conditions in $y$ and $p$

Substituting $u$ in the state equation by $p = \alpha u$, we arrive at the following system of two variational equations:

$$\begin{align*}
    a(y, v) &= \alpha^{-1}(p, v)_{0, \Omega} = \ell_1(v), \quad v \in V := H^1_0(\Omega), \\
    a(p, w) + (y, w)_{0, \Omega} &= \ell_2(w), \quad w \in V,
\end{align*}$$

where the functionals $\ell_\nu : V \to \mathbb{R}, 1 \leq \nu \leq 2$, are given by

$$
\ell_1(v) := 0, \quad v \in V, \quad \ell_2(w) := (y^d, w)_{0, \Omega}, \quad w \in V.
$$

The operator-theoretic formulation reads

$$
\mathcal{L}(y, p) = (\ell_1, \ell_2)^T,
$$

where the operator $\mathcal{L} : V \times V \to V^* \times V^*$ is defined according to

$$
(\mathcal{L}(y, p))(v, w) := a(y, v) - \alpha^{-1}(p, v)_{0, \Omega} + a(p, w) + (y, w)_{0, \Omega}.
$$
Operator Theoretic Formulation of the Optimality System I

**Theorem.** The operator $\mathcal{L}$ is a continuous, bijective linear operator. Hence, for any $(\ell_1, \ell_2) \in V^* \times V^*$ the system admits a unique solution $(y, p) \in V \times V$. The solution depends continuously on the data according to

$$\| (y, p) \|_{V \times V} \leq C \| (\ell_1, \ell_2) \|_{V^* \times V^*}.$$

**Proof.** The linearity and continuity are straightforward. For the proof of the inf-sup condition, we choose $v = \alpha y - p$ and $w = p + y$. It follows that

$$(\mathcal{L}(y, p))(\alpha y - p, y + p) = \alpha \ a(y, y) + a(p, p) + (y, y)_{0, \Omega} + \alpha^{-1} (p, p)_{0, \Omega},$$

which allows to conclude.
Operator Theoretic Formulation of the Optimality System II

Corollary. Let \((y_h, p_h) \in V_h \times V_h, V_h \subset V\), be an approximate solution of \((y, p) \in V \times V\).

Then, there holds
\[
\| (y - y_h, p - p_h) \|_{V \times V} \leq C \| (Res_1, Res_2) \|_{V^* \times V^*},
\]
where the residuals \(Res_1 \in V^*, Res_2 \in V^*\) are given by
\[
Res_1(v) := \ell_1(v) - a(y_h, v) + \alpha^{-1}(p_h, v)_{0, \Omega}, \quad v \in V,
\]
\[
Res_2(w) := \ell_2(w) - a(p_h, w) - (y_h, w)_{0, \Omega}, \quad w \in W.
\]

Proof. The assertion is an immediate consequence of the previous theorem.
Using Galerkin orthogonality and Clément’s quasi-interpolation operator $P_C$, for the first residual $\text{Res}_1$ we find

$$\text{Res}_1(v) = \sum_{T \in T_h(\Omega)} (f, v - P_C v)_0,T - \sum_{T \in T_h(\Omega)} \left( a(u_h, v - P_C v) + \alpha^{-1}(p_h, v - P_C v)_0,T \right).$$

By an elementwise application of Green’s formula and the local approximation properties of $P_C$ it follows that

$$\|\text{Res}_1\|_{V^*} \leq C \left( \sum_{T \in T_h(\Omega)} \eta_{T,1}^2 + \sum_{E \in \partial_h(\Omega)} \eta_{E,1}^2 \right)^{1/2},$$

The local residuals are given by

$$\eta_{T,1} := h_T \|\Delta y_h + u_h\|_{0,T},$$

$$\eta_{E,1} := h_E^{1/2} \| n \cdot [\nabla y_h] \|_{0,E}. $$
Likewise, for the second residual $\text{Res}_2$ we obtain
\[
\|\text{Res}_2\|_{V^*} \leq C \left( \sum_{T \in T_h(\Omega)} \eta_{T,2}^2 + \sum_{E \in \mathcal{E}_h(\Omega)} \eta_{E,2}^2 \right)^{1/2},
\]
where the local residuals are given by
\[
\eta_{T,2} := h_T \| y^d + \Delta p_h - y_h \|_{0,T}, \quad T \in T_h(\Omega),
\]
\[
\eta_{E,2} := h_E^{1/2} \| n \cdot [\nabla p_h] \|_{0,E}, \quad E \in \mathcal{E}_h(\Omega).
\]
Reliability of the Residual-Type A Posteriori Error Estimator

Theorem. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there holds
\[
\|(y - y_h, p - p_h)\|_{V \times V} \leq C\eta_h,
\]
where the estimator \(\eta_h\) is given by
\[
\eta_h := \left( \sum_{T \in T_h(\Omega)} (\eta_{T,1}^2 + \eta_{T,2}^2) + \sum_{E \in \mathcal{E}_h(\Omega)} (\eta_{E,1}^2 + \eta_{E,2}^2) \right)^{1/2}.
\]
Efficiency of the Residual-Type A Posteriori Error Estimator I

Lemma. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant \(c\) depending only on the shape regularity of \(\{\mathcal{T}_h(\Omega)\}\) such that for \(T \in \mathcal{T}_h(\Omega)\)

\[
\eta^2_{T, 1} \leq c \left( |y - y_h|_{1,T}^2 + h_T^2 \|u - u_h\|_{0,T}^2 \right).
\]

Proof. Setting \(z_h := u_h|_T \psi_T\) and observing that \(\Delta y_h|_T = 0\), Green’s formula and the fact that \(z_h\) is an admissible test function imply

\[
\eta^2_{T, 1} = h_T^2 \|u_h\|_{0,T}^2 \leq c \ h_T^2 \ (u_h + \Delta y_h, z_h)_{0,T} = c \ h_T^2 \ (-a(y_h, z_h) + (u, z_h)_{0,T} + (u_h - u, z_h)_{0,T}) \leq c (h_T^2 |y - y_h|_{1,T} |z_h|_{1,T} + h_T^2 \|u - u_h\|_{0,T} \|z_h\|_{0,T}).
\]
Proof cont’d. By the property of the element bubble function

\[ | p_h \psi_T |_{1,T} \leq c h_T^{-1} \| p_h \|_{0,T}, \quad p_h \in P_1(T), \]

and Young’s inequality we obtain

\[ h_T^2 \| u_h \|_{0,T}^2 \leq c (|y - y_h|_{1,T}^2 + h_T^2 \| u - u_h \|_{0,T}^2) + \frac{1}{2} h_T^2 \| u_h \|_{0,T}^2, \]

which gives the assertion.
Efficiency of the Residual-Type A Posteriori Error Estimator II

Lemma. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant \(c\) depending only on the shape regularity of \(\{T_h(\Omega)\}\) such that for \(T \in T_h(\Omega)\)

\[
\eta_{T,2}^2 \leq c \left( |p - p_h|_{1,T}^2 + h_T^2 \|y - y_h\|_{0,T}^2 + \text{osc}_T^2 \right),
\]

where

\[
\text{osc}_T := h_T \|y_d - y_h^d\|_{0,T}, \quad T \in T_h(\Omega).
\]

Proof. The assertion can be proved along the same lines as in the previous lemma.
Efficiency of the Residual-Type A Posteriori Error Estimator III

Lemma. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant \(c\) depending only on the shape regularity of \(\{T_h(\Omega)\}\) such that for \(E \in \mathcal{E}_h(\Omega)\)

\[
\eta_{E,1}^2 \leq c \left( |y - y_h|_{1,\omega_E}^2 + h_E^2 \|u - u_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^2 \eta_{T_{\nu},1}^2 \right).
\]

Proof. We set \(\zeta_E := (n_E \cdot [\nabla y_h])|_E\) and \(z_h := \tilde{\zeta}_E \psi_E\). Then, applying Green’s formula and observing that \(z_h\) is an admissible test function, we find

\[
\eta_{E,1}^2 = h_E \|n_E \cdot [\nabla y_h]\|_{0,E}^2 \leq c \ h_E \ (n_E \cdot [\nabla y_h], \zeta_E \psi_E)|_{0,E} = c \ h_E \ \sum_{\nu=1}^2 (n_{\partial T_{\nu}} \cdot [\nabla y_h], z_h)_{0,\partial T_{\nu}}
\]

\[
= c \ h_E \ (a(y_h - y, z_h) + (u - u_h, z_h)_{0,\omega_E} + (f + u_h, z_h)_{0,\omega_E})
\]

\[
\leq c \ h_E^{1/2} \|\nu_E \cdot [\nabla y_h]\|_{0,E} (|y - y_h|_{1,\omega_E}(h_E \|u - u_h\|_{0,\omega_E} + (\sum_{\nu=1}^2 \eta_{T_{\nu},1}^2)^{1/2})),
\]

which allows to conclude.
Efficiency of the Residual-Type A Posteriori Error Estimator IV

Lemma. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there exists a positive constant \(c\) depending only on the shape regularity of \(\{T_h(\Omega)\}\) such that for \(E \in \mathcal{E}_h(\Omega)\)

\[
\eta^2_{E,2} \leq c (|p - p_h|_{1,\omega_E}^2 + h_E^2 \|y - y_h\|_{0,\omega_E}^2 + \sum_{\nu=1}^{2} \eta_{T\nu,2}^2).
\]

Proof. The proof is similar to the one in the previous lemma.
Efficiency of the Residual-Type A Posteriori Error Estimator V

Theorem. Let \((y, p) \in V \times V\) and \((y_h, p_h) \in V_h \times V_h\) be the solutions of the continuous and discrete optimality system, respectively. Then, there exist positive constants \(C\) and \(c\) depending only on \(\Omega\) and the shape regularity of the triangulations such that

\[
\| (y - y_h, p - p_h) \|_{V \times V}^2 + \| u - u_h \|_{0, \Omega}^2 \geq C \eta_h^2 - c \text{osc}_h^2.
\]

where

\[
\text{osc}_h^2 := \sum_{T \in \mathcal{T}_h(\Omega)} \text{osc}_T^2.
\]

Proof. Combining the results of the previous four lemmas gives the assertion.