Numerical analysis of a control and state constrained elliptic control problem with piecewise constant control approximations

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Abstract We consider an elliptic optimal control problem with control and pointwise state constraints. The cost functional is approximated by a sequence of functionals which are obtained by discretizing the state equation with the help of linear finite elements and enforcing the state constraints in the nodes of the triangulation. Controls are discretized piecewise constant on every simplex of the triangulation. Error bounds for control and state are obtained both in two and three space dimensions.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ $(d = 2,3)$ be a bounded domain with a smooth boundary $\partial\Omega$ and consider the differential operator

$$Ay := -\sum_{i,j=1}^{d} \partial_{x_j}\left(a_{ij} y_{x_i}\right) + \sum_{i=1}^{d} b_i y_{x_i} + cy,$$

where for simplicity the coefficients a_{ij}, b_i and c are supposed to be smooth functions on $\bar{\Omega}$. Furthermore, the operator A is assumed to be uniformly elliptic. We associate with A the bilinear form

$$a(y,z) := \int_{\Omega} \big(\sum_{i,j=1}^{d} a_{ij}(x) y_{x_i} z_{x_j} + \sum_{i=1}^{d} b_i(x) y_{x_i} z + c(x) yz \big) dx, \qquad y,z \in H^1(\Omega)$$

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and suppose that this form is coercive on $H^1(\Omega)$ with constant $c_1 > 0$. From the above assumptions it follows that for a given $f \in (H^1(\Omega))'$ the elliptic boundary value problem

$$\begin{aligned} Ay &= f \text{ in } \Omega \\ \textstyle\sum_{i,j=1}^d a_{ij} y_{x_i} \nu_j &= 0 \text{ on } \partial\Omega \end{aligned} \tag{1}$$

has a unique weak solution $y \in H^1(\Omega)$ which we denote by $y = \mathscr{G}(f)$. Here, ν is the unit outward normal to $\partial\Omega$. Furthermore, if $f \in L^2(\Omega)$, then the solution y belongs to $H^2(\Omega)$ and satisfies

$$\|y\|_{H^2} \le C\|f\|. \tag{2}$$

In the above, $\|\cdot\| = \|\cdot\|_{L^2}$ and $\|\cdot\|_{H^m} = \|\cdot\|_{W^{m,2}}$, where $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{m,p}}$ denote the norms in $L^p(\Omega)$ and $W^{m,p}(\Omega)$ respectively.
We are interested in the following control problem

$$\begin{aligned} &\min_{u \in U_{ad}} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u|^2 \\ &\text{subject to } y = \mathscr{G}(u) \text{ and } y(x) \le b(x) \text{ in } \Omega. \end{aligned} \tag{3}$$

Here, $U_{ad} := \{v \in L^2(\Omega) \,|\, a_l \le v \le a_u \text{ a.e. in } \Omega\} \subseteq L^2(\Omega)$ denotes the set of admissible controls, where $\alpha > 0$ and $a_l < a_u$ are given constants. Furthermore, we suppose that $y_0 \in H^1(\Omega)$ and $b \in W^{2,\infty}(\Omega)$ are given functions.
For the case without control constraints the finite element analysis of problem (3) is carried out in [7]. In the present work we extend the analysis to the case of control and pointwise state constraints using techniques which are applicable to a wider class of control problems.

From here onwards we impose the following assumption which is frequently referred to as *Slater condition* or interior point condition.

Assumption 1 $\exists \tilde{u} \in U_{ad} \quad \mathscr{G}(\tilde{u}) < b$ *in* $\bar{\Omega}$.

Since the state constraints form a convex set and the set of admissible controls is closed and convex it is not difficult to establish the existence of a unique solution $u \in U_{ad}$ to this problem. In order to characterize this solution we introduce the space $\mathscr{M}(\bar{\Omega})$ of Radon measures which is defined as the dual space of $C^0(\bar{\Omega})$ and endowed with the norm

$$\|\mu\|_{\mathscr{M}(\bar{\Omega})} = \sup_{f \in C^0(\bar{\Omega}), |f| \le 1} \int_{\bar{\Omega}} f \, d\mu.$$

Using [3, Theorem 5.2] we then infer (compare also [2, Theorem 2])

Theorem 1. *Let $u \in U_{ad}$ denote the unique solution of* (3)*. Then there exist $\mu \in \mathscr{M}(\bar{\Omega})$ and $p \in L^2(\Omega)$ such that with $y = \mathscr{G}(Bu)$ there holds*

$$\int_{\Omega} pAv = \int_{\Omega} (y - y_0)v + \int_{\bar{\Omega}} v d\mu \;\forall v \in H^2(\Omega) \; with \; \sum_{i,j=1}^{d} a_{ij} v_{x_i} \nu_j = 0 \; on \; \partial\Omega \quad (4)$$

$$\int_{\Omega} (p + \alpha u)(v - u) \geq 0 \qquad \forall v \in U_{ad} \quad (5)$$

$$\mu \geq 0, \; y(x) \leq b(x) \; in \; \Omega \; and \; \int_{\bar{\Omega}} (b - y) d\mu = 0. \quad (6)$$

Our aim is to develop and analyze a finite element approximation of problem (3). We start by approximating the cost functional J by a sequence of functionals J_h where h is a mesh parameter related to a sequence of triangulations. The definition of J_h involves the approximation of the state equation by linear finite elements. The controls are discretized by piecewise constant functions which satisfy the constraints elementwise. Denoting by u_h the corresponding minimum of J_h with associate state y_h we shall prove the following error bounds,

$$\|u - u_h\|_{L^2}, \; \|y - y_h\|_{H^1} \leq \begin{cases} Ch|\log h|, & \text{if } d = 2 \\ C\sqrt{h}, & \text{if } d = 3. \end{cases}$$

To the authors knowledge only few attempts have been made to develop a finite element analysis for elliptic control problems in the presence of control and state constraints. In [4] Casas proves convergence of finite element approximations to optimal control problems for semi-linear elliptic equations with finitely many state constraints. Casas and Mateos extend these results in [5] to a less regular setting for the states and prove convergence of finite element approximations to semi-linear distributed and boundary control problems. In [9] Meyer considers the same discrete strategy as in the present note to approximate an elliptic control problem with pointwise state and control constraints and proves

$$\|u - u_h\|_{L^2}, \; \|y - y_h\|_{H^1} \leq \begin{cases} C_\varepsilon h^{1-\varepsilon}, & \text{if } d = 2 \\ C_\varepsilon h^{\frac{1}{2}-\varepsilon}, & \text{if } d = 3 \end{cases}$$

($\varepsilon > 0$ arbitrary). Our analysis differs from the one presented in [9] and we obtain a slighly better approximation order for the state and the control. Moreover we prove bounds on the discrete multipliers. For numerical tests we also refer to [9]. Numerical analysis for elliptic control problems with pointwise bounds on the state and general constraints on the control are presented by the authors in [8], and for pointwise bounds on the gradient of the state in [6].

The paper is organized as follows: in §2 we describe our discretization and establish bounds on the relevant discrete quantities which are uniform in the discretization parameter. These bounds are used in §3 in order to carry out the error analysis. Roughly speaking, the idea is to test (5) with u_h and (13), the discrete counterpart of (5), with a suitable projection of the continuous solution u. An important tool in the analysis is the use of L^∞–error estimates for finite element approximations of the Neumann problem developed in [10]. The need for uniform estimates is due to the presence of the measure μ in (4).

2 Finite element discretization

Let $\mathscr{T}_h$ be a triangulation of Ω with maximum mesh size $h := \max_{T\in\mathscr{T}_h} \mathrm{diam}(T)$ and vertices $x_1, \dots, x_m$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathscr{T}_h$ so that element edges lying on the boundary are curved. In addition, we assume that the triangulation is quasi-uniform in the sense that there exists a constant $\kappa > 0$ (independent of h) such that each $T \in \mathscr{T}_h$ is contained in a ball of radius $\kappa^{-1}h$ and contains a ball of radius κh. Let us define the space of linear finite elements

$$X_h := \{v_h \in C^0(\bar{\Omega}) \,|\, v_h \text{ is a linear polynomial on each } T \in \mathscr{T}_h\}$$

as well as the space of piecewise constant functions

$$Y_h := \{v_h \in L^2(\Omega) \,|\, v_h \text{ is constant on each } T \in \mathscr{T}_h\}.$$

Let $Q_h : L^2(\Omega) \to Y_h$ be the orthogonal projection onto Y_h so that

$$(Q_h v)(x) := \fint_T v, \quad x \in T, T \in \mathscr{T}_h,$$

where $\fint_T v$ denotes the average of v over T. In what follows it is convenient to introduce a discrete approximation of the operator $\mathscr{G}$. For a given function $v \in L^2(\Omega)$ we denote by $z_h = \mathscr{G}_h(v) \in X_h$ the solution of the discrete Neumann problem

$$a(z_h, v_h) = \int_\Omega v v_h \;\text{ for all } v_h \in X_h.$$

It is well–known that for all $v \in L^2(\Omega)$

$$\|\mathscr{G}(v) - \mathscr{G}_h(v)\| \le Ch^2 \|v\|. \tag{7}$$

The corresponding estimate in L^∞ will be crucial for our analysis.

Lemma 1. *There exists a constant C which only depends on the data such that*

$$\|\mathscr{G}(v) - \mathscr{G}_h(v)\|_{L^\infty} \le Ch^2 |\log h|^2 \qquad \textit{for all } v \in U_{ad}.$$

Proof. Let $v \in U_{ad}, z = \mathscr{G}(v), z_h = \mathscr{G}_h(v)$. Since $U_{ad} \subset L^\infty(\Omega)$ with $\|v\|_{L^\infty} \le \max(|a_l|, |a_u|)$, elliptic regularity theory implies that $z \in W^{2,q}(\Omega)$ for all $1 < q < \infty$. In addition, it is well–known that one has

$$\|z\|_{W^{2,q}} \le Cq \|v\|_{L^q} \qquad (C \text{ independent of } q)$$

by tracking the constants in the analysis. As a result we have

$$\|z\|_{W^{2,q}} \le Cq \qquad \text{for all } v \in U_{ad}. \tag{8}$$

Using Theorem 2.2 and the following Remark in [10] we have

$$\|z - z_h\|_{L^\infty} \le C|\log h| \inf_{\chi \in X_h} \|z - \chi\|_{L^\infty}, \tag{9}$$

which, combined with a well–known interpolation estimate and (8), yields

$$\|z - z_h\|_{L^\infty} \le Ch^{2-\frac{d}{q}}|\log h| \|z\|_{W^{2,q}} \le Cqh^{2-\frac{d}{q}}|\log h|$$

for all $v \in U_{ad}$. Choosing $q = |\log h|$ gives the result. $\square$

In order to approximate (3) we introduce a discrete counterpart of U_{ad},

$$U_{ad}^h := \{v_h \in Y_h \,|\, a_l \le v_h \le a_u \text{ in } \Omega\}.$$

Note that $U_{ad}^h \subset U_{ad}$ and that $Q_h v \in U_{ad}^h$ for $v \in U_{ad}$. Since $Q_h v \rightharpoonup v$ in $L^2(\Omega)$ as $h \to 0$ we infer from (2), the continuous embedding $H^2(\Omega) \hookrightarrow C^0(\bar{\Omega})$ and Lemma 1 that

$$\mathscr{G}_h(Q_h v) \to \mathscr{G}(v) \text{ in } L^\infty(\Omega) \text{ for all } v \in U_{ad}. \tag{10}$$

Problem (3) is approximated by the following sequence of control problems depending on the mesh parameter h:

$$\begin{aligned} &\min_{u \in U_{ad}^h} J_h(u) := \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u|^2 \\ &\text{subject to } \; y_h = \mathscr{G}_h(u) \text{ and } y_h(x_j) \le b(x_j) \text{ for } j = 1, \dots, m. \end{aligned} \tag{11}$$

Problem (11) represents a convex finite-dimensional optimization problem of similar structure as problem (3), but with only finitely many equality and inequality constraints for state and control, which form a convex admissible set. Again we can apply [3, Theorem 5.2] which together with [2, Corollary 1] yields (compare also the analysis of problem (P) in [4])

Lemma 2. *Problem (11) has a unique solution $u_h \in U_{ad}^h$. There exist $\mu_1, \dots, \mu_m \in \mathbb{R}$ and $p_h \in X_h$ such that with $y_h = \mathscr{G}_h(u_h)$ and $\mu_h = \sum_{j=1}^m \mu_j \delta_{x_j}$ we have*

$$a(v_h, p_h) = \int_\Omega (y_h - y_0)v_h + \int_{\bar{\Omega}} v_h d\mu_h \qquad \forall v_h \in X_h, \tag{12}$$

$$\int_\Omega (p_h + \alpha u_h)(v_h - u_h) \ge 0 \qquad \forall v_h \in U_{ad}^h, \tag{13}$$

$$\mu_j \ge 0, y_h(x_j) \le b(x_j), j = 1, \dots, m \text{ and } \int_{\bar{\Omega}} \big(I_h b - y_h\big) d\mu_h = 0. \tag{14}$$

Here, δ_x denotes the Dirac measure concentrated at x and I_h is the usual Lagrange interpolation operator.

As a first result for (11) we prove bounds on the discrete states and the discrete multipliers.

Lemma 3. *Let $u_h \in U_{ad}^h$ be the optimal solution of (11) with corresponding state $y_h \in X_h$ and adjoint variables $p_h \in X_h$ and $\mu_h \in \mathscr{M}(\bar{\Omega})$. Then there exists $\bar{h} > 0$ such that*

$$\|y_h\|, \|\mu_h\|_{\mathscr{M}(\bar{\Omega})} \le C, \quad \|p_h\|_{H^1} \le C\gamma(d,h) \qquad \textit{for all } 0 < h \le \bar{h},$$

where $\gamma(2,h) = \sqrt{|\log h|}$ *and* $\gamma(3,h) = h^{-\frac{1}{2}}$.

Proof. Since $\mathscr{G}(\tilde{u}) \in C^0(\bar{\Omega})$, Assumption 1 implies that there exists $\delta > 0$ such that

$$\mathscr{G}(\tilde{u}) \le b - \delta \qquad \text{in } \bar{\Omega}. \tag{15}$$

It follows from (10) that there is $\bar{h} > 0$ with

$$\mathscr{G}_h(Q_h\tilde{u}) \le b - \frac{\delta}{2} \quad \text{in } \bar{\Omega} \text{ for all } 0 < h \le \bar{h}. \tag{16}$$

Since $Q_h\tilde{u} \in U_{ad}^h$, (13), (12) and (16) imply

$$\begin{aligned}
0 &\le \int_\Omega (p_h + \alpha u_h)(Q_h\tilde{u} - u_h) = \int_\Omega p_h(Q_h\tilde{u} - u_h) + \alpha \int_\Omega u_h(Q_h\tilde{u} - u_h) \\
&= a(\mathscr{G}_h(Q_h\tilde{u}) - y_h, p_h) + \alpha \int_\Omega u_h(Q_h\tilde{u} - u_h) \\
&= \int_\Omega (\mathscr{G}_h(Q_h\tilde{u}) - y_h)(y_h - y_0) + \int_{\bar{\Omega}} (\mathscr{G}_h(Q_h\tilde{u}) - y_h) d\mu_h + \alpha \int_\Omega u_h(Q_h\tilde{u} - u_h) \\
&\le C - \frac{1}{2}\|y_h\|^2 + \sum_{j=1}^m \mu_j \big(b(x_j) - \frac{\delta}{2} - y_h(x_j)\big) = C - \frac{1}{2}\|y_h\|^2 - \frac{\delta}{2}\sum_{j=1}^m \mu_j
\end{aligned}$$

where the last equality is a consequence of (14). It follows that $\|y_h\|, \|\mu_h\|_{\mathscr{M}(\bar{\Omega})} \le C$. In order to bound $\|p_h\|_{H^1}$ we insert $v_h = p_h$ into (12) and deduce with the help of the coercivity of A, a well–known inverse estimate and the bounds we have already obtained that

$$\begin{aligned}
c_1\|p_h\|_{H^1}^2 &\le a(p_h, p_h) = \int_\Omega (y_h - y_0)p_h + \int_{\bar{\Omega}} p_h d\mu_h \\
&\le \|y_h - y_0\| \, \|p_h\| + \|p_h\|_{L^\infty} \|\mu_h\|_{\mathscr{M}(\bar{\Omega})} \le C\|p_h\| + C\gamma(d,h)\|p_h\|_{H^1}.
\end{aligned}$$

Hence $\|p_h\|_{H^1} \le C\gamma(d,h)$ and the lemma is proved. □

3 Error analysis

An important ingredient in our analysis is an error bound for a solution of a Neumann problem with a measure valued right hand side. Let A be as above and consider

$$\begin{aligned}
A^*q &= \tilde{\mu}\llcorner\Omega \quad \text{in } \Omega \\
\textstyle\sum_{i=1}^d \big(\sum_{j=1}^d a_{ij}q_{x_j} + b_i q\big)\nu_i &= \tilde{\mu}\llcorner\partial\Omega \text{ on } \partial\Omega.
\end{aligned} \tag{17}$$

Theorem 2. Let $\tilde{\mu} \in \mathscr{M}(\bar{\Omega})$. Then there exists a unique weak solution $q \in L^2(\Omega)$ of (17), i.e.

$$\int_\Omega qAv = \int_{\bar\Omega} v d\tilde\mu \qquad \forall v \in H^2(\Omega) \text{ with } \sum_{i,j=1}^d a_{ij} v_{x_i} \nu_j = 0 \text{ on } \partial\Omega.$$

Furthermore, q belongs to $W^{1,s}(\Omega)$ for all $s \in (1, \frac{d}{d-1})$. For the finite element approximation $q_h \in X_h$ of q defined by

$$a(v_h, q_h) = \int_{\bar\Omega} v_h d\tilde\mu \qquad \text{for all } v_h \in X_h$$

the following error estimate holds:

$$\|q - q_h\| \leq Ch^{2-\frac{d}{2}} \|\tilde\mu\|_{\mathscr{M}(\bar\Omega)}. \tag{18}$$

Proof. A corresponding result is proved in [1] for the case of an operator A without transport term subject to Dirichlet conditions, but the arguments can be adapted to our situation. We omit the details. □

We are now prepared to prove our main result.

Theorem 3. *Let u and u_h be the solutions of (3) and (11) respectively. Then we have for* $0 < h \leq \bar h$

$$\|u - u_h\| + \|y - y_h\|_{H^1} \leq \begin{cases} Ch|\log h|, & \textit{if } d = 2 \\ C\sqrt{h}, & \textit{if } d = 3. \end{cases}$$

Proof. We test (5) with u_h, (13) with $Q_h u$ and add the resulting inequalities. Keeping in mind that $u - Q_h u \perp Y_h$ we obtain

$$\begin{aligned} &\int_\Omega \big(p - p_h + \alpha(u - u_h)\big)(u_h - u) \\ &\geq \int_\Omega \big(p_h + \alpha u_h\big)(u - Q_h u) = \int_\Omega (p_h - Q_h p_h)(u - Q_h u). \end{aligned}$$

As a consequence,

$$\alpha\|u - u_h\|^2 \leq \int_\Omega (u_h - u)(p - p_h) - \int_\Omega (p_h - Q_h p_h)(u - Q_h u) \equiv I + II. \tag{19}$$

Let $y^h := \mathscr{G}_h(u) \in X_h$ and denote by $p^h \in X_h$ the unique solution of

$$a(w_h, p^h) = \int_\Omega (y - y_0) w_h + \int_{\bar\Omega} w_h d\mu \qquad \text{for all } w_h \in X_h.$$

Applying Theorem 2 with $\tilde\mu = (y - y_0)dx + \mu$ we infer

$$\|p - p^h\| \leq Ch^{2-\frac{d}{2}} \big(\|y - y_0\| + \|\mu\|_{\mathscr{M}(\bar\Omega)}\big). \tag{20}$$

Recalling that $y_h = \mathscr{G}_h(u_h), y^h = \mathscr{G}_h(u)$ and observing (12) as well as the definition of p^h we can rewrite the first term in (19)

$$
\begin{aligned}
I &= \int_\Omega (u_h - u)(p - p^h) + \int_\Omega (u_h - u)(p^h - p_h) \\
&= \int_\Omega (u_h - u)(p - p^h) + a(y_h - y^h, p^h - p_h) \qquad (21)\\
&= \int_\Omega (u_h - u)(p - p^h) + \int_\Omega (y - y_h)(y_h - y^h) + \int_{\bar\Omega} (y_h - y^h) d\mu - \int_{\bar\Omega} (y_h - y^h) d\mu_h \\
&= \int_\Omega (u_h - u)(p - p^h) - \|y - y_h\|^2 + \int_\Omega (y - y_h)(y - y^h) \\
&\quad + \int_{\bar\Omega} (y_h - y^h) d\mu + \int_{\bar\Omega} (y^h - y_h) d\mu_h.
\end{aligned}
$$

Applying Young's inequality we deduce

$$
\begin{aligned}
|I| \le \frac{\alpha}{4}\|u - u_h\|^2 - \frac{1}{2}\|y - y_h\|^2 + C\big(\|p - p^h\|^2 + \|y - y^h\|^2\big) \\
+ \int_{\bar\Omega} (y_h - y^h) d\mu + \int_{\bar\Omega} (y^h - y_h) d\mu_h. \qquad (22)
\end{aligned}
$$

Let us estimate the integrals involving the measures μ and μ_h. Since $y_h - y^h \le (I_h b - b) + (b - y) + (y - y^h)$ in $\bar\Omega$ we deduce with the help of (6), Lemma 1 and an interpolation estimate

$$
\int_{\bar\Omega} (y_h - y^h) d\mu \le \|\mu\|_{\mathscr{M}(\bar\Omega)} \Big(\|I_h b - b\|_\infty + \|y - y^h\|_\infty\Big) \le Ch^2 |\log h|^2.
$$

On the other hand $y^h - y_h \le (y^h - y) + (b - I_h b) + (I_h b - y_h)$, so that (14), Lemma 1 and Lemma 3 yield

$$
\int_{\bar\Omega} (y^h - y_h) d\mu_h \le \|\mu_h\|_{\mathscr{M}(\bar\Omega)} \Big(\|b - I_h b\|_\infty + \|y - y^h\|_\infty\Big) \le Ch^2 |\log h|^2.
$$

Inserting these estimates into (22) and recalling (7) as well as (18) we obtain

$$
|I| \le \frac{\alpha}{4}\|u - u_h\|^2 - \frac{1}{2}\|y - y_h\|^2 + Ch^{4-d} + Ch^2|\log h|^2. \qquad (23)
$$

Let us next examine the second term in (19). Since $u_h = Q_h u_h$ and Q_h is stable in $L^2(\Omega)$ we have

$$
\begin{aligned}
|II| &\le 2\|u - u_h\| \, \|p_h - Q_h p_h\| \le \frac{\alpha}{4}\|u - u_h\|^2 + Ch^2 \|p_h\|_{H^1}^2 \\
&\le \frac{\alpha}{4}\|u - u_h\|^2 + Ch^2 \gamma(d,h)^2
\end{aligned}
$$

using an interpolation estimate for Q_h and Lemma 3. Combining this estimate with (23) and (19) we finally obtain

$$
\|u - u_h\|^2 + \|y - y_h\|^2 \le Ch^{4-d} + Ch^2|\log h|^2 + Ch^2\gamma(d,h)^2
$$

which implies the estimate on $\|u-u_h\|$. In order to bound $\|y-y_h\|_{H^1}$ we note that

$$a(y-y_h,v_h)=\int_\Omega (u-u_h)v_h$$

for all $v_h \in X_h$, from which one derives the desired estimate using standard finite element techniques and the bound on $\|u-u_h\|$. $\square$

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