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Discrete concepts for elliptic optimal control problems with constraints on the gradient of the state

Klaus Deckelnick^{*1}**, Andreas Günther**^{**2}**,** and **Michael Hinze**^{***2}

[1] Institut für Analysis und Numerik, Otto–von–Guericke–Universität Magdeburg, Universitätsplatz 2, 39106 Magdeburg, Germany

[2] Schwerpunkt Optimierung und Approximation, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

We consider elliptic optimal control problems with constraints on the gradient of the state and propose two distinguish concepts for their discretization. The first concept uses piecewise linear, continuous finite element Ansatz functions for the state, while the second concept uses the lowest order Raviart-Thomas mixed finite element. In both cases variational discretization from [5] is used for the controls. We present optimal finite element error estimates for the numerical solutions and confirm our theoretical findings by a numerical experiment.

1 The optimal control problem

Let $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) be a bounded domain with a smooth boundary $\partial\Omega$ and consider an uniformly elliptic, coercive differential operator $\mathcal{A}y := -\sum_{i,j=1}^{d} \partial_{x_j}(a_{ij} y_{x_i}) + a_0 y$, with associated bilinear form a defined on $H^1(\Omega)$. We are interested in finite element analysis of the following control problem

$$\min_{u \in U_{\text{ad}}} J(u) = \frac{1}{2}\int_\Omega |y - y_0|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \text{ subject to } y = \mathcal{G}(u) \text{ and } \nabla y \in \mathcal{K}. \tag{1}$$

Here $y = \mathcal{G}(u)$ denotes the unique solution $y \in W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)$ for the elliptic boundary value problem

$$\begin{array}{rcll} \mathcal{A}y & = & u + f & \text{in} \quad \Omega, \\ y & = & 0 & \text{on} \quad \partial\Omega, \end{array} \tag{2}$$

where $u, f \in L^r(\Omega)$ ($1 < r < \infty$) are given functions. The further quantities in problem (1) are given by $y_0 \in L^2(\Omega)$, $\alpha > 0$ and $\mathcal{K} := \{\vec{z} \in C^0(\bar{\Omega})^2 \,|\, |\vec{z}(x)| \le \delta, x \in \bar{\Omega}\}$ for a fixed $\delta > 0$. We consider two different scenarios, namely

i) $r > d, U_{\text{ad}} = L^r(\Omega)$, and

ii) $r = 2, U_{\text{ad}} = \{u \in L^2(\Omega) \,|\, a \le u \le b \text{ a.e. in } \Omega\}$, where $a < b$ are fixed constants.

In the following let $f = 0$. Since $\mathcal{K}$ is convex, problem (1) admits a unique solution $u \in U_{\text{ad}}$. For our numerical analysis it is sufficient to suppose the following Slater condition: $\exists \hat{u} \in U_{\text{ad}} : |(\nabla\mathcal{G}(\hat{u}))(x)| < \delta \,\forall x \in \bar{\Omega}$. The analysis is carried out in [2].

2 Structure exploiting discretization concepts

We use variational discretization [5] for the controls and in the following subsections we present two approaches for the discretization of the states. In both cases we end up with a convex, infinite-dimensional optimal control problem with finitely many constraints on the gradient of the state. Now let $\mathcal{T}_h$ be a quasi-uniform triangulation of Ω with maximum mesh size $h := \max_{T \in \mathcal{T}_h} \text{diam}(T)$. We suppose that $\bar{\Omega}$ is the union of the elements of $\mathcal{T}_h$ so that element edges lying on the boundary are curved.

2.1 Linear finite elements

In [4] scenario i) is investigated and the state equation is discretized with piecewise linear finite elements in the space $X_h := \{v_h \in C^0(\bar{\Omega}) \,|\, v_{h|T} \in \mathcal{P}^1(T) \,\forall\, T \in \mathcal{T}_h\}$ due to a discretized solution operator $\mathcal{G}_h$. Now the variational discretized optimal control problem reads

$$\min_{u \in U_{\text{ad}}} J(u) = \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{r}\int_\Omega |u|^r \text{ subject to } y_h = \mathcal{G}_h(u) \text{ and } (\nabla y_{h|T})_{T \in \mathcal{T}_h} \in \mathcal{K}_h, \tag{3}$$

where $\mathcal{K}_h := \{\vec{z}_h : \bar{\Omega} \to \mathbb{R}^d \,|\, \vec{z}_{h|T} = const \text{ and } |\vec{z}_{h|T}| \le \delta, T \in \mathcal{T}_h\}$. In [4] we present the proof of

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[*] E-mail: klaus.deckelnick@mathematik.uni-magdeburg.de

[**] E-mail: andreas.guenther@math.uni-hamburg.de

[***] E-mail: michael.hinze@uni-hamburg.de

Theorem 2.1 *Let $u \in L^r(\Omega)$ and $u_h \in L^r(\Omega)$ be the solutions of* (1) *and* (3) *with corresponding states y and y_h.*

$$\exists\, h_1 > 0: \quad \|y - y_h\| \le C h^{\frac{1}{2}(1-\frac{d}{r})} \quad \textit{and} \quad \|u - u_h\|_{L^r(\Omega)} \le C h^{\frac{1}{r}(1-\frac{d}{r})} \quad \textit{for all } 0 < h \le h_1. \tag{4}$$

2.2 Mixed finite elements

In [3] scenario ii) is investigated and the state equation (2) is written in a mixed formulation by introducing a new variable $\vec{v} = A\nabla y$, where $A = (a_{ij}(x))_{i,j=1}^d$. Then for a given function $u \in L^2(\Omega)$ $\mathcal{G}_h(u) = (y_h, \vec{v}_h) \in Y_h \times RT_{0,h}$ is given as a solution of a coupled system, where $Y_h := \{z_h \in L^2(\Omega) \,|\, z_{h|T} = const, T \in \mathcal{T}_h\}$ and $RT_{0,h}$ denotes the space of lowest order Raviart-Thomas elements. Now the variational discretized optimal control problem reads

$$\min_{u \in U_{\text{ad}}} J(u) = \frac{1}{2}\int_\Omega |y_h - y_0|^2 + \frac{\alpha}{2}\int_\Omega |u|^2 \text{ subject to } (y_h, \vec{v}_h) = \mathcal{G}_h(u) \text{ and } \left(\fint_T A^{-1}\vec{v}_h\right)_{T \in \mathcal{T}_h} \in \mathcal{K}_h. \tag{5}$$

In [3] we present the proof of

Theorem 2.2 *Let $u \in U_{ad}$ and $u_h \in U_{ad}$ be the solutions of* (1) *and* (5) *with corresponding states y and y_h.*

$$\exists\, h_0 > 0: \quad \|u - u_h\| + \|y - y_h\| \le C h^{\frac{1}{2}} |\log h|^{\frac{1}{2}} \quad \textit{for all } 0 < h \le h_0. \tag{6}$$

3 Numerical example

We investigate an example considered in [3] with domain $\Omega = B_2(0) \subset \mathbb{R}^2$ and differential operator $\mathcal{A} = -\Delta$. The further parameters and given functions are $\alpha = 1$, $\delta = \frac{1}{2}$,

$$y_0(x) = \begin{cases} \frac{1}{4} + \frac{1}{2}\log 2 - \frac{1}{4}|x|^2, & 0 \le |x| \le 1, \\ \frac{1}{2}\log 2 - \frac{1}{2}\log|x|, & 1 < |x| \le 2 \end{cases}$$

and $f = 2\,\mathbb{1}_{B_1(0)}$. For both scenarios i) with $r = 4, U_{\text{ad}} = L^4(\Omega)$ and ii) with $r = 2, U_{\text{ad}} = \{u \in L^2(\Omega) \,|\, -2 \le u \le 2$ a.e. in $\Omega\}$, the optimal control is given by $u = -\mathbb{1}_{B_1(0)}$ with corresponding state $y = y_0$ and adjoint state $p = -u$. For numerical results of scenario i) with linear finite elements we refer to [4]. The numerical solution of scenario ii) is obtained in Matlab with the help of a toolbox provided in [1]. Figures 1 and 2 show the optimal control u_h and corresponding state y_h at refinement level RL $= 4$ with 1089 gridpoints. In Table 1 we investigate the experimental order of convergence EOC $= (\log E(h_1) - \log E(h_2))/(\log h_1 - \log h_2)$ for the error functionals $E_u(h) := \|u - u_h\|$ and $E_y(h) := \|y - y_h\|$. The error in the control behaves as predicted by Theorem 2.2, whereas the L^2 norm of the state seems to converge linearly.

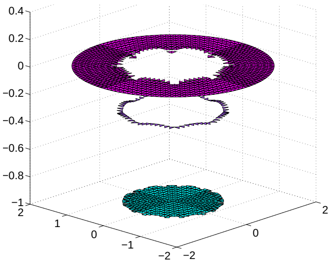

Fig. 1 Optimal control u_h.

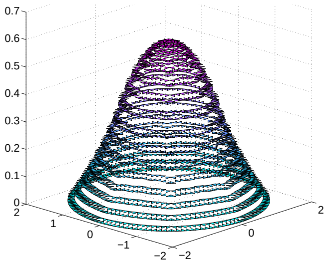

Fig. 2 Optimal state y_h.

Table 1 EOC for the error in the control and state.

RL	$E_u(h)$	$E_y(h)$
1-2	0.98576	1.06726
2-3	0.51814	1.02547
3-4	0.50034	1.01442

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