# On the tree-likeness of hyperbolic graphs

Matthias Hamann

Universität Hamburg

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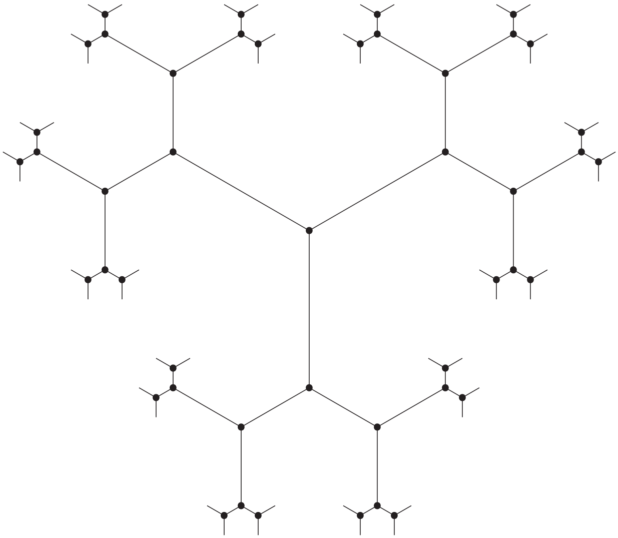
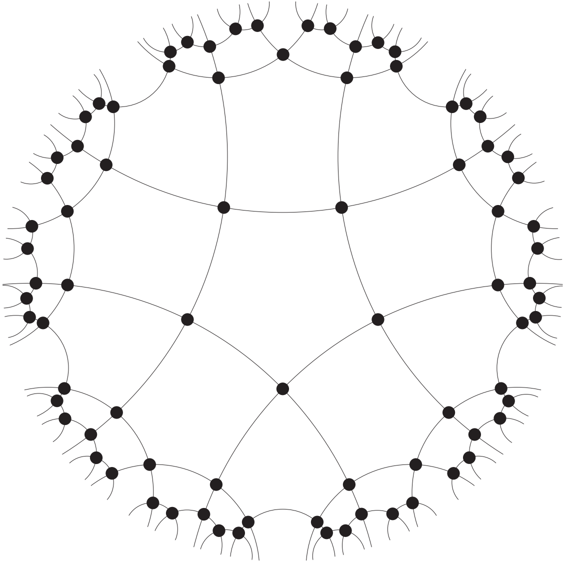
# Aim of the talk

Exhibit the tree-likeness of hyperbolic graphs

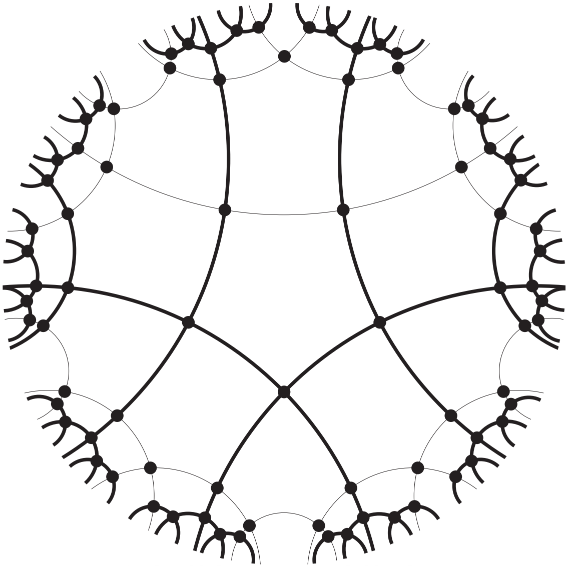
# Hyperbolic graphs: Example 1



# Hyperbolic graphs: Example 2



# Hyperbolic graphs: Example 2



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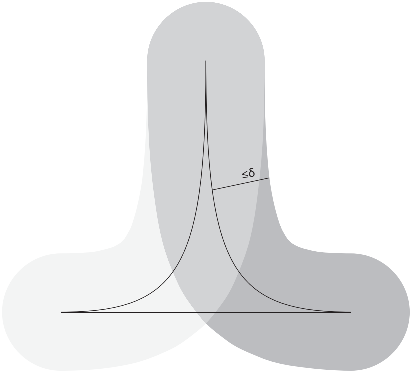

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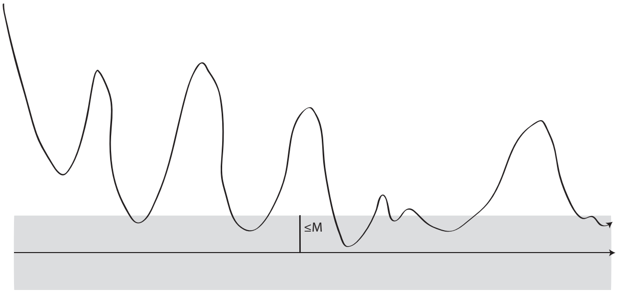

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**Proof.**

Reflexivity and symmetry: $\surd$
Transitivity: Two equivalent geodetic rays are eventually $2\delta$-close to each other. $\square$

# The hyperbolic boundary $\partial G$: Definition

The *hyperbolic boundary* $\partial G$ of a hyperbolic graph $G$ is the set of equivalence classes of geodetic rays.
Let $\widehat{G} := G \cup \partial G$.

# Ends of graphs

Two rays in a graph $G$ are *equivalent* if for any finite set $S$ of vertices they lie eventually in the same component of $G - S$.

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**Remark**

The equivalence of rays is an equivalence relation.

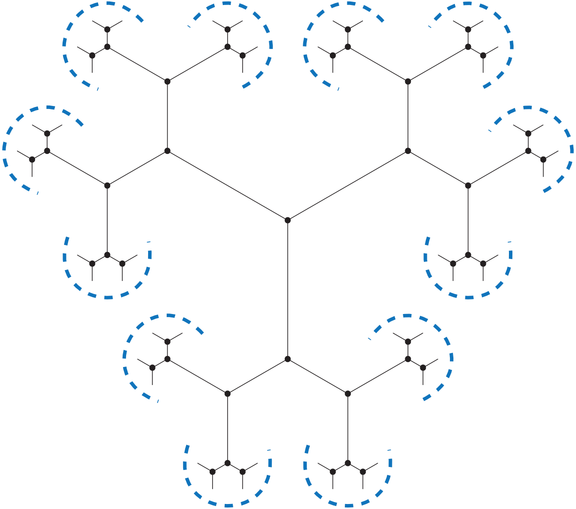
The equivalence classes of this relation are the *ends* of $G$.

# Hyperbolic boundary versus ends

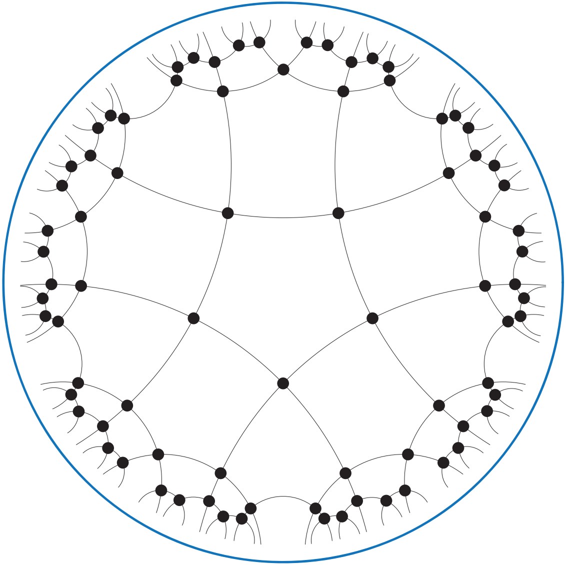
**Remark**

The hyperbolic boundary of a locally finite hyperbolic graph is a refinement of its end space.

# The hyperbolic boundary: Example 1



# The hyperbolic boundary: Example 2



# Hyperbolic metric

**Theorem (Gromov, 1987)**

*Let $G$ be a locally finite hyperbolic graph. Then there exists a metric $d_\varepsilon$ such that $(\widehat{G}, d_\varepsilon)$ is a compact metric space.*

# End-faithful spanning trees

**Theorem (Halin, 1964)**

*Every countable connected graph has an end-faithful spanning tree.*

# Spanning trees in hyperbolic graphs

For a subtree $T$ of a hyperbolic graph $G$, we say that the *canonical map $\partial T \to \partial G$ exists* if the identity $T \to G$ extends to a continuous map $\widehat{T} \to \widehat{G}$.

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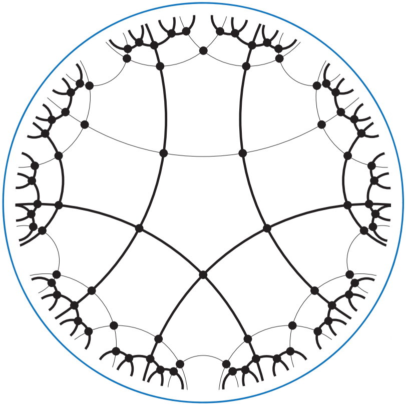

# Spanning-trees in hyperbolic graphs

**Theorem (H.)**

*For every locally finite hyperbolic graph $G$ whose hyperbolic boundary has topological dimension $n$ and for every spanning tree $T$ of $G$ such that the canonical map $\varphi : \partial T \to \partial G$ exists and is onto, there is an $\eta \in \partial G$ such that $|\varphi^{-1}(\eta)| \geq n + 1$.*

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3. *every ray in $T$ is eventually quasi-geodetic for some global constant depending only on $\dim_A(\partial G)$ and $\delta$;*
4. *there exists a constant $\Delta = \Delta(\dim_A(\partial G), \delta)$ such that for the subtree $T' \subseteq T$ that consists of all rays in $T$ that starts at the root every geodetic ray of $G$ lies eventually in $B_\Delta(T')$.*

# Hyperbolic graphs whose boundary is finite-dimensional

Every hyperbolic graph with bounded degree satisfies the assumptions of the theorem (Bonk & Schramm, 2000).

These are in particular all Cayley graphs of hyperbolic groups.

# Final remark

**Question**

*Does there exists a dimension concept that offers a lower and an upper bound for the canonical map?*