Homogeneity in graphs and digraphs

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Highly symmetric objects

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- Are they classifiable?

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Rado graph

There is a unique countable graph $\mathcal{R}$ such that for any two finite disjoint $A, B \in V(\mathcal{R})$ there is a vertex x with $A \subseteq N(x)$ and $B \cap N(x) = \emptyset$.
This graph is called Rado graph.

Highly symmetric objects

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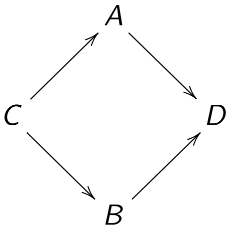
An age $\mathcal{R}$ is amalgamable if for any $A, B, C \in \mathcal{R}$ with embeddings $f\colon C \to A$ and $g\colon C \to B$ there is some $D \in \mathcal{R}$ and embeddings $f'\colon A \to D$ and $g'\colon B \to D$ with $cff' = cgg'$ for all $c \in C$.



Examples of highly symmetric structures: graphs

Theorem (Gardiner 1976, Lachlan&Woodrow 1980)

A countable graph is homogeneous iff it or its complement belongs to the following list:

- *disjoint union of cliques of the same cardinality,*
- *Rado graph,*
- *generic K_r-free graphs,*
- C_5, $L(K_{3,3})$.

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Theorem (Cameron&Johnson 1987, Cameron 2000, Cherlin 2014)

Every countable homogeneous graph is a Cayley graph.

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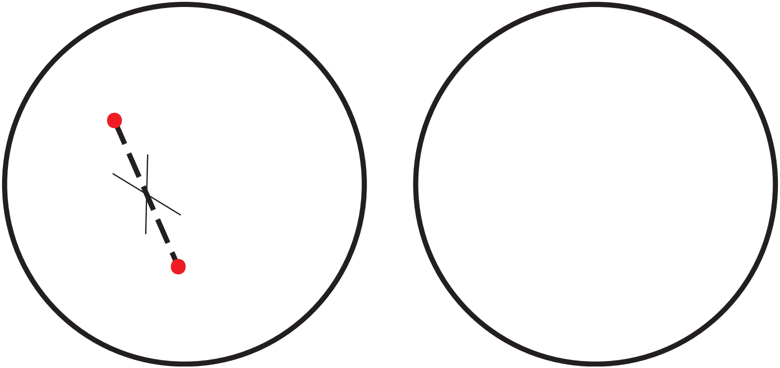

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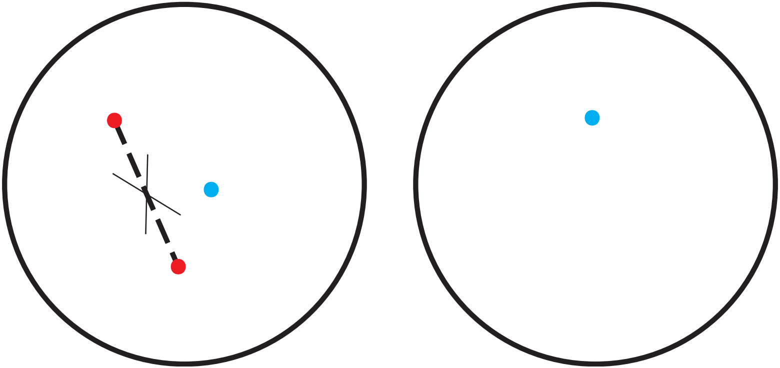

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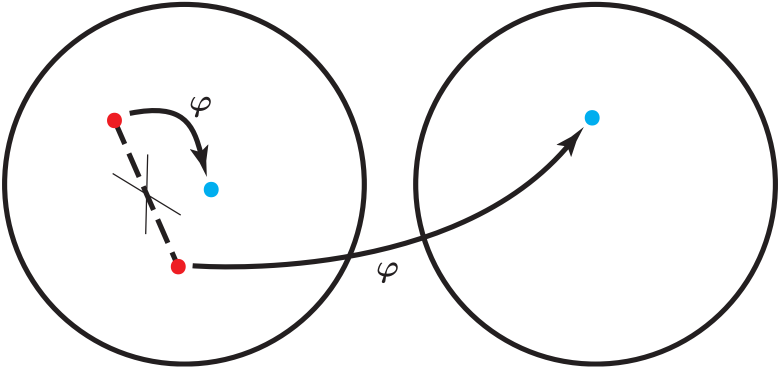

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Definition

A graph G is connected-homogeneous (or C-homogeneous) if every isomorphism between any two finite connected induced subgraphs extends to an automorphism of the whole graph.

- The notion of C-homogeneity carries over verbatim to digraphs, where a digraph is connected if its underlying undirected graph is.

homogeneous vs C-homogeneous

Remark

If a (di-)graph is homogeneous, then it is C-homogeneous

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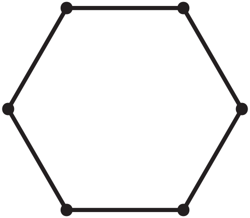

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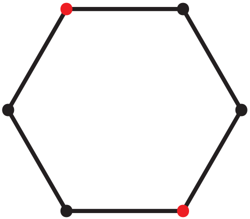

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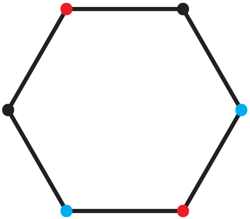

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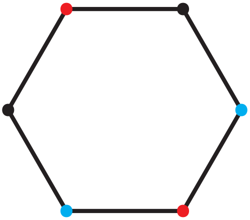

Remark

C-homogeneous (di-)graphs need not be homogeneous.

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- countable C-homogeneous digraphs:
  - connected with precisely two ends: Gray&Möller 2011,
  - connected with at least two ends: H&Hundertmark 2012,
  - finite and locally finite: H '15^{+}, and
  - all: H '15^{++}

Strategy

Strategy

Question

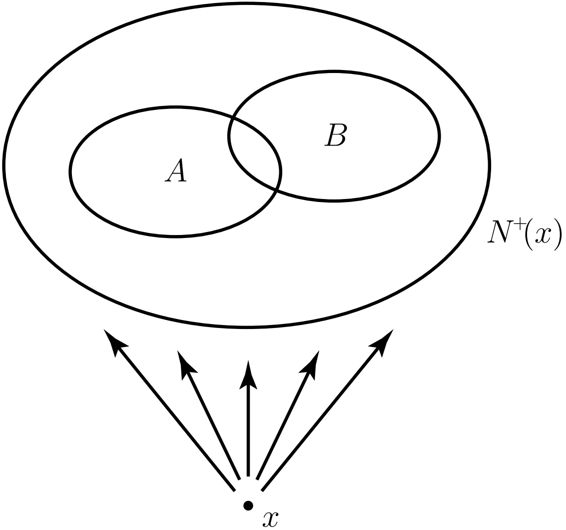
How can we obtain structural facts from the property 'C-homogeneous' that will help us in the proof?

First structural fact

Lemma

1. *The out-neighbourhood of some (and hence every) vertex of a C-homogeneous digraphs induces a homogeneous digraph.*
2. *The in-neighbourhood of some (and hence every) vertex of a C-homogeneous digraphs induces a homogeneous digraph.*

First structural fact (proof)



1^{st} result

Theorem

For every countable C-homogeneous digraph D one of the following statements is true:

1. *D is a blow-up of a homogeneous digraph;*
2. *D has more than one end;*
3. *every vertex of D has an independent out- and an independent in-neighbourhood.*

2^{nd} result

Theorem (Dunwoody&Krön 2015)

For transitive graphs G with more than one end there is an $\mathrm{Aut}(G)$*-invariant nested set of vertex cuts distinguishing some ends.*

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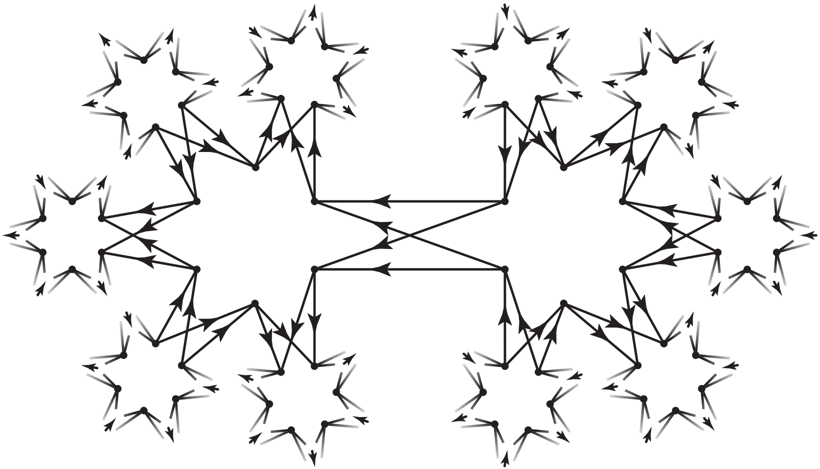
Theorem

Connected C-homogeneous digraphs with at least two ends have connectivity 1 *or* 2 *and are tree-like.*

There are five classes of such digraphs.

An infinitely ended C-homogeneous digraph

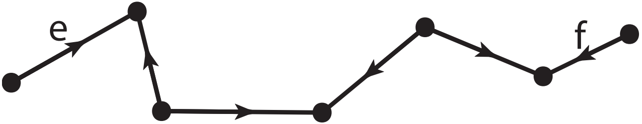


Reachability

Definition

An edge e is reachable from an edge f if there is some walk $x_1 \ldots x_n$ containing e and f such that:

$$x_{i-1} \in N^+(x_i) \Leftrightarrow x_{i+1} \in N^-(x_i).$$



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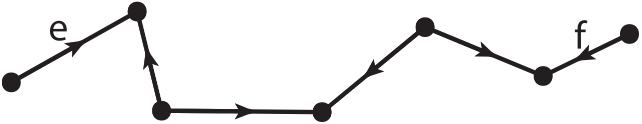

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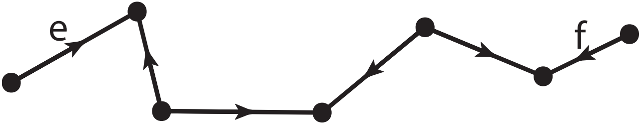


Remark

Reachability is an equivalence relation.

Lemma (Cameron&Praeger&Wormald 1993)

In edge-transitive digraphs either the reachability relation is universal or one (and hence every) equivalence class forms a bipartite digraph.

Reachability

Lemma (Gray&Möller 2011)

In C-homogeneous digraphs whose reachability relation is not universal and with independent out- and in-neighbourhood for every vertex, the equivalence classes of the reachability relation form C-homogeneous bipartite digraphs.

3^{rd} result

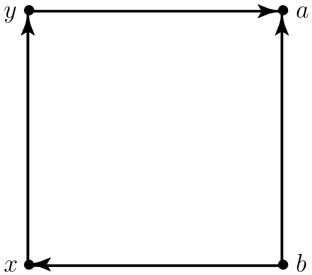
Theorem

For every countable C-homogeneous digraph with at most end whose reachability relation is not universal and with independent out- and in-neighbourhood for every vertex one of the following statements is true:

1. *essentially, the digraph is a blow-up of a directed cycles or double ray.*
2. *it is a quotient digraph of D^*.*

Final situation

If the reachability relation is universal, then the digraph contains the following *induced* subdigraph:



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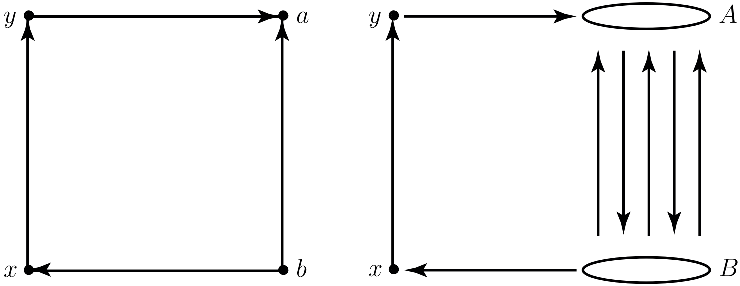

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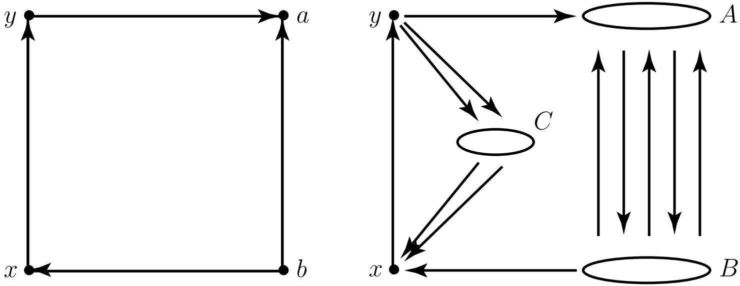
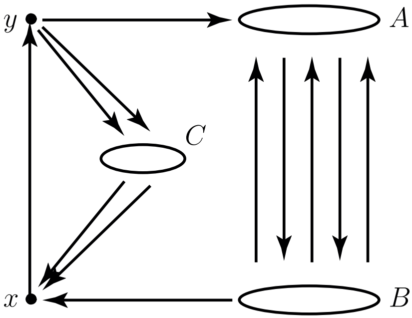


Final situation

Lemma

If D is a countable C-homogeneous digraph with universal reachability relation and with independent out- and in-neighbourhood for every vertex, then with
$A := N^+(y) \smallsetminus N^-(x)$ *and*
$B := N^-(x) \smallsetminus N^+(y)$
for $xy \in E(D)$ the digraph induced by $A \cup B$ is a non-empty homogeneous 2-partite digraph.



4^{th} result

Theorem

A countable C-homogeneous digraph with universal reachability relation and with independent out- and in-neighbourhood for every vertex is either

1. *homogeneous or*
2. *the generic orientation of the generic bipartite graph.*

The classification

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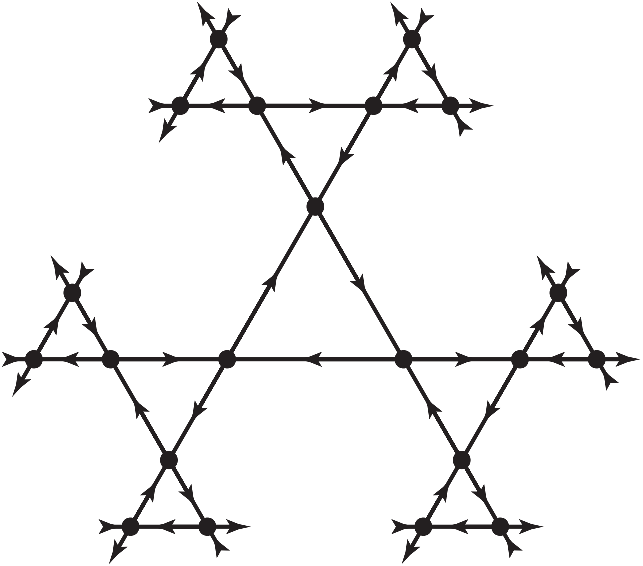
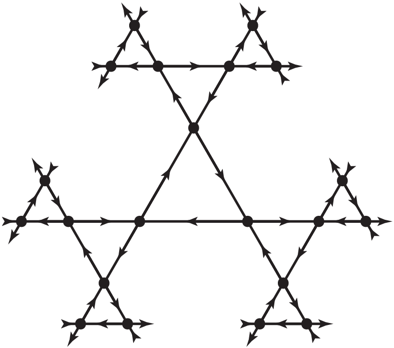
- Eleven of these classes have explicit constructions but
- one does not!

A C-homogeneous digraph: D^*



One particular class



the digraph D^*

One class of connected C-homogeneous digraph of degree 4 are quotient digraphs of D^*, where the quotient is built using some $\mathrm{Aut}(D^*)$-invariant equivalence relation on $V(D^*)$.

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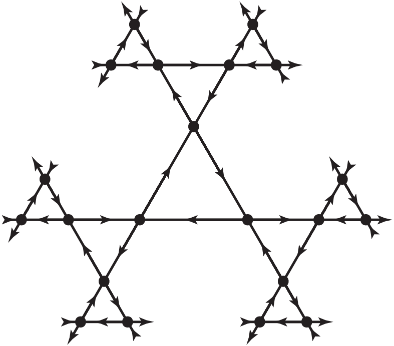

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Theorem

There is a canonical bijection from this class of C-homogeneous digraphs to those subgroups of the modular group $C_2 * C_3$ that contain a fixed involution.

The classification

Theorem

A countable digraph is C-homogeneous if and only if it is a disjoint union of countably many copies of one of the following digraphs:

(i) *a countable homogeneous digraph;*

(ii) $H[I_n]$ *for some* $n \in \mathbb{N}^\infty$ *and with either* $H = S(3)$ *or* $H = T^\wedge$ *for some countable homogeneous tournament* $T \neq S(2)$*;*

(iii) $X_\lambda(T)$ *for some countable homogeneous tournament* T *and* $\lambda \in \mathbb{N}^\infty$*;*

(iv) *a regular tree;*

(v) $DL(\Delta)$*, where* Δ *is a bipartite digraph such that* $G(\Delta)$ *is one of*

- C_{2m} *for some integer* $m \geq 2$,
- CP_k *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$,
- $K_{k,l}$ *for* $k, l \in \mathbb{N}^\infty$, $k, l \geq 2$, *or*
- *the countable generic bipartite graph;*

(vi) $M(k, m)$ *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$ *and some integer* $m \geq 2$*;*

(vii) $M'(2m)$ *for some integer* $m \geq 2$*;*

(viii) Y_k *for some* $k \in \mathbb{N}^\infty$ *with* $k \geq 3$*;*

(ix) $C_m[I_k]$ *for some* $k, m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(x) $\mathcal{R}_m$ *for some* $m \in \mathbb{N}^\infty$ *with* $m \geq 3$*;*

(xi) $X_2(C_3)_\sim$*, where* $\sim$ *is a non-universal* $Aut(X_2(C_3))$*-invariant equivalence relation on* $V(X_2(C_3))$*; or*

(xii) *the generic orientation of the countable generic bipartite graph.*