Hyperbolic graphs and trees

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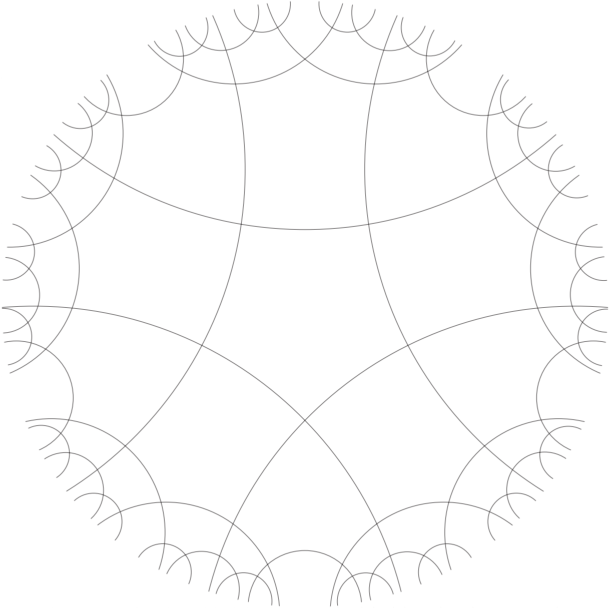
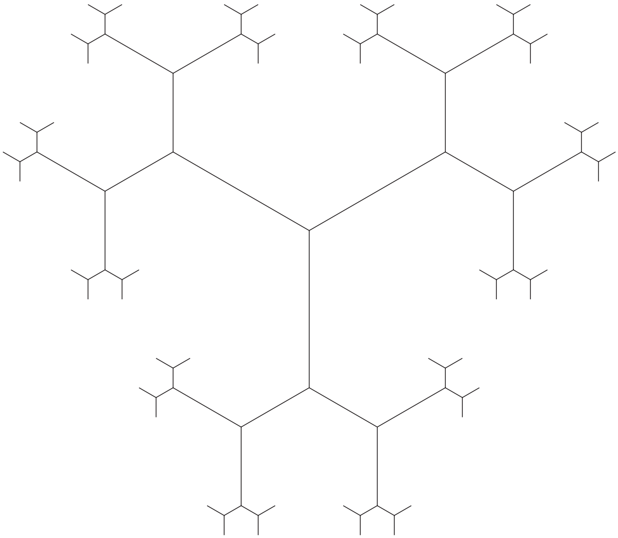
Hyperbolic graphs: Example 1



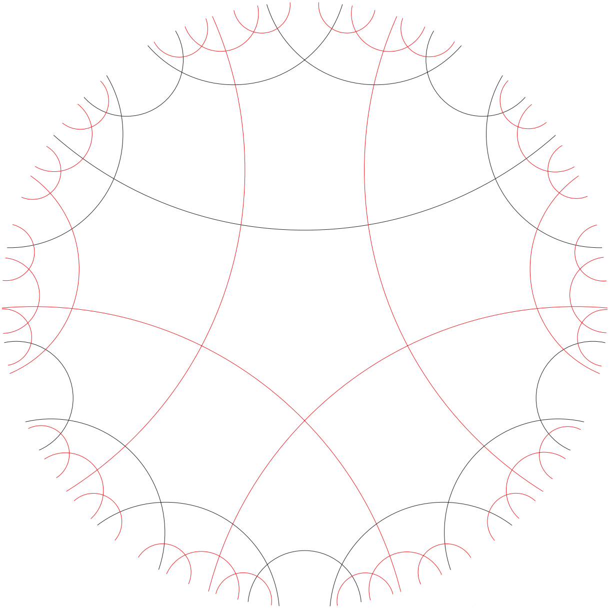
Hyperbolic graphs: Example 2



Aim of the talk

Exhibit the tree-likeness of hyperbolic graphs

Hyperbolic graphs: Example 2



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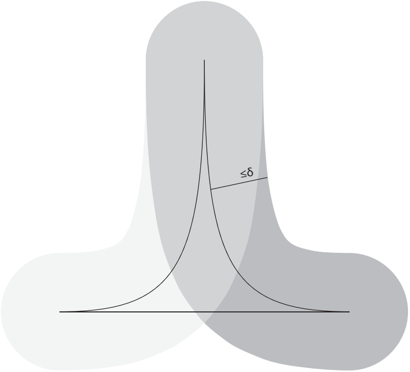

Observation

A graph is 0-hyperbolic if and only if it is a tree.

Trees as hyperbolic graphs

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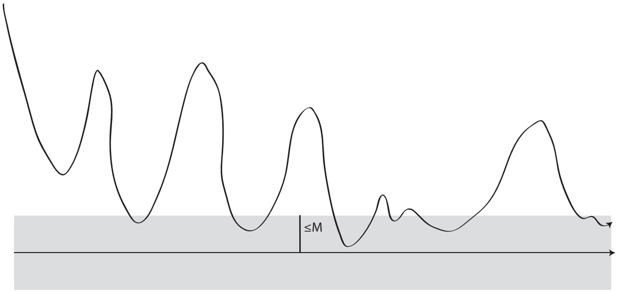

The hyperbolic boundary ∂G: Definition

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The equivalence of geodetic rays is an equivalence relation.

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The hyperbolic boundary ∂G: Definition

The *hyperbolic boundary* ∂G of a hyperbolic graph G is the set of equivalence classes of geodetic rays and the *hyperbolic compactification* $\widehat{G}$ is $G \cup \partial G$.

Ends of graphs

Two rays in a graph G are *equivalent* if for any finite set S of vertices they lie eventually in the same component of $G - S$.

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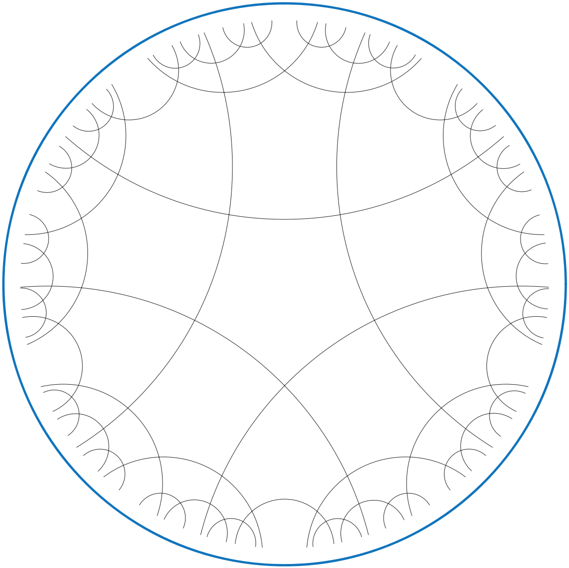
The equivalence classes of this relation are the *ends* ΩG of G. Let $|G| := G \cup \Omega G$.

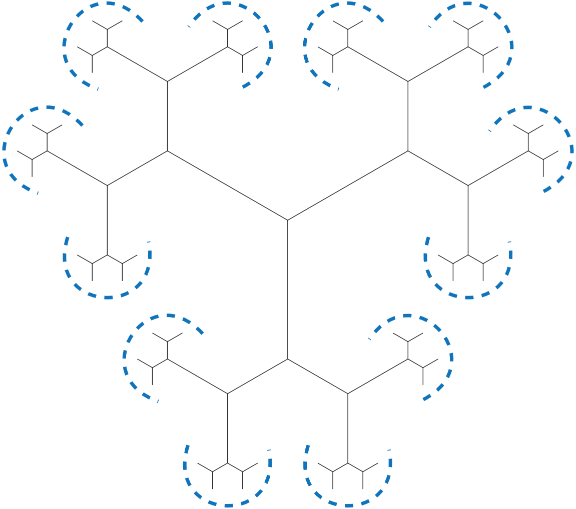
Observation

The hyperbolic boundary of a locally finite hyperbolic graph is a refinement of its end space.

The hyperbolic boundary: Example 1



The hyperbolic boundary: Example 2

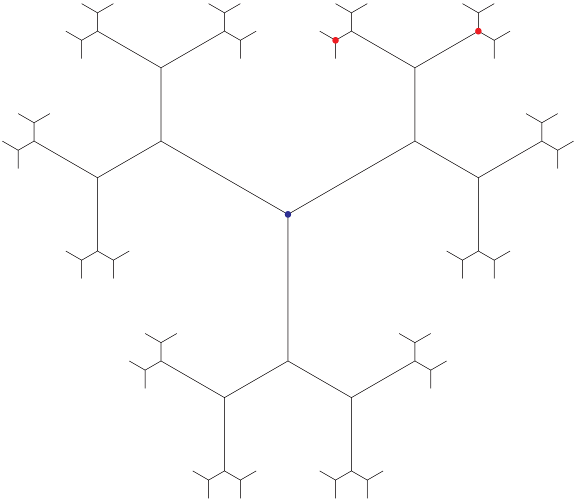


Hyperbolic metric

Theorem (Gromov, 1987)

Let G be a locally finite hyperbolic graph. Then there exists a metric d_ε such that $(\widehat{G}, d_\varepsilon)$ is a compact metric space.

Hyperbolic metric



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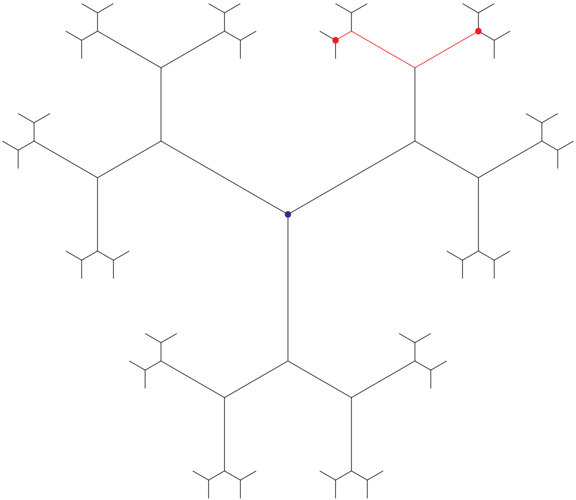

Spanning trees in hyperbolic graphs

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End-faithful spanning trees

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Theorem (Halin, 1964)

Every countable graph has an end-faithful spanning tree.

Spanning trees in hyperbolic graphs

For a subtree T of a hyperbolic graph G, we say that the *canonical map $\partial T \to \partial G$ exists* if the identity $T \to G$ extends to a continuous map $\widehat{T} \to \widehat{G}$.

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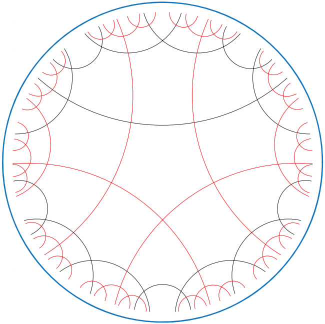

Theorem (H., 2011)

For every locally finite hyperbolic graph G whose hyperbolic boundary has topological dimension n and for every spanning tree T of G such that the canonical map $\varphi : \partial T \to \partial G$ exists and is onto, there is an $\eta \in \partial G$ such that $|\varphi^{-1}(\eta)| \geq n + 1$.

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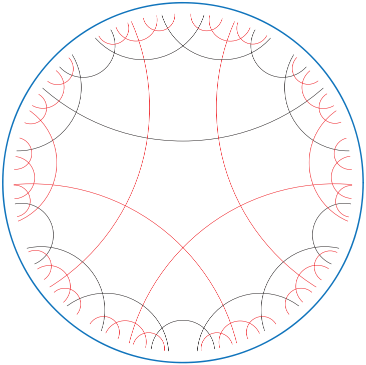

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2. *there exists a constant $\Delta(\dim_A(\partial G), \delta)$ such that for the subtree $T' \subseteq T$ that consists of all rays in T that starts at the root the graph $G - B_\Delta(T')$ contains no geodetic ray;*

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Hyperbolic graphs whose boundary is finite-dimensional

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These are in particular all Cayley graphs of hyperbolic groups.

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Theorem (Assouad)

A metric space is doubling if and only if it has finite Assouad dimension.

Sketch of the proof

Theorem

G locally finite δ-hyperbolic graph, ∂G has finite Assouad dimension.

$\exists$ spanning tree T of G s.t.

1. *rays in T are eventually quasi-geodetic for a constant $c(\dim_A(\partial G), \delta)$;*
2. *$\exists \Delta(\dim_A(\partial G), \delta)$ s.t. $G - B_\Delta(T')$ contains no geodetic ray;*
3. *$\exists \varphi : \partial T \to \partial G$ that is onto;*
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The proof is done constructively.

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- Add the remaining vertices to T' appropriately to obtain a spanning tree T with all the properties. □

Final remarks I

There is an analogue result in the case of proper hyperbolic geodetic spaces.

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Theorem (H., 2011)

Let X be a proper δ-hyperbolic geodetic space with boundary ∂X that has finite Assouad dimension.
Then there exists an $\mathbb{R}$-tree T in X such that

1. *every ray in T is eventually quasi-geodetic for some global constant depending only on $\dim_A(\partial X)$ and δ;*
2. *there is a constant $\Delta(\dim_A(\partial G), \delta)$ such that $X \setminus B_\Delta(T)$ contains no geodetic ray;*
3. *the canonical map $\varphi : \partial T \to \partial X$ exists and is surjective;*
4. *$|\varphi^{-1}(\eta)|$ is bounded in terms of $\dim_A(\partial X)$.*

Final remarks II

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Question

Does there exists a dimension concept that offers a lower and an upper bound for the canonical map?