Hyperbolic graphs

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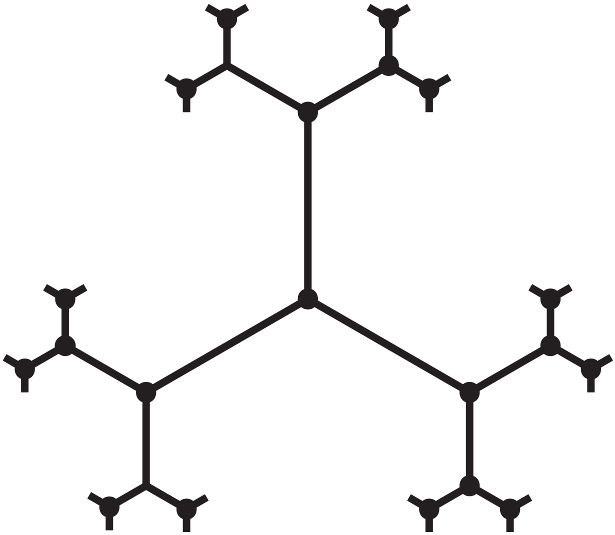
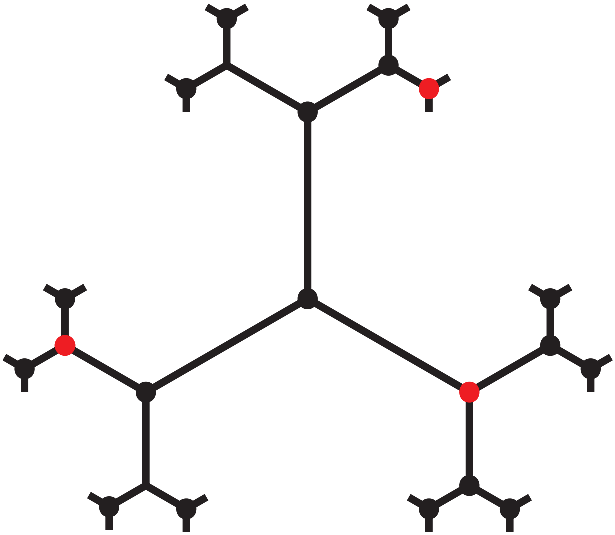
A property of trees





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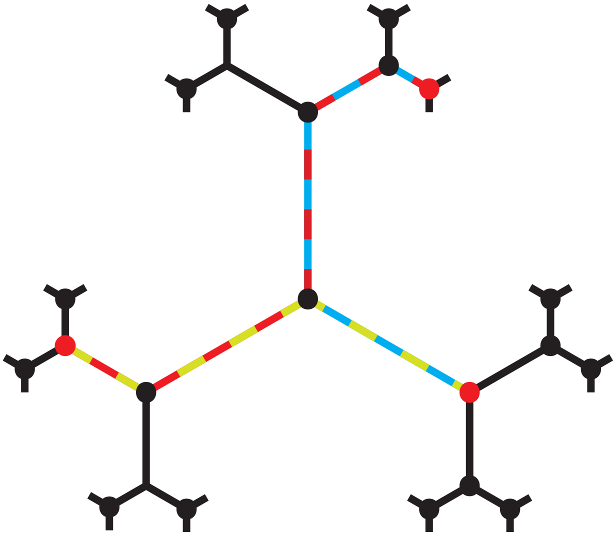
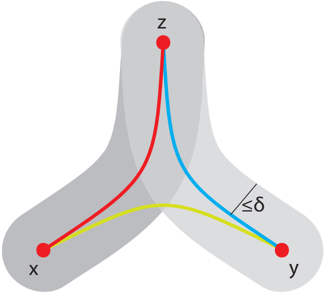


Definition

A connected graph G is called hyperbolic if there exists $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three geodesics (that are any shortest paths), one between each two of the vertices, each of the geodesics lies in the δ-neighbourhood of the union of the other two.



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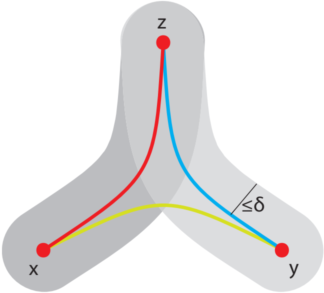
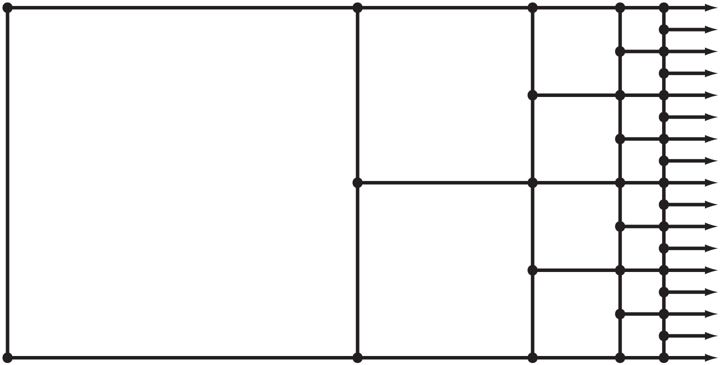


Question

Is every connected hyperbolic graph treelike?

Hyperbolic graphs: an example



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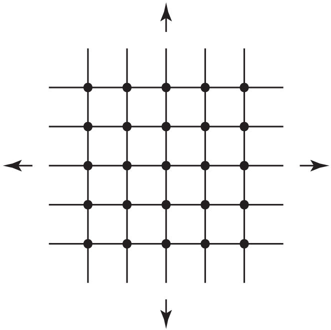
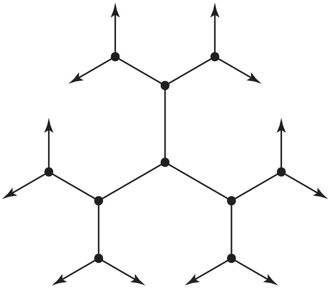
- Two rays in a graph G are *equivalent* if for any finite vertex set $S \subseteq V(G)$ both rays lie eventually in the same component of $G - S$.
- The equivalence classes of this relation are the *ends* of the graph.

Ends of graphs: examples



one end

two ends



infinitely many ends

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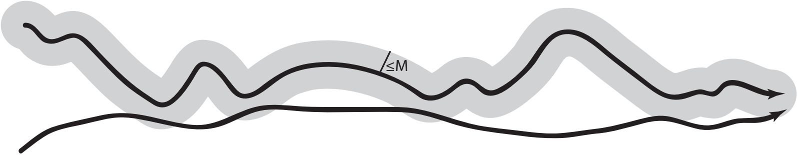
In a hyperbolic graph, two geodesic rays $x_1x_2\dots$ and $y_1y_2\dots$ are equivalent if there exists $M \in \mathbb{N}$ such that for infinitely many $i \in \mathbb{N}$ there exists $j \in \mathbb{N}$ with $d(x_i, y_j) \leq M$.



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Lemma

In hyperbolic graphs, this relation on geodesic rays in an equivalence relation.

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Lemma

In hyperbolic graphs, this relation on geodesic rays in an equivalence relation.

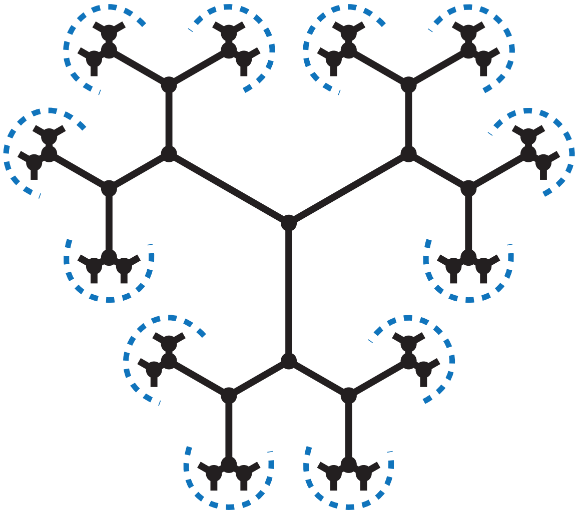
Definition

The hyperbolic boundary ∂G of a hyperbolic graph G is the set of all equivalence classes of this equivalence relation.

Remark

The hyperbolic boundary of a locally finite hyperbolic graph is a refinement of the ends.

Hyperbolic boundary: examples



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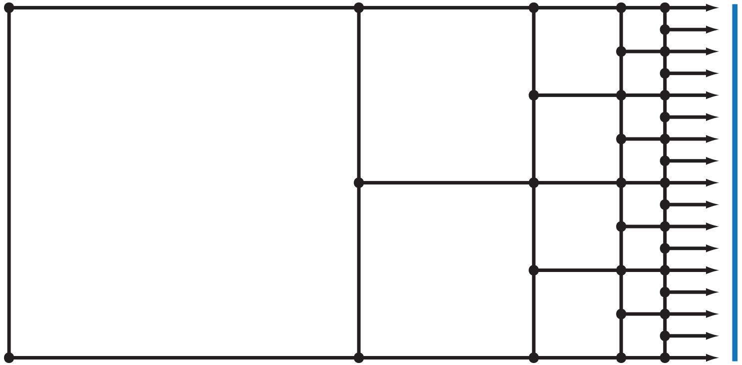

A metric for the boundary

Theorem (Gromov, 1987)

For every locally finite connected hyperbolic graph G, there is a metric d_h on $\widehat{G} := G \cup \partial G$ such that $(\widehat{G}, d_h)$ is a compact metric space.

Theorem (Halin, 1964)

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- The hyperbolic boundary of a tree is totally disconnected.

Spanning trees

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Every countable connected graph has an end-faithful spanning tree.

Remark

- There is a locally finite hyperbolic graph whose hyperbolic boundary is homeomorphic to the unit interval.
- The hyperbolic boundary of a tree is totally disconnected.

$\Rightarrow$ In general, hyperbolic graphs do not have spanning trees that are faithful with respect to the hyperbolic boundary.

Two aims for a spanning tree T in a connected locally finite hyperbolic graph G:

1. T should represent G well;
2. ∂T should represent ∂G well.

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4. *$\partial G = \hat{\iota}(\partial T)$;*
5. *there exists $M \in \mathbb{N}$ such that every $\eta \in \partial G$ has at most M inverse images under $\hat{\iota}$.*

Which graphs satisfy the assumptions?

Theorem (Bonk & Schramm, 2000)

The hyperbolic boundary of any connected hyperbolic graph of bounded degree has finite Assouad dimension.

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Let G be a locally finite connected hyperbolic graph and let T be a spanning tree of G such that the embedding $\iota : T \to G$ extends continuously on the hyperbolic boundary.

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Let G be a locally finite connected hyperbolic graph and let T be a spanning tree of G such that the embedding $\iota : T \to G$ extends continuously on the hyperbolic boundary.
Then there exists $\eta \in \partial G$ with at least $M + 1$ inverse images, where M is the topological dimension of the hyperbolic boundary.