Hyperbolic monoids and digraphs

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March 2024

Outline

1. motivation
2. hyperbolic metric spaces
3. hyperbolic digraphs and monoids
4. quasi-isometries
5. hyperbolic boundary ∂D
6. free submonoids
7. final remarks

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Motivation

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Problem (Gray and Kambites)

If one Cayley digraph (wrt a finite generating set) of a finitely generated semigroup is hyperbolic, then is every such Cayley digraph hyperbolic?

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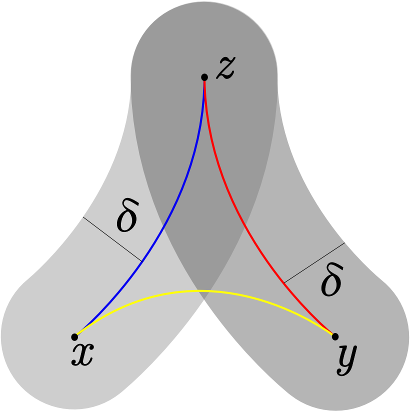
A metric space is hyperbolic if $\exists \delta \geq 0$ such that for all points x, y, z every geodesic between x and y lies in the δ-neighbourhood of the union of any geodesic between y and z and any geodesic between x and z.



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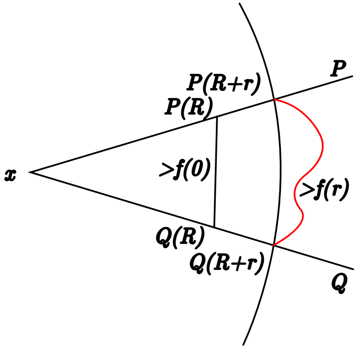
They imply the most important property for hyperbolic spaces: being invariant under quasi-isometries.

A strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ is a divergence function of a metric space X if for all $x \in X$, for all geodesics P, Q starting at x and for all $r, R \in \mathbb{R}$ with $r + R \leq \min\{\ell(P), \ell(Q)\}$ and $d(P(R), Q(R)) > f(0)$ every path in $X \setminus B_{R+r}(x)$ from $P(R+r)$ to $Q(R+r)$ has length more than $f(r)$.



Let X, Y be metric spaces and let $\gamma \geq 1$ and $c \geq 0$. A map $f\colon X \to Y$ is a (γ, c)-quasi-isometry if the following hold:

1. for all $x, x' \in X$ we have
$$\frac{1}{\gamma} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq \gamma d_X(x, x') + c;$$
2. for every $y \in Y$ there exists $x \in X$ with $d_Y(f(x), y) \leq c$.

Quasi-geodesics

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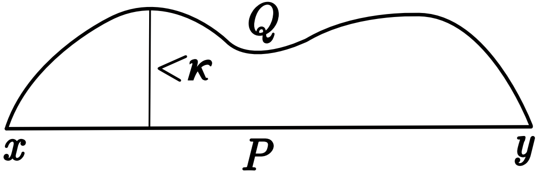
For $\gamma \geq 1$ and $c \geq 0$, a path P in a metric space is a (γ, c)-quasi-geodesic if it is the (γ, c)-quasi-isometric image of an interval.

A metric space X satisfies geodesic stability if for all $\gamma \geq 1$ and $c \geq 0$ there exists $\kappa \geq 0$ such that for all $x, y \in X$, all x-y geodesics P and all (γ, c)-quasi-geodesics Q from x to y we have $P \subseteq B_\kappa(Q)$ and $Q \subseteq B_\kappa(P)$.



Hyperbolic metric spaces

Theorem (Gromov, Bonk)

The following are equivalent for proper metric spaces X.

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Proof idea.

Let $f\colon X \to Y$ be a quasi-isometry and assume that Y is hyperbolic.

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Proof idea.

Let $f\colon X \to Y$ be a quasi-isometry and assume that Y is hyperbolic. Then Y satisfies geodesic stability. Apply the quasi-isometry to deduce geodesic stability and hence hyperbolicity for X. □

Outline

A digraph D is a pair $(V(D), E(D))$ of a vertex set $V(D)$ and an edge set $E(D)$ with $E(D) \subseteq V(D) \times V(D)$.

Digraphs

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We denote the edges by uv instead of (u, v).

A directed path in D is a (finite) sequence $v_0, \ldots, v_n$ of distinct vertices with $v_i v_{i+1} \in E(D)$. We call n its length.

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For $k \in \mathbb{N}$, the k-in-ball of a vertex u is the set

$$B_k^-(u) := \{v \in V(D) \mid d(v, u) \leq k\}$$

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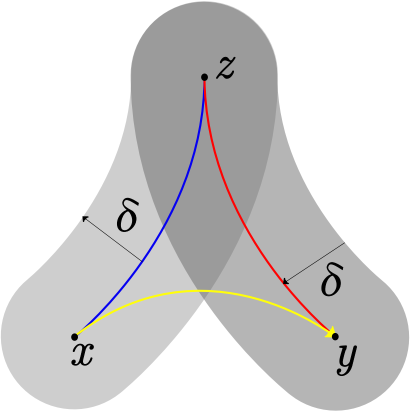
and the k-out-ball of u is the set

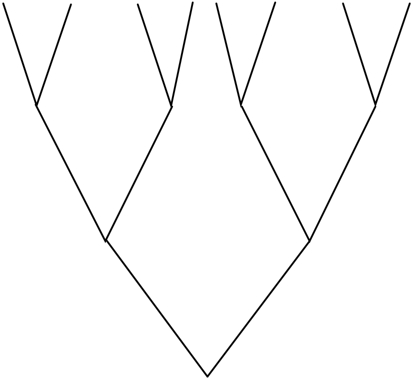
$$B_k^+(u) := \{v \in V(D) \mid d(u, v) \leq k\}.$$

A digraph is hyperbolic if $\exists\delta \geq 0$ such that for all vertices x, y, z and geodesics between them every geodesic from x to y lies in the union of the δ-out-ball of any geodesic between x and z and of the δ-in-ball of any geodesic between y and z.



Example

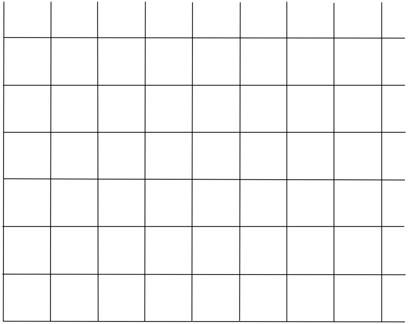
1. Oriented trees are examples for hyperbolic digraphs.



Examples

Example

2. $\mathbb{N} \times \mathbb{N}$ is not a hyperbolic digraph.



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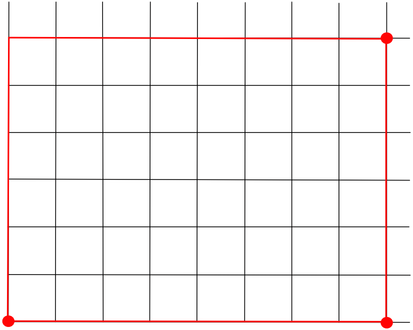
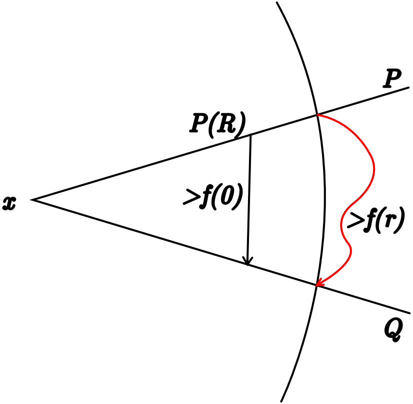


Divergence of geodesics

A strictly increasing function $f\colon \mathbb{N} \to \mathbb{N}$ is a divergence function of a digraph D if for all $x \in V(D)$, for all geodesics P, Q that start or end at x and for all $r, R \in \mathbb{N}$ with $r + R \leq \min\{\ell(P), \ell(Q)\}$ and $d(P(R), Q) > f(0)$ every directed P-Q path that lies outside of $B^+_{R+r}(x) \cup B^-_{R+r}(x)$ has length more than $f(r)$.



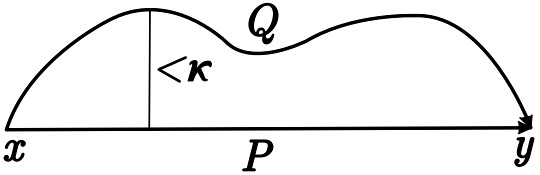
Quasi-geodesics

For $\gamma \geq 1$ and $c \geq 0$, a directed u-v path P in a digraph is a (γ, c)-quasi-geodesic if

$$d_P(x, y) \leq \gamma d(x, y) + c$$

for all x, y on P with $x \in V(uPy)$.

A digraph D satisfies geodesic stability if for all $\gamma \geq 1$ and $c \geq 0$ there exists $\kappa \geq 0$ such that for all $x, y \in V(D)$, all geodesics P and all (γ, c)-quasi-geodesics Q from x to y we have $P \subseteq B^+_\kappa(Q) \cap B^-_\kappa(Q)$ and $Q \subseteq B^+_\kappa(P) \cap B^-_\kappa(P)$.



Question

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1. There is a digraph that satisfies geodesic stability but neither is hyperbolic nor has a divergence function.
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Theorem (H.)

Every hyperbolic digraph of bounded degree has an exponential divergence function and satisfies geodesic stability.

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Remark

The original result is more general but also more technical: we can replace *bounded degree* by a condition of end vertices of geodesics. Thereby, we can generalize the result for spaces that have a suitable distance function and satisfy this condition.

A monoid is a set with an associative binary function and an identity element.

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The (right) Cayley digraph of a monoid M with generating set A has M as vertex set and edges (m, ma) for all $m \in M$ and $a \in A$.

A finitely generated monoid is hyperbolic if it has a Cayley digraph with respect to a finite generating set that is hyperbolic.

Problem (Gray and Kambites)

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Remark (Gray and Kambites)

Hyperbolic groups considered as monoids are hyperbolic.

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Let D_1, D_2 be digraphs and let $\gamma \geq 1$ and $c \geq 0$. A map $f\colon V(D_1) \to V(D_2)$ is a (γ, c)-quasi-isometry if the following hold:

1. for all $x, y \in V(D_1)$ we have
$$\frac{1}{\gamma} d_{D_1}(x, y) - c \leq d_{D_2}(f(x), f(y)) \leq \gamma d_{D_1}(x, y) + c;$$
2. for every $x \in V(D_2)$ there exists $y \in V(D_1)$ with $d_{D_2}(f(x), y) \leq c$ and $d_{D_2}(y, f(x)) \leq c$.

Quasi-isometries

Theorem (H.)

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Corollary (H.)

1. *Quasi-isometries between finitely generated right cancellative monoids preserve hyperbolicity.*

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COROLLARY (H.)

1. *Quasi-isometries between finitely generated right cancellative monoids preserve hyperbolicity.*
2. *For a finitely generated right cancellative hyperbolic monoid each of its locally finite Cayley digraphs is hyperbolic.*

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In a digraph D, a geodesic ray is a sequence $R = v_0 v_1 \dots$ such that $d(v_i, v_j) = j - i$ for all $i \leq j \in \mathbb{N}$ and a geodesic anti-ray is a sequence $Q = \dots v_{-1} v_0$ such that $d(v_i, v_j) = j - i$ for all $i \leq j \leq 0 \in \mathbb{Z}$.

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In hyperbolic digraphs of bounded degree, we can define an equivalence relation $\approx$ on the geodesic rays and anti-rays as follows:
$R_1 \approx R_2$ for geodesic rays or anti-rays R_1, R_2 if there exists $m \in \mathbb{N}$ and infinitely many pairwise disjoint R_1-R_2 and R_2-R_1 paths of length at most m.

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The equivalence classes of $\approx$ are the hyperbolic boundary points of D. We denote by ∂D the hyperbolic boundary of D, i. e. the set of hyperbolic boundary points.

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Let X be a set. A pseudo-semimetric is a function $d\colon X \times X \to [0, \infty]$ that satisfies the following properties

- $d(x, x) = 0$ for all $x \in X$ and
- $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

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Here, being a visual pseudo-semimetric means roughly that $d_h(x, y)$ is about $e^{-\varepsilon d^{\leftrightarrow}(o,P)}$, where P is any x-y geodesic, o is the root and

$$d^{\leftrightarrow}(o, P) = \min\{d(o, P), d(P, o)\}.$$

The pseudo-semimetric defines two topologies: one wrt open out-balls, the other wrt open in-balls.

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Theorem (H.)

Quasi-isometries $D_1 \to D_2$ between hyperbolic digraphs of bounded degree extend to homeomorphisms $D_1 \cup \partial D_1 \to D_2 \cup \partial D_2$ (wrt to both topologies).

Outline

Free submonoids

Theorem (H.)

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No infinite finitely generated hyperbolic cancellative monoid contains $\mathbb{N} \times \mathbb{N}$ as a submonoid.

Corollary (H.)

Every infinite finitely generated hyperbolic cancellative monoid with infinite hyperbolic boundary has exponential growth.

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Theorem (Gray and Kambites)

1. *Left cancellative finitely generated hyperbolic semigroups are finitely presentable.*

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2. *Right cancellative finitely generated hyperbolic semigroups need not be recursively presentable.*

Our results hold in a more general case than bounded degree.

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $x \in V(D)$, for every $n \in \mathbb{N}$ and for all $y, z \in B_n^+(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(n)$.

There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every $x \in V(D)$, for every $n \in \mathbb{N}$ and for all $y, z \in B_n^-(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(n)$.