Hyperbolic digraphs

Matthias Hamann

University of Warwick

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Outline

1. motivation
2. hyperbolic digraphs
3. quasi-isometries
4. geodesic boundary ∂D
5. topological properties of $D \cup \partial D$
6. final remarks

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Motivation

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Problem (Gray and Kambites)

If one Cayley digraph (wrt a finite generating set) of a semigroup is hyperbolic, then is every such Cayley digraph hyperbolic?

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A good notion should have the following properties:

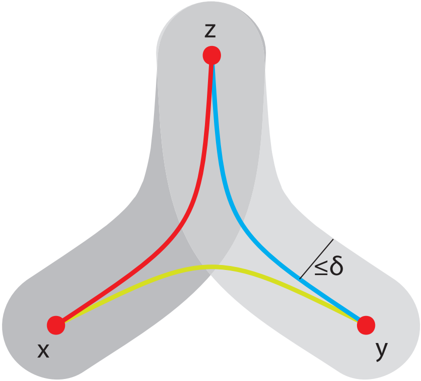
1. it is stable with respect to quasi-isometries
2. most of the theory of hyperbolic graphs should (more or less) carry over to hyperbolic digraphs

Outline

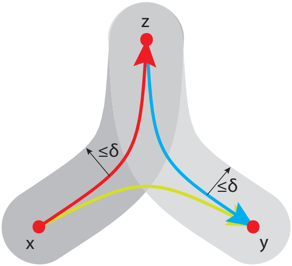
Hyperbolic graphs have many equivalent definitions.
Most important ones:

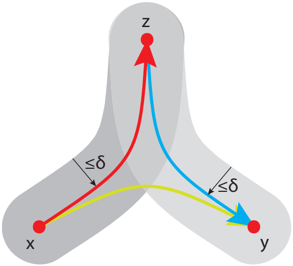
1. thin triangles
2. diverging geodesics
3. geodesic stability

A graph is hyperbolic if $\exists\delta \geq 0$ such that for all vertices x, y, z every shortest path (=geodesic) between x and y lies in the δ-neighbourhood of the union of any geodesic between y and z and any geodesic between x and z.

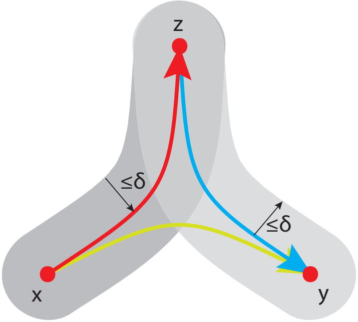


First idea: if $\exists\delta \geq 0$ such that for all vertices x, y, z every shortest directed path (=geodesic) from x to y lies in the δ-out-neighbourhood and in the δ-in-neighbourhood of the union of any geodesic between y and z and any geodesic between x and z.



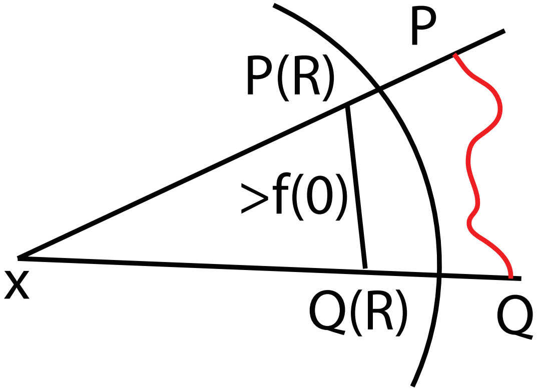


A digraph is hyperbolic if $\exists\delta \geq 0$ such that for all vertices x, y, z every shortest directed path (=geodesic) from x to y lies in the union of the δ-out-neighbourhood of any geodesic between y and z and of the δ-in-neighbourhood of any geodesic between x and z.



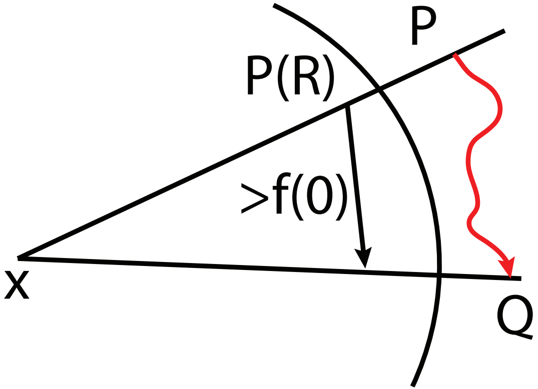
Divergence of geodesics

A function $f : \mathbb{N} \to \mathbb{N}$ is a divergence function of a graph G if for all $x \in V(G)$, for all geodesics P, Q starting at x and for all $r, R \in \mathbb{N}$ with $r + R \leq \min\{d(x, y), d(x, z)\}$ and $d(P(R), Q(R)) > f(0)$ every path in $G - B_{R+r}(x)$ from $P(R + r)$ to $Q(R + r)$ has length more than $f(r)$.



Divergence of geodesics

A function $f : \mathbb{N} \to \mathbb{N}$ is a divergence function of a digraph D if for all $x \in V(D)$, for all geodesics P, Q that start or end at x and for all $r, R \in \mathbb{N}$ with $r + R \leq \min\{\ell(P), \ell(Q)\}$ and $d(P(R), Q) > f(0)$ every directed P-Q path that lies outside of $B^+_{R+r}(x) \cup B^-_{R+r}(x)$ has length more than $f(r)$.



For $\gamma \geq 1$ and $c \geq 0$, a path P in a graph is a (γ, c)-quasi-geodesic if

$$\frac{1}{\gamma} d(x, y) - c \leq d_P(x, y) \leq \gamma d(x, y) + c$$

for all x, y on P.

Quasi-geodesics

For $\gamma \geq 1$ and $c \geq 0$, a path P in a graph is a (γ, c)-quasi-geodesic if

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A graph G satisfies geodesic stability if for all $\gamma \geq 1$ and $c \geq 0$ there exists $\kappa \geq 0$ such that for all $x, y \in V(G)$, all x-y geodesics P and all x-y (γ, c)-quasi-geodesics Q we have $P \subseteq B_\kappa(Q)$ and $Q \subseteq B_\kappa(P)$.

For $\gamma \geq 1$ and $c \geq 0$, a directed a-b path P in a digraph is a (γ, c)-quasi-geodesic if

$$\frac{1}{\gamma} d(x,y) - c \leq d_P(x,y) \leq \gamma d(x,y) + c$$

for all x, y on P with $x \in V(aPy)$.

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for all x, y on P with $x \in V(aPy)$.

A digraph D satisfies geodesic stability if for all $\gamma \geq 1$ and $c \geq 0$ there exists $\kappa \geq 0$ such that for all $x, y \in V(D)$, all x-y geodesics P and all x-y (γ, c)-quasi-geodesics Q we have $P \subseteq B^+_\kappa(Q) \cap B^-_\kappa(Q)$ and $Q \subseteq B^+_\kappa(P) \cap B^-_\kappa(P)$.

Theorem (Gromov, Bonk, ?)

For a graph G the following are equivalent.

1. *G is hyperbolic.*
2. *G has an exponential divergence function.*
3. *G satisfies geodesic stability.*

Remark

1. There is a digraph that satisfies geodesic stability but neither is hyperbolic nor has a divergence function.

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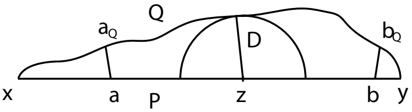
The original result is more general but also more technical: we can replace *bounded degree* by a condition of end vertices of geodesics.

Let us prove that (hyperbolicity and) exponential divergence of geodesics implies that geodesics lie close to quasi-geodesics with the same end vertices.

Let G be a graph. Let $x, y \in V(G)$ and P be an x-y geodesic and Q be an x-y (γ, c)-quasi-geodesic.
Let $z \in V(P)$ with $D := d(z, Q)$ maximum.
Let $a \in V(xPz)$, $b \in V(zPy)$ with $d(a, z) = 2D = d(z, b)$.
There are $a_Q, b_Q \in V(Q)$ with $d(a, a_Q), d(b, b_Q) \leq D$.



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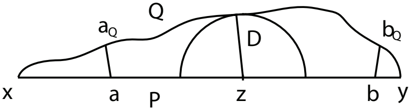

Then $d_Q(a_Q, b_Q) \leq 6\gamma D + c$ and there is an a-b path of length $\leq 6\gamma D + c + 2D$ outside of $B_{D-1}(z)$.

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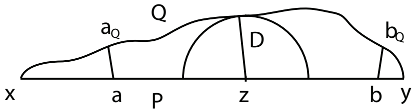

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If assumption is false, D may be arbitrarily large. In particular, we may choose $D > f(0)$, where f is an exponential divergence function of G. Since the above path outside of $B_{D-1}(z)$ is linear in D, this contradicts divergence of the geodesics zPx and zPy.

Let D be a digraph. Let $x, y \in V(D)$ and P be an x-y geodesic and Q be an x-y (γ, c)-quasi-geodesic.
Let $z \in V(P)$ with $D := \min\{d(z, Q), d(Q, z)\}$ maximum.
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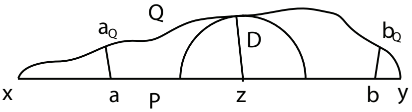

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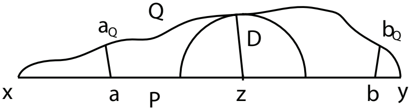

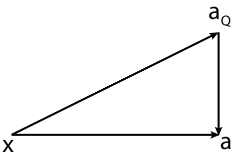

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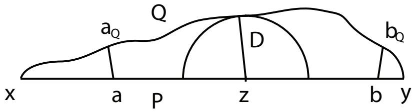

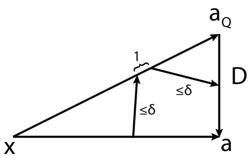

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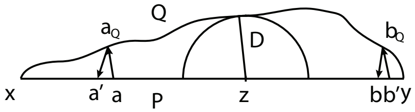

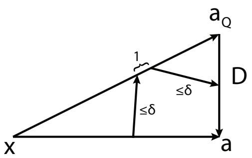

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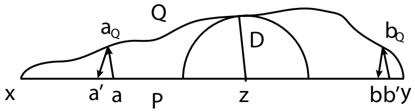

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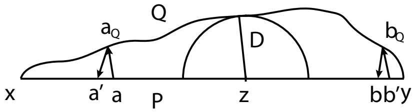

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6. final remarks

Let D_1, D_2 be digraphs and let $\gamma \geq 1$ and $c \geq 0$. A map $f\colon V(D_1) \to V(D_2)$ is a quasi-isometry if the following hold:

1. for all $x, y \in V(D_1)$ we have
$$\frac{1}{\gamma} d_{D_1}(x,y) - c \leq d_{D_2}(f(x), f(y)) \leq \gamma d_{D_1}(x,y) + c;$$
2. for every $x \in V(D_2)$ there exists $y \in f(V(D_1))$ with $d_{D_2}(x,y) \leq c$ and $d_{D_2}(y,x) \leq c$.

Theorem

Quasi-isometries between digraphs of bounded degree preserve geodesic stability

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Question

Do quasi-isometries between digraphs of bounded degree preserve divergence of geodesics?

Let $\varphi\colon D_1 \to D_2$ be a quasi-isometry and let D_1 be hyperbolic.
There exists a quasi-isometry $\psi\colon D_2 \to D_1$ such that $\varphi \circ \psi = id$.

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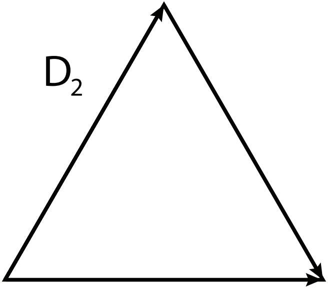

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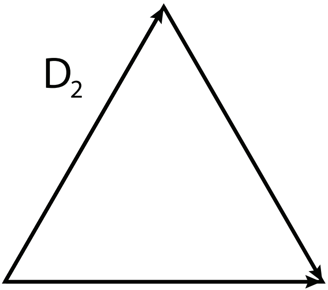

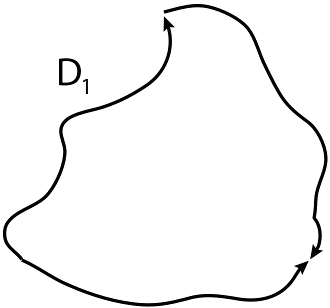

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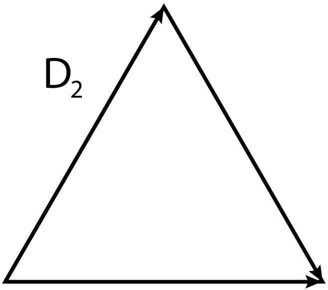

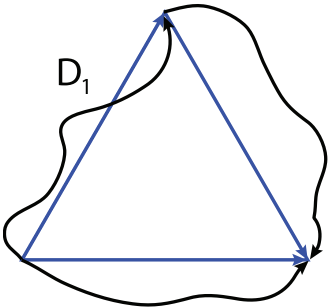

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Let $\mathcal{R}$ be the set of geodesic rays and anti-rays in D.
We write $R_1 \leq R_2$ for $R_1, R_2 \in \mathcal{R}$ if there exists $m \geq 0$ such that for all $x \in V(D)$ and all $r \in \mathbb{N}$ there is a directed R_1-R_2 path of length $\leq m$ outside of $B_r^+(x) \cup B_r^-(x)$.

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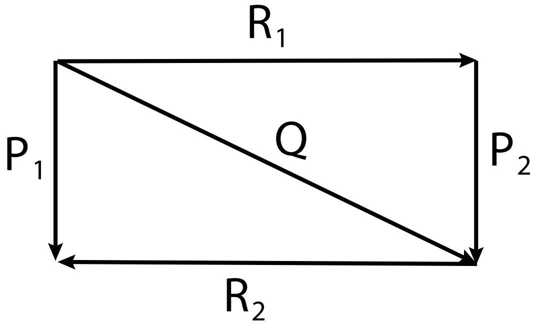

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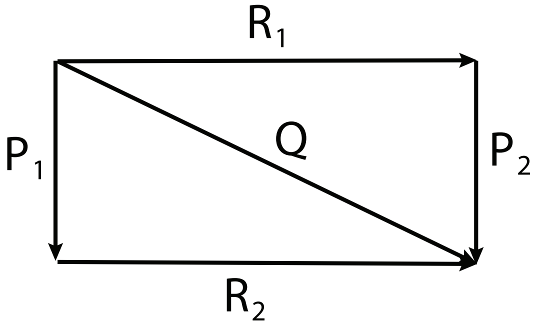

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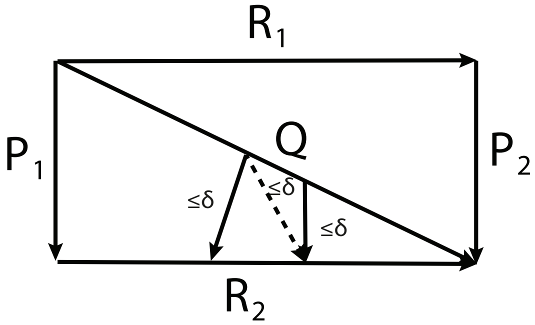

Set $R_1 \approx R_2$ for $R_1, R_2 \in \mathcal{R}$ if $R_1 \leq R_2$ and $R_2 \leq R_1$.
Then $\approx$ is an equivalence relation whose classes are the geodesic boundary points of D.
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Remark

1. ∂D is a refinement of the ends in the sense of Zuther.
2. There are geodesic boundary points that lie in no end in the sense of Bürger and Melcher

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Topologies of D

The distance function of digraphs D induce two different topologies:

- the forward topology $\mathcal{O}_f$ has the balls $\{y \in V(D) \mid d(x,y) < r\}$ for all $r \geq 0$ and $x \in V(D)$ as base
- the backward topology $\mathcal{O}_b$ has the balls $\{y \in V(D) \mid d(y,x) < r\}$ for all $r \geq 0$ and $x \in V(D)$ as base

Let $x \in V(D)$, $r \geq 0$ and $\omega \in \partial D$. Set

$$C^+(\omega, x, r) := \{y \in V(D) \mid \exists R \in \omega \forall z \in V(R) \\ \exists z\text{-}y \text{ geodesic outside of } B_r^+(x) \cup B_r^-(x)\}$$

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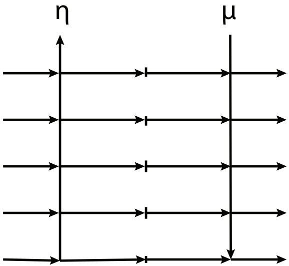

the forward topology of $D \cup \partial D$ has a base that consists of the following set:

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Theorem

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2. *Quasi-isometries* $f\colon D_1 \to D_2$ *between digraphs of bounded degree induce maps* $\widehat{f}\colon \partial D_1 \to \partial D_2$ *that are homeomorphisms with respect to both topologies.*

Let X be a set. A pseudo-semimetric is a function $d\colon X \times X \to [0, \infty]$ that satisfies the following properties

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Theorem

Let D be a digraph of bounded degree with finite base S. Then there is a visual pseudo-semimetric d_h on $D \cup \partial D$ that induces the same topologies that we defined earlier.

Here, visual means roughly that $d_h(x, y)$ is about $e^{-\varepsilon d^{\leftrightarrow}(S,P)}$, where P is any x-y geodesic and

$$d^{\leftrightarrow}(S, P) = \min\{d(S, P), d(P, S)\}.$$

The space ∂D

Question

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- If $d_h(x_1, x_2) = 0 = d_h(x_2, x_3)$, then either $x_1 = x_2$ or $x_2 = x_3$.
- $D \cup \partial D$ is f-complete and b-complete:

A sequence $(x_i)_{i \in \mathbb{N}}$ in $D \cup \partial D$ is f-Cauchy if for every $\varepsilon > 0$ there exists some $N \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $m \geq n \geq N$.
$D \cup \partial D$ is f-complete if every f-Cauchy sequence converges with respect to the backward topology.
b-Cauchy sequence and b-complete are defined analogously.

Outline

1. motivation
2. hyperbolic digraphs
3. quasi-isometries
4. geodesic boundary ∂D
5. topological properties of $D \cup \partial D$
6. final remarks

Replacing bounded degree condition

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E.g. we do not know whether the geodesic boundary is preserved by quasi-isometries. Instead, we just consider the quasi-geodesic boundary that is defined by the analogous relation on quasi-geodesic rays and anti-rays.

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Theorem

For every finitely generated hyperbolic right cancellative semigroup, each of its Cayley digraphs (wrt finite generating sets) is hyperbolic.

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2. *Right cancellative finitely generated semigroups need not be recursively presentable.*