Cuts, cycles and accessibility

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April 18, 2016

Topic

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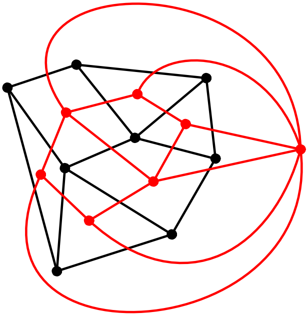

Folklore

The cycles of a planar graph are the minimal cuts of its dual.

Cut space

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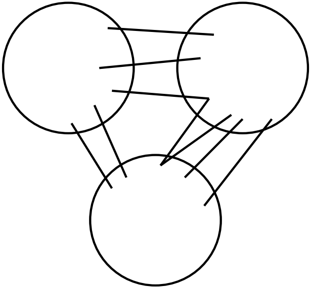

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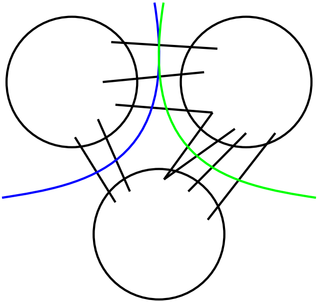

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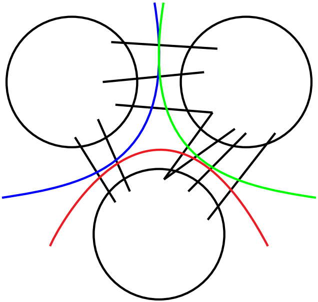
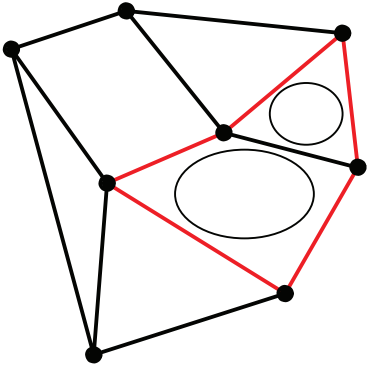


Cycle space

Definition

- The cycle space of a graph is the set of all finite sums (over GF(2)) of edge sets of finite cycles.



The case: finite graphs

Remark

(1) In a finite graph the cut space is the orthogonal space of the cycle space and vice versa.

(2) In a finite graph with n vertices and m edges, the cut space has dimension $n-1$ and the cycle space has dimension $m-n+1$.

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Is (2) interesting for infinite graphs?

Can we make infinite dimensions finite?

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Generally, the cut space or cycle space are not finitely generated since the graph need not have a rich automorphism group.
$\Rightarrow$ we restrict ourselves to transitive graphs

Theorem

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If the cycle space has a generating set of n $\mathrm{Aut}(G)$-orbits and every generator has length at most ℓ, then the cut space has a generating set of at most $2^{\ell+1}n$ orbits.

Brief sketch of the proof

Theorem (Dicks & Dunwoody 1989)

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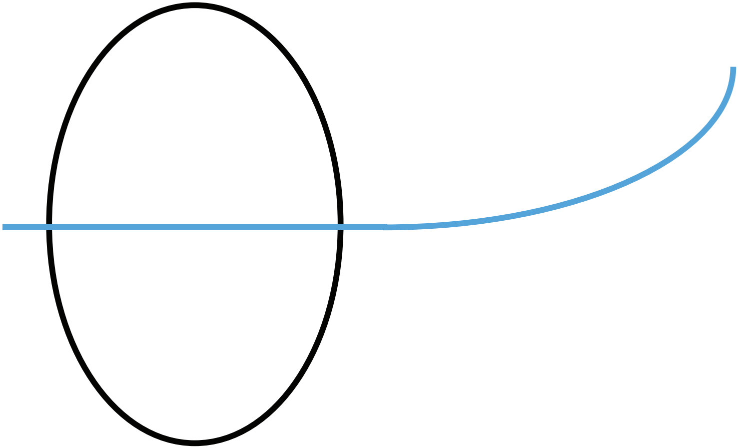
Let $\mathcal{C}$ be a set of finitely many cycles with their $\mathrm{Aut}(G)$-images that generates the cycle space.
If $\mathcal{E}'$ has *many* orbits, one of them has never a minimal or maximal element of any such chain with $C \in \mathcal{C}$.

But such a bipartition cannot exist:



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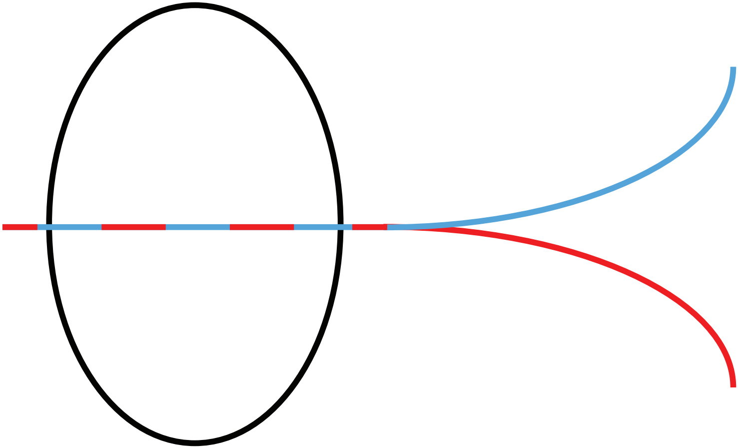

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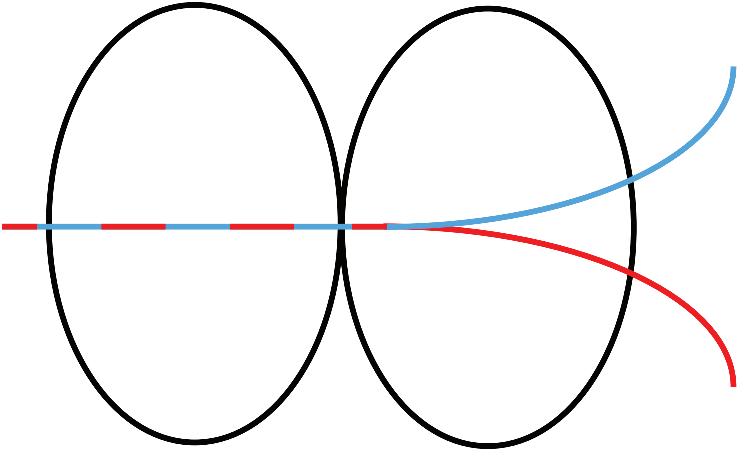

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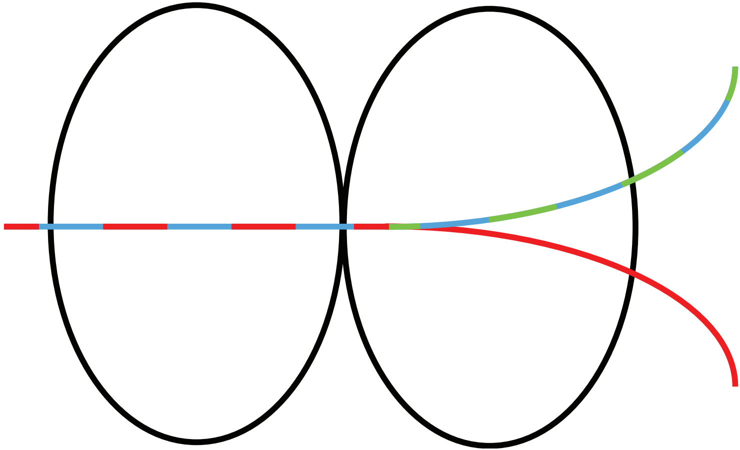

Brief sketch of the proof (cont'd)

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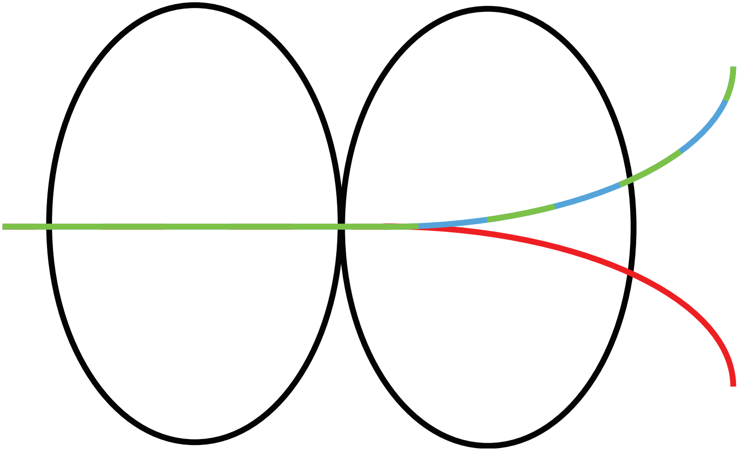

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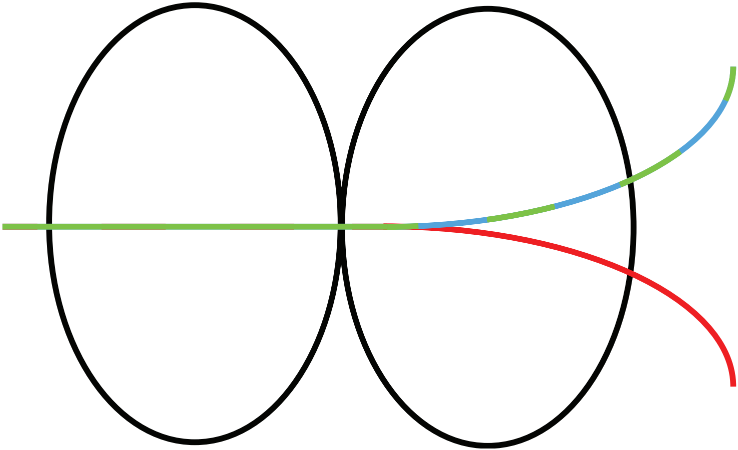

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Can we ask for an 'if and only if' in our theorem?
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I guess not.

Going to infinity: ends

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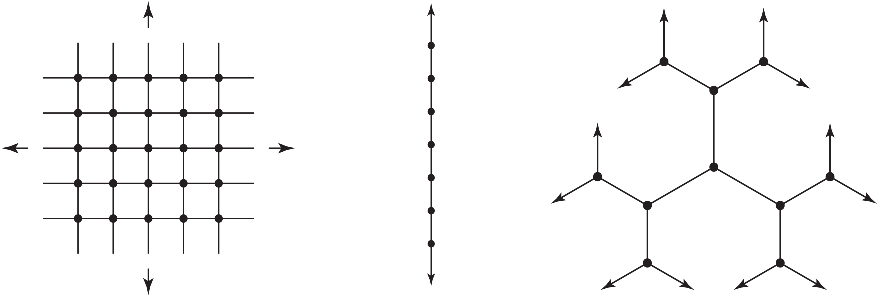

Going to infinity: accessibility

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A graph is *accessible* if there is some $k \in \mathbb{N}$ such that for any two distinct ends, there an edge set of size at most k separating them.

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Theorem (Thomassen & Woess 1993)

A locally finite connected transitive graph G is accessible if and only if its cut space is a finitely generated $\mathrm{Aut}(G)$*-module.*

A conjecture

Conjecture (Diestel 2010)

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

A conjecture is confirmed

Theorem

Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.

Applications

We obtain a combinatorial proof of

Theorem (Dunwoody 1985)

Finitely presented groups are accessible.

Stallings's structure theorem

Theorem (Stallings 1971)

Every finitely generated group G with more than one end splits non-trivially over a finite subgroup C, that is, $G = *_C A$ or $G = A *_C B$ for some subgroups $A \neq C \neq B$.

Splitting recursively

G

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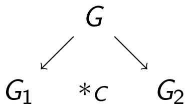

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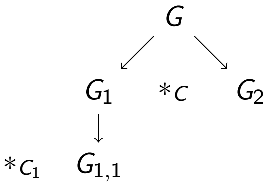

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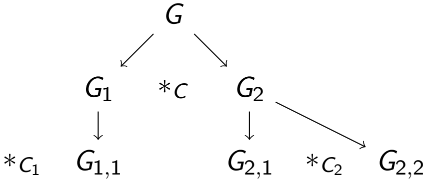

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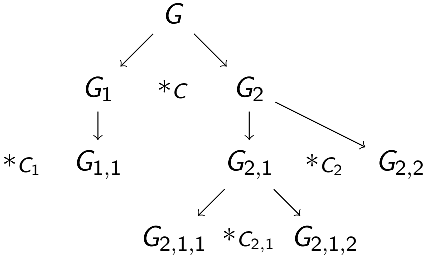

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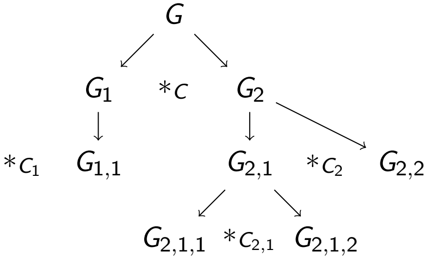

Definition

A finitely generated group is *accessible* if this process of successively decomposing factors with more than one end terminates after finitely many steps.

Conjecture (Wall 1971)

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- Disproved by Dunwoody 1993.

THEOREM (THOMASSEN & WOESS 1993)

A finitely generated group is accessible if and only one (and hence every) of its locally finite Cayley graphs is accessible.

Remark

The class of transitive graphs is much larger than the class of Cayley graphs (even in terms of quasi-isometry).

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The followings theorem answers a question of Woess and verifies a conjecture of Diestel and Leader:

Theorem (Eskin, Fisher, Whyte 2012)

There are locally finite transitive graphs not quasi-isometric to any finitely generated group.

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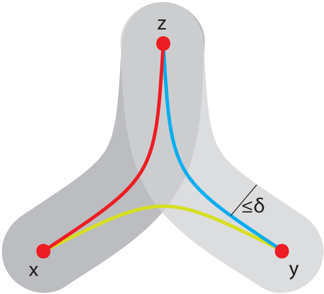
I guess that one-ended finitely generated groups that are not finitely presentable give rise to counterexamples.
(E.g. the lamplighter groups.)

Definition

A connected graph G is called hyperbolic if there exists some $\delta \geq 0$ such that for any three vertices x, y, z of G and for any three shortest paths, one between every two of the vertices, each of those paths lies in the δ-neighbourhood of the union of the other two.



Application II: hyperbolic graphs

Theorem (Gromov 1987)

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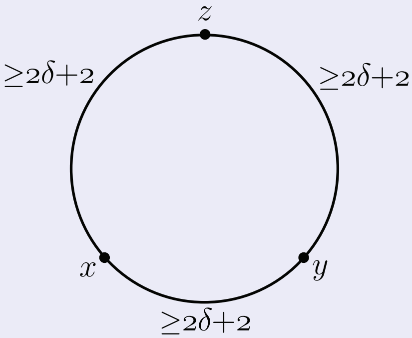

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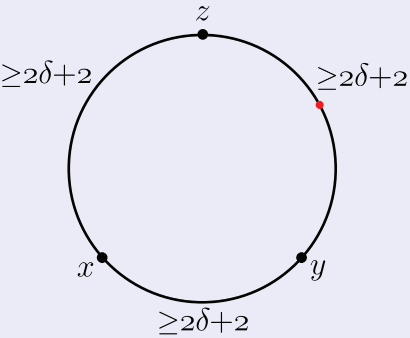

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2. There are planar Cayley graphs without any finite face boundaries.

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Basically the same proof for closed walks with a bit more complicated notion of generation yields a combinatorial proof of

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