

QUASI-ISOMETRY INVARIANCE OF HYPERBOLICITY IN SEMIMETRIC SPACES, DIGRAPHS AND SEMIGROUPS

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ABSTRACT. Gray and Kambites introduced a notion of hyperbolicity in the setting of semimetric spaces like digraphs or semigroups. We will prove that under a small additional geometric assumption their notion of hyperbolicity is preserved by quasi-isometries. Applied to semigroups, this will partially solve a problem of Gray and Kambites.

1. INTRODUCTION

Since Gromov's paper [12] on hyperbolic groups, hyperbolicity is a topic that received a lot of interest, mostly for geodesic metric spaces, graphs or groups, see e. g. [1, 2, 6, 9, 14]. Occasionally, some of these assumptions are dropped. One such example is the paper of Väisälä [16], where the metric spaces under consideration need not be geodesic. In the present paper, we try to extend the perhaps most important result, quasi-isometry invariance, to a generalisation of hyperbolicity in spaces, where the distance function need not be symmetric. The particular notion we are interested in was defined by Gray and Kambites [11] in order to have a geometric notion of hyperbolicity for monoids.

Gray and Kambites were not the first to look for a notion of hyperbolicity for monoids but their notion is the first geometric one that takes directions into account. Previous definitions either were based on context-free rewriting systems, see e. g. [3, 4, 7, 8, 13], or looked at the underlying undirected graph of Cayley digraphs of semigroups, see e. g. [3, 5, 8]. Portilla et al. [15] considered a notion of hyperbolicity in (a certain class of) digraphs that is also defined via hyperbolicity of their underlying undirected graphs. The advantage of a geometric notion for monoids that takes directions into account is that it also applies to other objects such as directed graphs, digraphs for short, or geodesic semimetric spaces. The latter are spaces where the distance function has $[0, \infty]$ as its range and we do ask for all condition needed for a metric except for $d(x, y) = d(y, x)$ for distinct x and y , see Section 2.1 for the formal definition. In the literature, these spaces are also known as quasi-metric or asymmetric metric spaces, but we use the term "semimetric" here. We refer to Gray and Kambites [10, Section 2] for a discussion about these terms.

In Section 4, we will see that the notion of Gray and Kambites indeed generalizes the one for geodesic metric spaces. Their notion is based on a thinness condition for geodesic triangles, see Section 3 for the precise definitions.

While Gray and Kambites were mostly interested in properties of hyperbolic monoids, such as the word problem or other semigroup-theoretic decision problems and the question of finite presentability of those monoids, we focus on a geometric

aspect of the spaces themselves. We consider the question whether this notion of hyperbolicity is preserved by quasi-isometries.

In the situation of metric spaces, in order to prove that quasi-isometries preserve hyperbolicity, first, it is shown that hyperbolicity is equivalent with geodesic stability and the latter is easily seen to be preserved by quasi-isometries. In order to prove that equivalence, one uses that both properties are also equivalent to exponential divergence of geodesics.

In the situation of semimetric spaces, we do no longer have equivalence between these three properties. More precisely, we will prove with a small additional geometric assumption that hyperbolicity implies both other properties but neither of the other two implies hyperbolicity nor do they imply each other, see Sections 5 and 6. Still, this is enough to prove that hyperbolicity (and geodesic stability) is preserved by quasi-isometries, see Section 7. It stays an open problem whether (exponential) divergence of geodesics is preserved by quasi-isometries.

Let us note that the definitions of e. g. quasi-isometries and geodesics are slightly different in semimetric spaces compared to metric spaces. For the introduction and as intuition, it is fine to just think of these notions as being similar to the ones in metric spaces and that they carry over the important properties that we will need. We omit these definitions here in the introduction and instead refer to the later sections for these definitions.

Let us briefly talk about the geometric assumption on the semimetric spaces that we need. In geodesic metric spaces X , we find for every point $x \in X$ and points y, z in the ball of radius r around x a y - z geodesic of length at most $2r$. For semimetric spaces Y , this is no longer true: if $x \in Y$ and $y, z \in Y$ with $d(x, y) < r$ and $d(x, z) < r$, then $d(y, z)$ need not be bounded or even be finite; it may also happen that the out-ball of radius r around x contains an infinite geodesic. The same also holds for the in-balls. We will ask our semimetric spaces to satisfy a condition that bounds the lengths of geodesics in out-balls or in-balls of finite radius in terms of the radius. So our results still cover a large proper subclass of the geodesic semimetric spaces that contain all geodesic metric spaces. For example, we will see that all digraphs of bounded degree satisfy this condition. As a consequence, when applying our results to semigroups, we only obtain results for right cancellative semigroups.

In the final section, Section 8, we apply our results to semigroups. Gray and Kambites [11] defined hyperbolicity for finitely generated semigroups by asking one of the Cayley digraphs be hyperbolic for a finite generating set. They ask whether for a finitely generated hyperbolic semigroup each of its Cayley digraphs with respect to finite generating sets is hyperbolic. As corollary from our result on quasi-isometry invariance of hyperbolicity, we answer this positively for right cancellative semigroups.

2. PRELIMINARIES

In this section, we define all basic notions related to pseudo-semimetric spaces and to digraphs.

2.1. Semimetric spaces. Let X be a set. A map $d: X \times X \rightarrow [0, \infty]$ is a *pseudo-semimetric* if

- (i) $d(x, x) = 0$ for all $x \in X$ and
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

We then call (X, d) a *pseudo-semimetric space*.

A pseudo-semimetric is a *semimetric* if $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$. Then we call (X, d) a *semimetric space*.

For x, y in a (pseudo-)semimetric space X , we set

$$d^{\leftrightarrow}(x, y) := \min\{d(x, y), d(y, x)\}.$$

(Pseudo-)Semimetric spaces X come along with two natural topologies that are defined via the out-balls and the in-balls. For $r \geq 0$ and $x \in X$, we set the *out-ball* and the *open out-ball* of radius r around x as

$$\mathcal{B}_r^+(x) := \{y \in X \mid d(x, y) \leq r\}, \quad \mathring{\mathcal{B}}_r^+(x) := \{y \in X \mid d(x, y) < r\}$$

and the *in-ball* and the *open in-ball* of radius r around x as

$$\mathcal{B}_r^-(x) := \{y \in X \mid d(y, x) \leq r\}, \quad \mathring{\mathcal{B}}_r^-(x) := \{y \in X \mid d(y, x) < r\},$$

respectively.

The open out-balls $\mathring{\mathcal{B}}_r^+(x)$ for all $r \geq 0$ and $x \in X$ generate the *forward* topology \mathcal{O}_f and the open in-balls $\mathring{\mathcal{B}}_r^-(x)$ for all $r \geq 0$ and $x \in X$ generate the *backward* topology \mathcal{O}_b .

For $a, b \in \mathbb{R}$ with $a < b$, we consider the interval $[a, b]$ with the above two topologies. A function $P: [a, b] \rightarrow X$ is a *directed path* if it is continuous with respect to the forward topologies of $[a, b]$ and X and if it is continuous with respect to the backward topologies of $[a, b]$ and X . Then $P(a)$ is the *starting point* and $P(b)$ is the *end point* of P . The *length* $\ell(P)$ of P is defined as

$$\ell(P) := \lim_{N \rightarrow \infty} \sum_{i=1}^N d(P(t_{i-1}), P(t_i))$$

with $t_i := a + i(b - a)/N$ for all $0 \leq i \leq N$. For $x, y \in X$, a *directed x - y path* is a directed path with starting point x and end point y . For $U, V \subseteq X$, a *directed U - V path* is a directed path with starting point in U and end point in V .

Let $P: [a, b] \rightarrow X$ and $Q: [a', b'] \rightarrow X$ be directed paths. They are *parallel* if $P(a) = Q(a')$ and $P(b) = Q(b')$ and they are *composable* if $P(b) = Q(a')$. If they are composable, then we denote by PQ the resulting directed path, their *composition*.

We say that $x \in X$ *lies on* P if it is in the image of P . For x, y on P with $x = P(r)$ and $y = P(R)$ such that $r < R$, we denote by xPy the subpath of P between x and y . With a slight abuse of notation, we denote for $r \geq 0$ by $\mathcal{B}_r^+(P)$, by $\mathcal{B}_r^-(P)$, the out-ball, the in-ball, of radius r around the image of P , respectively.

Let (X, d_X) and (Y, d_Y) be semimetric spaces. A map $f: X \rightarrow Y$ is a *quasi-isometric embedding* if there are constants $\gamma \geq 1$ and $c \geq 0$ such that

$$\gamma^{-1}d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq \gamma d_X(x, x') + c$$

for all $x, x' \in X$. It is a *quasi-isometry* if additionally for every $x \in X$ there is $y \in Y$ such that $d(f(x), y) \leq c$ and $d(y, f(x)) \leq c$. Then we say that X is *quasi-isometric* to Y . If we want to emphasize on the particular constants γ and c , we talk about (γ, c) -*quasi-isometries*. By Gray and Kambites [10, Proposition 1], being quasi-isometric is an equivalence relation. An *isometry* is a $(1, 0)$ -quasi-isometry.

A *geodesic* from $x \in X$ to $y \in X$, also called an *x - y geodesic*, is a directed path P from x to y with $d(u, v) = \ell(uPv)$ for all u, v on P with v on uPy . For subsets

$U, V \subseteq X$, a U - V *geodesic* is a directed U - V path that is a geodesic. For $\gamma \geq 1$ and $c \geq 0$, a (γ, c) -*quasi-geodesic* from x to y is a directed path from x to y with

$$\ell(uPv) \leq \gamma d(u, v) + c$$

for all u, v on P with v on uPy . We call X *geodesic* if there exists an x - y geodesic for all $x, y \in X$ with $d(x, y) < \infty$.

Note that x - y geodesics are not isometric images of $[0, d(x, y)]$ since we make no restrictions on $d(P(b), P(a))$ for $0 \leq a \leq b \leq 1$. Similarly, quasi-geodesics are not quasi-isometric images of $[0, d(x, y)]$.

2.2. Digraphs. Here, a digraph $D = (V(D), E(D))$ is an oriented multigraph. In particular, we allow our digraphs to have loops and multiple edges between the same two vertices. For a subset X of $V(D)$, we denote by $D[X]$ the digraph *induced by* X , i. e. the digraph with vertex set X and all edges of D both of whose incident vertices lie in X .

A *directed path* is a sequence $x_0 \dots x_n$ of vertices such that $x_i x_{i+1} \in E(D)$ for all $0 \leq i < n$. A *proper directed path* P is a sequence $x_0 \dots x_n$ of pairwise distinct vertices such that $x_i x_{i+1} \in E(D)$ for all $0 \leq i < n$.¹ The *length* $\ell(P)$ of P is n , the number of edges of the path.

If x_0, x_1, \dots are distinct vertices in D with $x_i x_{i+1} \in E(D)$ we call $x_0 x_1 \dots$ a *ray*. If we have $x_{i+1} x_i \in E(D)$ instead we say that $x_0 x_1 \dots$ is an *anti-ray*.

For $x, y \in V(D)$, let $d(x, y)$ be the length of a shortest directed path from x to y or ∞ if no such path exists. Note that in contrast to graphs, this distance function is not a metric but a semimetric. The *out-degree* of a vertex $x \in V(D)$ is the number of vertices $y \in V(D)$ with $d(x, y) = 1$ and the *in-degree* of x is the number of vertices $y \in V(D)$ with $d(y, x) = 1$. A digraph is *locally finite* if its in- and out-degrees are all finite.

We consider the the following semimetric space associated to a digraph D : we consider the 1-complex of the underlying undirected (multi-)graph of D . The distance between two points x, y corresponding to vertices u, v , respectively, is set as follows: $d(x, y) := d(u, v)$. For an inner point x of an edge uv , its distance to a point y corresponding to a vertex w or an inner point of an edge ww' is set as $d(x, y) := d(x, v) + d(v, w) + d(w, y)$, where we consider the edges e to be directed paths of length 1 in the semimetric space.

3. THIN TRIANGLES

Let X be a geodesic semimetric space. A *triangle* consists of three points of X and three directed paths, one between every two of those points. We call these paths the *sides* and the three point the *end points* of the triangle. The triangle is *geodesic* if all three sides are geodesics and it is *transitive* if two of its sides are composable and the resulting directed path is parallel to the third side.

¹With our terminology, we differ from the usual one for digraphs: what we call directed paths are usually directed walks and our proper directed paths are directed paths. We chose this different terminology because now directed paths in digraphs are the same regardless whether we consider it as a digraph or as a semimetric space.

We consider thin triangles as defined by Gray and Kambites in [11]. Let $\delta \geq 0$. A geodesic triangle is δ -thin if the following holds:

if P, Q, R are the sides of the triangle and the starting point of P is either the starting or the end point of Q and the last point of P is either the starting or the end point of R , then P is contained in $\mathcal{B}_\delta^+(Q) \cup \mathcal{B}_\delta^-(R)$.

If all geodesic triangles in X are δ -thin then X is δ -hyperbolic. We call X hyperbolic if it is δ' -hyperbolic for some $\delta' \geq 0$.

We note that this definition differs slightly from the definition of Gray and Kambites in that they only ask transitive geodesic triangles to be δ -thin. However, our first result is that – up to the constant δ – both definitions are equivalent.

Proposition 3.1. *Let X be a geodesic semimetric space and $\delta \geq 0$. If all transitive geodesic triangles are δ -thin, then all geodesic triangles are 3δ -thin.*

Proof. Let $x, y \in X$ such that there are geodesics P and Q from x to y and from y to x , respectively. Let R be the trivial directed path with image x . Then P, Q, R form the sides of a transitive geodesic triangle. So P lies in $\mathcal{B}_\delta^+(R) \cup \mathcal{B}_\delta^-(Q)$. Since P is a geodesic, only its directed subpath with starting point x and end point $P(\delta)$ is contained in $\mathcal{B}_\delta^+(R)$. The remaining part of P lies in $\mathcal{B}_\delta^-(Q)$ and thus all of P lies in $\mathcal{B}_{2\delta}^-(Q)$. Similarly, if we take the third side as trivial directed path y , we obtain that P lies in $\mathcal{B}_{2\delta}^+(Q)$.

Now let us consider an arbitrary geodesic triangle that is not transitive and let P, Q, R be its sides and x, y, z be its end points such that P is an x - y geodesic, Q is a y - z geodesic and R is a z - x geodesic. Then P and Q are composable and hence there is an x - z geodesic S . By our previous argumentation, we know that R lies in $\mathcal{B}_{2\delta}^-(S) \cap \mathcal{B}_{2\delta}^+(S)$. Since the geodesic triangle with sides P, Q, S is δ -thin, we obtain that R lies in $\mathcal{B}_{3\delta}^-(P) \cap \mathcal{B}_{3\delta}^+(Q)$. \square

Contrary to metric spaces, in a semimetric space X , the lengths of geodesics with starting and end point in $\mathcal{B}_r^+(x)$ for $x \in X$ and $r \in \mathbb{R}$ need not be bounded. (In metric spaces, this is bounded by $2r$.) However, we will often restrict ourselves to situations, where this is satisfied. We define the following two properties.

(B1) *There exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for every $x \in X$, for every $r \geq 0$ and for all $y, z \in \mathcal{B}_r^+(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(r)$.*

(B2) *There exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that for every $x \in X$, for every $r \geq 0$ and for all $y, z \in \mathcal{B}_r^-(x)$ the distance $d(y, z)$ is either ∞ or bounded by $f(r)$.*

These properties are satisfied in several applications as we will see now. As mentioned above, if X is also a metric space, then it satisfies (B1) and (B2) for the function $f(r) = 2r$.

Lemma 3.2. *Every hyperbolic digraph of bounded in- and bounded out-degree satisfies (B1) and (B2).*

Proof. Let D be a δ -hyperbolic digraph of in- and out-degree at most ρ and let $x \in V(D)$. Then the number of vertices in $\mathcal{B}_k^+(x)$ and the number of vertices in $\mathcal{B}_k^-(x)$ is bounded by $f_k := \sum_{i=0}^k (\rho - 1)^i$. Let P be a geodesic with starting and

end vertex in $\mathcal{B}_k^+(x)$. Let Q be a geodesic from x to the starting vertex of P and let R be a geodesic from x to the end vertex of P . So P, Q, R form the sides of a geodesic triangles. Since this is δ -thin, each vertex of P lies either in a ball $\mathcal{B}_\delta^+(y)$ for some vertex y on Q or in $\mathcal{B}_\delta^-(z)$ for some vertex z on R . In these balls, there are at most $kf_\delta + kf_\delta$ many vertices. Thus, this number is also an upper bound on the length of P . \square

It follows from Lemma 3.2 that locally finite transitive² hyperbolic digraphs satisfy (B1) and (B2). Another class of digraphs satisfying (B1) and (B2) are Cayley digraphs of finitely generated right cancellative hyperbolic semigroups, see Section 8 for more details.

A major property that follows from (B1) and (B2) is formulated in our next propositions. Note that, by Lemma 3.2, that proposition is a generalization of a result of Gray and Kambites [11, Lemma 3.1].

Proposition 3.3. *Let $\delta \geq 0$ and let X be a δ -hyperbolic geodesic semimetric space that satisfies (B1) for the function $f: \mathbb{R} \rightarrow \mathbb{R}$ and (B2) for the function $g: \mathbb{R} \rightarrow \mathbb{R}$.*

- (i) *If P, Q, R are the sides of a geodesic triangle such that the starting point of P is either the starting or the end point of Q and the end point of P is either the starting or the end point of R , then we have*

$$\ell(P) \leq (\ell(Q)/\varepsilon)f(\delta + \varepsilon) + (\ell(R)/\varepsilon)g(\delta + \varepsilon)$$

for all $\varepsilon > 0$.

- (ii) *If $x, y \in X$ with $d(x, y) \neq \infty$ and $d(y, x) \neq \infty$, then we have*

$$d(x, y) \leq (d(y, x)/\varepsilon)f(\delta + \varepsilon) + g(\delta)$$

and

$$d(x, y) \leq (d(y, x)/\varepsilon)g(\delta + \varepsilon) + f(\delta)$$

for all $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$. By assumption, P lies in $\mathcal{B}_\delta^+(Q) \cup \mathcal{B}_\delta^-(R)$. Let a be a point on Q and let $u = P(r)$, $v = P(R)$ in $\mathcal{B}_{\delta+\varepsilon}^+(a)$ such that $r < R$. By (B1), we have $d(u, v) \leq f(\delta + \varepsilon)$. All points on P that lie in $\mathcal{B}_\delta^+(Q)$ lie in the union of all $\mathcal{B}_{\delta+\varepsilon}^+(Q(i\varepsilon))$ with $i \in \mathbb{N}$ and $i\varepsilon \leq \ell(Q)$. For each $i \leq \ell(Q)/\varepsilon$, let P_i be a smallest subpath of P containing all points of P that lie in $\mathcal{B}_{\delta+\varepsilon}^+(Q(i\varepsilon))$. The lengths of these paths sum up to at most $(\ell(Q)/\varepsilon)f(\delta + \varepsilon)$. Analogously, the lengths of the smallest subpaths of P that contain all point on P that lie in $\mathcal{B}_{\delta+\varepsilon}^-(R(i\varepsilon))$ sum up to at most $(\ell(R)/\varepsilon)g(\delta + \varepsilon)$. This proves (i).

Finally, (ii) follows directly from (i): we just choose P to be an x - y geodesic and in the first case R to be the trivial path y and Q a y - x geodesic and in the second case R to be a y - x geodesic and Q the trivial path x . Note that it suffices to take $g(\delta)$ or $f(\delta)$ for the trivial path. \square

In our last result in this section, we prove that we can control that side in a transitive triangle that is parallel to the composition of the other two even further than the definition of a δ -thin triangle indicates.

²Recall that a digraph is *transitive* if for all $x, y \in V(D)$ there is an automorphism α of D that maps x to y .

Lemma 3.4. *Let $\delta \geq 0$ and let X be a δ -hyperbolic geodesic semimetric space that satisfies (B1) and (B2) for the function $f: \mathbb{R} \rightarrow \mathbb{R}$. Let P, Q, R be the sides of a geodesic triangle such that P and Q are composable and their composition is parallel to R . Then R lies in the out-ball of radius $6\delta + 2\delta f(\delta + 1)$ around $P \cup Q$ and in the in-ball of the same radius around $P \cup Q$.*

Proof. Set $k^* := 4\delta + 2\delta f(\delta + 1)$ and $k := k^* + 2\delta$. Let a be the starting and b be the end point of R . Let x be a point on R . Then $x \in \mathcal{B}_\delta^+(P) \cup \mathcal{B}_\delta^-(Q)$. If $x \in \mathcal{B}_\delta^+(P)$, then $d(P, x) \leq \delta \leq k^* \leq k$. So we may assume that there is a point y on Q with $d(x, y) \leq \delta$. Then $y \in \mathcal{B}_\delta^+(P) \cup \mathcal{B}_\delta^-(R)$.

Let us first assume that there is a point z on P with $d(z, y) \leq \delta$. Let P' be a z - y geodesic, Q' an a - y geodesic and R' an x - y geodesic. Then every point of Q' that lies in $\mathcal{B}_\delta^-(P')$ lies in $\mathcal{B}_{2\delta}^-(y)$. Thus, Q' lies completely in $\mathcal{B}_{3\delta}^+(P)$. Using Proposition 3.3 with $\varepsilon = 1$, at most the subpath of length $2\delta f(\delta + 1)$ with end vertex x of aRx lies in $\mathcal{B}_\delta^-(R')$. Thus, there exists u on Q' with $d(u, x) \leq \delta + 2\delta f(\delta + 1)$ and hence we have

$$d(P, x) \leq d(P, u) + d(u, x) \leq 4\delta + 2\delta f(\delta + 1) = k^*.$$

Let us now assume that there is a point z on R with $d(y, z) \leq \delta$. If z lies on aRx , then Proposition 3.3 implies $d(z, x) \leq 2\delta f(\delta + 1)$. So we have

$$d(y, x) \leq \delta + 2\delta f(\delta + 1) \leq k^*.$$

Hence, we assume that z lies on xRb . If $d(a, x) \leq 2\delta$, we immediately obtain $d(P, x) \leq 2\delta \leq k$. Otherwise, let x' be on aRx with $d(x', x) = 2\delta$. By an analogous situation for x' as for x either we have $d(P \cup Q, x') \leq k^*$ or there exists a point v on $x'Rb$ with $d(Q, v) \leq \delta$ and $d(x', v) \leq 2\delta$. Thus, we obtain $d(P \cup Q, x) \leq k$.

By a symmetric argument, we also obtain that $d(x, P \cup Q) \leq k$. \square

4. EXAMPLES OF HYPERBOLIC SEMIMETRIC SPACES

We recall that for geodesic metric spaces X , the definition of hyperbolicity is as follows: X is *hyperbolic* if there exists $\delta \geq 0$ such that for all $x, y, z \in X$ and all x - y , y - z and x - z geodesics each of these geodesics lies in the δ -neighbourhood of the other two geodesics. It directly follows from the definitions that hyperbolic geodesic metric spaces are also hyperbolic when considering the space as a semimetric space and using the definition from Section 3. This is similar to the observation of Gray and Kambites [11, Section 2] that starting with a graph and replacing each edge by two directed edges that are oppositely oriented leads to a digraph that is hyperbolic if and only if the graph is hyperbolic when viewed as a metric space.

Two points x, y in a semimetric space X are *equivalent*, if $d(x, y) < \infty$ and $d(y, x) < \infty$. This is an equivalence relation whose classes are the strongly connected components of X . Gray and Kambites [11, Proposition 2.5] showed that in hyperbolic digraphs, the strongly connected components are hyperbolic, too. Their proof immediately carries over to semimetric spaces, so we obtain the following.

Proposition 4.1. *Let X be a δ -hyperbolic geodesic semimetric space. Then every strongly connected component of X is δ -hyperbolic.* \square

In the case of graphs, the 0-hyperbolic graphs are the trees. For digraphs, this is no longer the case: while Gray and Kambites [11, Proposition 2.4] noted that digraphs whose underlying undirected graphs are trees are 0-hyperbolic, there are

more 0-hyperbolic digraphs. We will characterise the 0-hyperbolic digraphs by using one property of trees: they have uniquely determined paths between every two vertices.

Proposition 4.2. *Let D be a digraph. Then D is 0-hyperbolic if and only if there are no $x, y \in V(D)$ with two distinct directed x - y paths (which may be trivial).*

Proof. First, let us assume that D is 0-hyperbolic. Let us suppose that there are $x, y \in V(D)$ with two distinct directed x - y paths P_1 and P_2 . First, let us assume that $x = y$. Then we may assume that P_1 is trivial and that P_2 contains a vertex distinct from x . Let z be an out-neighbour of x on P_2 and let a be a point on an edge $e = xz$ distinct from x and z . Let P be a z - x geodesic. Then P does not contain the edge e . In particular, a lies neither on P nor on the trivial directed path x . This is a contradiction to 0-hyperbolicity.

So let us now assume that $x \neq y$ and that all vertices of P_1 are distinct and all vertices of P_2 are distinct. Let P be an x - y geodesic. We may assume that $P_1 \neq P$. Let $e = uv$ be an edge on P_1 that does not lie on P . Let Q_u^1 and Q_v^1 be x - u and x - v geodesics, respectively, and let Q_u^2 and Q_v^2 be u - y and v - y geodesics, respectively. If e does not lie on Q_v^1 , then the geodesic triangle with end vertices x, u, v and sides Q_u^1, Q_v^1 and e contradicts 0-hyperbolicity. Thus, e lies on Q_v^1 and the geodesic triangle with end vertices x, v, y and sides Q_v^1, Q_v^2 and P contradicts 0-hyperbolicity. Thus, there exists a uniquely determined directed x - y path.

Let us now assume that for all $x, y \in V(D)$ there exists a unique directed x - y path. If a geodesic triangle is not transitive, then there are two distinct directed z - z paths for every end vertex of the triangle: one trivial one and the other one following all three sides of the triangle. Thus, all geodesic triangles are transitive. But then the composition of the composable sides must coincide with the third side and thus the triangle is 0-thin. \square

5. DIVERGENCE OF GEODESICS

For geodesic metric spaces, hyperbolicity is equivalent to divergence of geodesics. For semimetric spaces, the analogue is not true. But we will see that at least one direction holds under the assumptions of (B1) and (B2): hyperbolicity implies (exponential) divergence of geodesics.

Let us start by giving the definition of divergence of geodesics in the context of semimetric spaces. Let X be a geodesic semimetric space. Let P be an x - y geodesic in X and let $0 \leq R \leq d(x, y)$. We denote by $P^x(R)$ be the point u with $d(x, u) = R$ and we denote by $P^y(R)$ be the point v with $d(v, y) = R$.

A function $e: \mathbb{R} \rightarrow \mathbb{R}$ is a *divergence function* if for all $x \in X$, all geodesics P_1, P_2 that start or end at x and all $r, R \in \mathbb{R}$ the following holds: if $d(P_1^x(R), P_2) > e(0)$, then every directed P_1 - P_2 path that lies outside of $\mathcal{B}_{R+r}^+(x) \cup \mathcal{B}_{R+r}^-(x)$ has length more than $e(r)$. We say that geodesics *diverge (exponentially)* in X if there exists an (exponential) divergence function. They *diverge properly* if $e(r) \rightarrow \infty$ for $r \rightarrow \infty$.

Proposition 5.1. *Let X be a hyperbolic geodesic semimetric space that satisfies (B1) and (B2). Then the geodesics diverge exponentially in X .*

Proof. Let $\delta \geq 0$ such that X is δ -hyperbolic. Let $x \in X$, let P_1, P_2 be two geodesics that have x as their starting or end point and let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a function such that X satisfies (B1) and (B2) for f . Let $\varepsilon > 0$. Set

$$e(0) := (2\delta + \varepsilon + 1)f(\delta + 1) + f(\delta) + \delta$$

and, for $r > 0$ and $k := 6\delta + 2\delta f(\delta + 1)$, set

$$e(r) := 2^{\frac{r-2\delta-1}{k}} - 1.$$

Let $R, r \in \mathbb{R}$ with $d(P_1^x(R), P_2) > e(0)$. Let P be a directed path from P_1 to P_2 that lies completely outside of $\mathcal{B}_{R+r}^+(x) \cup \mathcal{B}_{R+r}^-(x)$ and let Q be a geodesic from the starting point a of P to its end point b .

We are going to show the following.

- (i) If x is the starting point of P_1 , then $d(x, Q) \leq R + 2\delta$.
- (ii) If x is the end point of P_1 , then $d(Q, x) \leq R + \delta$.

If P_1 ends at x , then there is a point v either on Q with $d(v, P_1^x(R)) \leq \delta$ or on P_2 with $d(P_1^x(R), v) \leq \delta$. By the choice of R , we know that v must lie on Q and this shows (ii). So let P_1 start at x and let v be a point either on Q with $d(P_1^x(R), v) \leq \delta$ or on P_2 with $d(v, P_1^x(R)) \leq \delta$. If v lies on Q , then we have (i). So let us assume that v lies on P_2 . If x is the end point of P_2 , then the directed path $P_1^x(R)P_1aQbP_2v$ shows that we find a $P_1^x(R)$ - v geodesic, whose length is at most $\delta f(\delta + 1) + f(\delta)$ by Proposition 3.3 (ii), a contradiction to the choice of R . So let us assume that x is the starting point of P_2 . Let us suppose that (i) is false. Since $d(P_2^x(R + \delta), Q) > \delta$, there is a point u on xP_1a with $d(u, P_2^x(R + \delta)) \leq \delta$. This must lie on $P_1^x(R)P_1a$ as P_2 is a geodesic. Now let v be on $xP_2P_2^x(R + \delta)$ such that there is a point v' on $P_1^x(R)P_1a$ with $d(v', v) \leq \delta$ but for no point w on xP_2v with $d(w, v) \geq \varepsilon$ there exists a point w' on $P_1^x(R)P_1a$ with $d(w', w) \leq \delta$. Note that we also have $d(v, Q) > \delta$ since (i) is false. Let w be on xP_2v with $d(w, v) = \varepsilon$ if it exists and $w = x$ otherwise. Then there is a point w' on P_1 with $d(w', w) \leq \delta$ since (i) is false. This point must lie on $xP_1P_1^x(R)$. Considering a geodesic triangle with end points w', v, v' where the side between w' and v' is $w'P_1v'$ and the other two sides are directed towards v , we obtain $d(w', v') \leq (2\delta + \varepsilon + 1)f(\delta + 1)$ by Proposition 3.3 (i). Since $P_1^x(R)$ lies on the side between w' and v' , this is a contradiction to the choice of R . This finishes the proof of (i) and (ii).

We consider Q as being indexed with the empty word. We define a set of directed paths Q_σ with starting and end points on P and points u_σ on P indexed by finite words over $\{0, 1\}$ such that the following holds.

- (I) If Q_σ is a directed path of length more than one, then u_i is a point on P such that u_i halves the length of the directed subpath of P between the starting and the end points of Q_σ .
- (II) $Q_{\sigma 0}$ is a geodesic with the same starting point as Q_σ and with end point u_σ .
- (III) $Q_{\sigma 1}$ is a geodesic with u_σ as starting point and the same end point as Q_σ .

It follows from the choice of the points u_σ that the length of any $\{0, 1\}$ -word used as index is at most $\ln(\ell(P))$. By applying Lemma 3.4 recursively, we obtain for every $n \geq 0$ a point v_n on some Q_σ , where σ has length n , such that $d(x, v_n) \leq R + 2\delta + nk$ if x is the starting point of P_1 or $d(v_n, x) \leq R + \delta + nk$ if x is the end point of P_1 . Note that for $n = \ln(\ell(P))$ the point v_n has distance at most 1 to and from P . So we found a point y on P with either $d(x, y) \leq R + 2\delta + nk + 1$ or $d(y, x) \leq R + 2\delta + nk + 1$. Since P lies outside of $\mathcal{B}_{R+r}^+(x) \cap \mathcal{B}_{R+r}^-(x)$, we have

$$R + r \leq R + 2\delta + nk + 1 \leq R + 2\delta + k \ln(\ell(P)) + 1.$$

Thus, we have

$$e(r) < 2^{\frac{r-2\delta-1}{k}} \leq \ell(P),$$

which shows the assertion. \square

As mentioned earlier, the reverse implication of Proposition 5.1 does not hold as the following example shows.

Example 5.2. Let D be the digraph whose vertex set is given by two directed rays $x_0x_1\dots$ and $y_1y_2\dots$ with additional edges x_0y_1 and x_iy_i for all $i \in \mathbb{N}$. Then there is no $\delta \geq 0$ such that all geodesic triangles with end vertices x_0, x_i, y_i are δ -thin. Furthermore, every function $e: \mathbb{R} \rightarrow \mathbb{R}$ with $e(r) > 2$ for all $r \in \mathbb{R}$ is a divergence function for D .

A property for geodesic spaces is that divergence of geodesics that is faster than linear already implies that it is exponential. For geodesic semimetric spaces, we pose it as an open problem.

Problem 5.3. *Do there exist geodesic semimetric spaces in which geodesics diverge exponentially but not superlinearly?*

6. GEODESIC STABILITY

Another property in geodesic spaces that is equivalent to hyperbolicity is *geodesic stability*, that is, for every two points, all (γ, c) -quasi-geodesics between those two points lie close to each other. This is the main step in showing that hyperbolicity is preserved under quasi-isometries.

A geodesic semimetric space X satisfies *geodesic stability* if for all $\gamma \geq 1$ and $c \geq 0$ there exists a $\kappa \geq 0$ such that, for all $x, y \in X$ and all (γ, c) -quasi-geodesics P and Q from x to y , every point of P lies in $\mathcal{B}_\kappa^+(Q) \cap \mathcal{B}_\kappa^-(Q)$.

As a first step, we prove that geodesics lie close to and from quasi-geodesics with the same starting and end points (Proposition 6.1) and then that quasi-geodesics lie close to and from geodesics with the same starting and end points (Proposition 6.2).

Proposition 6.1. *Let X be a δ -hyperbolic geodesic semimetric space that satisfies (B1) and (B2) for the function $f: \mathbb{R} \rightarrow \mathbb{R}$. Let $\gamma \geq 1$ and $c \geq 0$. Then there is a constant $\kappa = \kappa(\delta, \gamma, c, f)$ depending only on δ, γ, c and f such that for all $x, y \in X$ every x - y geodesic lies in the κ -out-ball and in the κ -in-ball of every (γ, c) -quasi-geodesic from x to y .*

Proof. By Proposition 5.1, we find a superlinear divergence function $e: \mathbb{R} \rightarrow \mathbb{R}$ for X .³ First, we will show the following.

- (1) *There exists κ_1 such that for all $x, y \in X$ and for all x - y geodesics P and (γ, c) -quasi-geodesics Q from x to y we have $d^{\leftrightarrow}(z, Q) \leq \kappa_1$ for all z on P .*

Let us suppose that (1) does not hold. Then there are $x, y \in X$, an x - y geodesic P and a (γ, c) -quasi-geodesic Q from x to y such that there exists z on P with $2d^{\leftrightarrow}(z, Q) > e(0)f(\delta + 1) + f(\delta)$. Let $\varepsilon > 0$. We choose z on P such that for no z' on P we have $d^{\leftrightarrow}(z', Q) \geq D := d^{\leftrightarrow}(z, Q) + \varepsilon$. Set

$$K := (D + \delta)f(\delta + 1) + \delta + \varepsilon$$

and

$$L := (K + D)(2 + f(\delta + 1)) + \varepsilon.$$

³Proposition 5.1 implies that e is exponential, but in this proof it suffices to know that e is superlinear.

In order to show (1) we will first show the following

- (2) *There are points a^+, a^- on xPz and a_Q on Q with $D - \varepsilon \leq d(a^+, z) \leq L$ and $D - \varepsilon \leq d(a^-, z) \leq L$, with $d(a^+, a_Q) \leq K$ and $d(a_Q, a^-) \leq K$ and such that all $a^+ - a_Q$ and $a_Q - a^-$ geodesics lie outside of $B := \mathcal{B}_D^+(z) \cup \mathcal{B}_D^-(z)$.*

This obviously holds if $d(x, z) \leq L$ by choosing $a^+ = a^- = a_Q = x$. Otherwise, let a on xPz with $d(a, z) = K + D + (K + D)f(\delta + 1) + \varepsilon$. By the choice of z , we have $d^{\leftrightarrow}(a, Q) \leq D$.

First, let us consider the case that $d(Q, a) < D$. Let a_Q be on Q with $d(Q, a) \leq d(a_Q, a) + \varepsilon$. We consider a geodesic triangle with end points x, a, a_Q such that the side P' between x and a is a directed subpath of P . Let Q_1 be the side from x to a_Q and let Q_2 be the side from a_Q to a . Then there are u, v on Q_1 with $d(u, v) \leq \varepsilon$ such that u lies in $\mathcal{B}_\delta^+(P')$ and v lies in $\mathcal{B}_\delta^-(Q_2)$. Let w be on P' with $d(w, u) \leq \delta$ and let w' be on Q_2 with $d(v, w') \leq \delta$. Applying Proposition 3.3 to the geodesic triangle with end vertices v, a_Q, w' , we obtain $d(v, a_Q) \leq (D + \delta)f(\delta + 1)$ and thus

$$d(w, a_Q) \leq d(w, u) + d(u, v) + d(v, a_Q) \leq \delta + \varepsilon + (D + \delta)f(\delta + 1) = K.$$

It follows that

$$d(w, a) \leq d(w, a_Q) + d(a_Q, a) \leq K + D.$$

We set $a^+ := w$ and $a^- := a$. Then we have

$$d(a^+, z) = d(a^+, a^-) + d(a^-, z) \leq (K + D)(2 + f(\delta + 1)) + \varepsilon = L$$

and $d(a^+, a_Q) = d(w, a_Q) \leq K$. If there were a point on an $a^+ - a_Q$ or $a_Q - a^-$ geodesic inside B , then we apply Proposition 3.3 and obtain $d(a^+, z) \leq (K + D)f(\delta + 1)$ or $d(a^-, z) \leq 2Df(\delta + 1)$, which is impossible.

If $d(a, Q) \leq D$, then let a_Q be on Q with $d(a, Q) \leq d(a, a_Q) + \varepsilon$. Similarly as in the first case, we use the geodesic triangle with end points a, a_Q, y to find a point w on aPy with $d(a_Q, w) \leq K$ and $d(a, w) \leq K + D$. We set $a^+ := a$ and $a^- := w$. Again by Proposition 3.3, the $a^+ - a_Q$ and $a_Q - a^-$ geodesics lie outside of B . Thus, we have proved (2).

A completely symmetric argument shows the following.

- (3) *There are points b^+, b^- on zPy and b_Q on Q with $D - \varepsilon \leq d(z, b^+) \leq L$ and $D - \varepsilon \leq d(z, b^-) \leq L$, with $d(b^+, b_Q) \leq K$ and $d(b_Q, b^-) \leq K$ and such that all $b^+ - b_Q$ and $b_Q - b^-$ geodesics lie outside of B .*

Let z_a be on xPz and let z_b be on zPy with $d(z_a, z) = D = d(z, z_b)$. Then we have $d(z_a, z_b) = 2D > e(0)$ and

$$d(z_b, z_a) \geq \frac{d(z_a, z_b) - f(\delta)}{f(\delta + 1)} > e(0)$$

by Proposition 3.3. If a_Q lies on xQb_Q , then let R be the composition of an $a^+ - a_Q$ geodesic with $a_Q - b_Q$ and with a $b_Q - b^-$ geodesic. Then R is a directed $a^+ - b^-$ path outside of B . The composition of an $a_Q - a^-$ geodesic with $a^- - Pb^+$ and a $b^+ - b_Q$ geodesic shows that $d(a_Q, b_Q) \leq 2(K + L)$. So we have

$$\ell(R) \leq 2K + 2\gamma(K + L) + c,$$

which is linear in D .

Let us now assume that b_Q lies on xQa_Q . Let R be the composition of a $b^+ - b_Q$ geodesic with $b_Q - a_Q$ and with an $a_Q - a^-$ geodesic. Then R is a $b^+ - a^-$ path outside

of B . The composition of an $a_Q a^-$ geodesic with $a^- P b^+$ and a $b^+ b_Q$ geodesic shows $d(a_Q, b_Q) \leq 2(K+L)$. Since b_Q lies on the $x a_Q$ subpath of Q , there is a $b_Q a_Q$ geodesic, which has length at most $2(K+L)f(\delta+1) + f(\delta)$ by Proposition 3.3. Thus, we have

$$\ell(R) \leq 2K + 2\gamma(K+L)f(\delta+1) + \gamma f(\delta) + c,$$

which is also linear in D .

So in each case we obtain a contradiction to e being a superlinear divergence function. We thus proved (1).

Let

$$\kappa_2 > 2(\kappa_1 + 7\delta + (2\delta + \kappa_1)f(\delta+1))$$

and set

$$\lambda_1 := ((\kappa_2 + \kappa_1)f(\delta+1) + \kappa_1)f(\delta+1),$$

$$\lambda_2 := \gamma\lambda_1 + c,$$

$$\kappa_3 := \lambda_2 + 7\delta + 2\delta f(\delta+1) + \kappa_2/2.$$

We will prove that the assertion holds for $\kappa = \kappa_2 + \kappa_3$.

Let $x, y \in X$, let P be an x - y geodesic and let Q be a (γ, c) -quasi-geodesic from x to y . For all $0 \leq i$ with $i\kappa_2 \leq d(x, y)$ let x_i be the point on P with $d(x, x_i) = i\kappa_2$ and y_i be a point on Q with $d^{*+}(x_i, y_i) \leq \kappa_1$. We assume $x_0 = y_0$. We will show the following.

- (4) *For all $i \in \mathbb{N}$ such that $i\kappa_2 \leq d(x, y)$, we have $d(x_i, Q) \leq \kappa_3$ and $d(Q, x_i) \leq \kappa_3$.*

Note that the assertion follows for the constant κ from (4) directly.

Let $N \in \mathbb{N}$ be maximum such that $N\kappa_2 \leq d(x, y)$ and set $x_{N+1} := y_{N+1} := y$. For all $i \in \{0, \dots, N-1\}$ such that y_i lies on xQy_{i+1} , there is either a directed $y_i x_{i+1}$ path or a directed $x_i y_{i+1}$ path depending on the direction of the shortest path between x_i and y_i . Applying Proposition 3.3 twice to geodesic triangles with end points either x_i, y_i, x_{i+1} and y_i, x_{i+1}, y_{i+1} or x_i, x_{i+1}, y_{i+1} and x_i, y_i, y_{i+1} we obtain $d(y_i, y_{i+1}) \leq \lambda_1$.

Let $n \in \{1, \dots, N\}$ such that $d(x_n, y_n) \leq \kappa_2$ and let $i \in \{0, \dots, n-1\}$ be largest such that y_i lies on xQy_n . Then y_i lies on xQy_{i+1} and y_n lies on $y_i Q y_{i+1}$. Since Q is a (γ, c) -quasi-geodesic, the length of the path $y_i Q y_{i+1}$ is at most λ_2 and thus we have $d(y_i, y_n) \leq \lambda_2$.

We will show $d(y_i, x_n) \leq \kappa_3$. This holds trivially, if $d(y_i, x_i) \leq \kappa_1$. So let us assume that $d(x_i, y_i) \leq \kappa_1$. Note that there is a directed $x_i y_n$ path, so we find an $x_i y_n$ geodesic R_1 . Let P' the subpath of P between x_i and x_n and let Q' be a $y_i y_n$ geodesic. Let R_2 be a $x_n y_n$ geodesic and let R_3 be a geodesic between x_i and y_i . We consider two geodesic triangles: one with x_i, x_n, y_n as its end points and P', R_1, R_2 as its sides and the other with x_i, y_i, y_n as its end points and Q', R_1, R_3 as its sides. Let u be on P' with $d(u, x_n) = \kappa_2/2$. By hyperbolicity, there is either a point v on R_1 with $d(v, u) \leq \delta$ or a point v' on R_2 with $d(u, v') \leq \delta$. The latter case leads to a contradiction since then Proposition 3.3 implies $d(u, x_n) \leq (\delta + \kappa_1)f(\delta+1) < \kappa_2/2$. So we find a point v on R_1 with $d(v, u) \leq \delta$.

Applying Lemma 3.4, we find a point w on either Q' or R_3 with $d(w, v) \leq 6\delta + 2\delta f(\delta+1)$. If w is on R_3 , we find a path from x_i via w and v to u of length at most $\kappa_1 + 7\delta + 2\delta f(\delta+1)$, which contradicts $d(x_i, u) = \kappa_2/2$. Thus, w lies on Q'

and hence, the composition of $y_i Q' w$ with a w - v geodesic, a v - u geodesic and $u P x_n$ shows

$$d(y_i, x_n) \leq \lambda_2 + 7\delta + 2\delta f(\delta + 1) + \kappa_2/2 = \kappa_3.$$

Now let us consider the case that for $n \in \{1, \dots, N\}$ we have $d(y_n, x_n) \leq \kappa_1$. We follow a completely symmetric argument as in the previous case with reversed directions. Most importantly, in this case x_i is chosen so that $i \in \{n+1, \dots, N+1\}$ is smallest such that y_i lies on $y_n Q y$.

This finishes the proof of (4) and thus the proof of our assertion as mentioned earlier. \square

Proposition 6.2. *Let X be a δ -hyperbolic geodesic semimetric space satisfying (B1) and (B2) for the function $f: \mathbb{R} \rightarrow \mathbb{R}$. Let $\gamma \geq 1$ and $c \geq 0$. Then there is a constant $\lambda = \lambda(\delta, \gamma, c, f)$ depending only on δ, γ, c and f such that for all $x, y \in X$ every (γ, c) -quasi-geodesic from x to y lies in the λ -out-ball and in the λ -in-ball of every x - y geodesic.*

Proof. Let κ be the constant from Proposition 6.1. Let $x, y \in X$ and let P be an x - y geodesic and Q be a (γ, c) -quasi-geodesic from x to y . Let z be on Q such that either $d(z, P) > \kappa$ or $d(P, z) > \kappa$. Set $Q_1 := xQz$ and $Q_2 := zQy$. Let $\varepsilon > 0$ and let u be on P such that there exists w on Q_1 with $d(u, w) \leq \kappa$ and such that for no point u' on uPy with $d(u, u') \geq \varepsilon$ there is w' on Q_1 with $d(u', w') \leq \kappa$.

If there is w' on Q_2 with $d(w', u) \leq \kappa$, then $d(w', w) \leq 2\kappa$ and since there is a directed w - w' path, we have $d(w, w') \leq 2\kappa f(\delta + 1) + f(\delta)$ by Proposition 3.3. So the distance from w to w' on Q is at most $2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c$ and hence z and u lie on a closed directed walk of length at most $2\kappa + 2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c$.

Let us now assume that there is no w' on Q_2 with $d(w', u) \leq \kappa$. In particular, we have $u \neq y$. By Proposition 6.1, there is a point w' on Q_1 with $d(w', u) \leq \kappa$. Let v be on uPy with $d(u, v) = \varepsilon$. By the choice of u , there is a point v' on Q_2 with $d(v, v') \leq \kappa$. So we have $d(w', v') \leq 2\kappa + \varepsilon$ and hence the length of the directed w' - v' subpath of Q is at most $\gamma(2\kappa + \varepsilon) + c$. Note that this directed subpath contains z . We consider the geodesic triangle with end points v, v', y such that the side P' between v and y is vPy , the side R_1 is a v' - y geodesic and the side R_2 is a v - v' geodesic. Then there a, b on R_1 with a on $v'R_1b$ and $d(a, b) \leq \varepsilon$ such that $d(R_2, a) \leq \delta$ and $d(b, P') \leq \delta$. By Proposition 3.3, we conclude $d(v', a) \leq (\kappa + \delta)f(\delta + 1)$. So we have

$$d(v', P) \leq d(v', a) + d(a, b) + d(b, P) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta$$

and hence

$$d(z, P) \leq d(z, v') + d(v', P) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta + \gamma(2\kappa + \varepsilon) + c.$$

With a symmetric argument by considering a geodesic triangle with end vertices x, w', u , we conclude

$$d(P, z) \leq (\kappa + \delta)f(\delta + 1) + \varepsilon + \delta + \gamma(2\kappa + \varepsilon) + c.$$

So the assertion follows for

$$\lambda := \max\{2\kappa + 2\gamma\kappa f(\delta + 1) + \gamma f(\delta) + c, (\kappa + \delta)f(\delta + 1) + 1 + \delta + \gamma(2\kappa + 1) + c\}. \quad \square$$

We obtain the following corollary directly from Propositions 6.1 and 6.2 the following.

Corollary 6.3. *Every hyperbolic geodesic semimetric space that satisfies (B1) and (B2) satisfies geodesic stability. \square*

To compare geodesic stability with divergence of geodesics, we note that exponential divergence of geodesics does not imply geodesic stability as Example 5.2 shows. The reverse is also false as our next example shows. That example is a digraph that satisfies geodesic stability but whose geodesics do not diverge properly and that is not hyperbolic.

Example 6.4. Let x_0, x_1, \dots and y_0, y_1, \dots be two disjoint sequences of distinct vertices. For every i , we add an edge $x_i y_i$, a path from x_{i+1} to x_i of length $i+1$ and a path from y_i to y_{i+1} of length $i+1$ such that all these new paths are pairwise disjoint and meet the two sequences only in its end vertices. For $n \in \mathbb{N}$, we take two geodesics: one that ends at x_0 and passed through all x_i for $i \leq n$ and one that starts at x_0 and passes through all y_i for $i \leq n$. There are vertices on the first geodesic that have distance at least $n/2$ to the other geodesic, but the edge $x_n y_n$ shows that these two geodesics do not diverge properly. The geodesic triangles with end vertices x_n, x_{n+1} and y_{n+1} are not $(n/2 - 1)$ -thin. So D is not hyperbolic.

Let R_1 be the geodesic anti-ray that passes through all x_i and ends at x_0 and let R_2 be the geodesic ray that starts at y_0 and passes through all y_i . Then every directed path that is not a subpath of R_1 or R_2 can be obtained by composing a subpath of R_1 with a path consisting of an edge $x_i y_i$ and with a subpath of R_2 . Hence, if a geodesic and a (γ, c) -quasi-geodesic have the same first and the same last vertex, then they differ in that an edge $x_i y_i$ of the geodesic is replaced by $P = x_i R_1 x_j y_j R_2 y_i$ for some $j < i$. Note that we have $\ell(P) \leq \gamma + c$ as $d(x_i, y_i) = 1$. It follows that the distance of points on P from and to the geodesic is at most $\ell(P)$ and hence bounded by $\gamma + c$. Trivially, all points on the geodesic lie within 1 from and to the quasi-geodesic. It follows that the digraph satisfies geodesic stability.

7. QUASI-ISOMETRIES

In this section, we look at quasi-isometries of geodesic semimetric spaces and ask which of our properties are preserved by quasi-isometries. This is motivated by the fact that hyperbolicity of geodesic metric spaces is preserved by quasi-isometries, see e. g. [2, Theorem III.H.1.9]. We will prove that hyperbolicity (with the additional assumptions of (B1) and (B2)) and geodesic stability are preserved under quasi-isometries and pose it as open problem what happens for (superlinear/exponential) divergence of geodesics.

Proposition 7.1. *Geodesic stability is preserved by quasi-isometries.*

Proof. Let X and Y be geodesic semimetric spaces such that Y satisfies geodesic stability and let $f: X \rightarrow Y$ be a (λ, c) -quasi-isometry for some $\lambda \geq 1$ and $c \geq 0$. Let $x_1, x_2 \in X$, let P be an x_1 - x_2 geodesic and let Q be a (λ, ℓ) -quasi-geodesic from x_1 to x_2 for some $\lambda \geq 1$ and $\ell \geq 0$. Then there are constants $\gamma' \geq 1$ and $c' \geq 0$ such that the images of P and Q under f are (γ', c') -quasi-geodesics, see Gray and Kambites [10, Proposition 1]. By geodesic stability, there is a constant $\kappa \geq 0$ such that

$$f(P) \subseteq \mathcal{B}_\kappa^+(f(Q)) \cap \mathcal{B}_\kappa^-(f(Q))$$

and

$$f(Q) \subseteq \mathcal{B}_\kappa^+(f(P)) \cap \mathcal{B}_\kappa^-(f(P)).$$

Since f is a quasi-isometry, there is a constant $\kappa' \geq 0$ such that

$$P \subseteq \mathcal{B}_{\kappa'}^+(Q) \cap \mathcal{B}_{\kappa'}^-(Q)$$

and

$$Q \subseteq \mathcal{B}_{\kappa'}^+(P) \cap \mathcal{B}_{\kappa'}^-(P). \quad \square$$

The proof that hyperbolicity is preserved under quasi-isometries relies also on their geodesic stability.

Proposition 7.2. *Let X and Y be two geodesic semimetric spaces such that X is hyperbolic and satisfies (B1) and (B2). If X is quasi-isometric to Y , then Y is hyperbolic.*

Proof. Let X be δ -hyperbolic for some $\delta \geq 0$. Let $f: Y \rightarrow X$ be a (γ, c) -quasi-isometry for some $\gamma \geq 1$ and $c \geq 0$. We consider a geodesic triangle in Y with end points x, y, z and sides P_1, P_2, P_3 , where P_1 is an x - y geodesic, P_2 a geodesic between y and z and P_3 are geodesic between x and z . Then the images of the sides are (γ, c) -quasi-geodesics in X . Let κ be a constant such that geodesic stability holds for κ in X with respect to (γ, c) -quasi-geodesics, see Corollary 6.3.

Let u be on P_1 . By geodesic stability, we find u_1, u_2 on some $f(x)$ - $f(y)$ geodesic in X with $d(u, u_1) \leq \kappa$ and $d(u_2, u) \leq \kappa$. As geodesic triangles are δ -thin, there are v_1, v_2 on geodesics either from $f(x)$ to $f(z)$ or from $f(y)$ to $f(z)$ with $d(u_1, v_1) \leq \delta$ and $d(v_2, u_2) \leq \delta$. Applying geodesic stability once more, there are w_1, w_2 on the images under f of P_3 and P_2 with $d(v_1, w_1) \leq \kappa$ and $d(w_2, v_2) \leq \kappa$. So we have $d(u, w_1) \leq 2\kappa + \delta$ and $d(w_2, u) \leq 2\kappa + \delta$. Since f is a quasi-isometry, it follows that there exists $\delta' \geq 0$, depending only on κ, δ, γ and c such that the geodesic triangle is δ' -thin. \square

We end this section by a stating the corresponding problem for divergence of geodesics.

Problem 7.3. *Is proper divergence (or superlinear/exponential divergence) of geodesics preserved under quasi-isometries?*

8. HYPERBOLIC SEMIGROUPS

Let S be a semigroups and let A be a finite generating set of S . The (right) Cayley digraph of S (with respect to A) has S as its vertex set and edges from x to xa for all $x \in S$ and $a \in A$. This way, S is a semimetric space. A straight-forward argument shows that different finite generating sets define quasi-isometric Cayley digraphs, see Gray and Kambites [10, Proposition 4].

A finitely generated semigroup is *hyperbolic* if it has a finite generating set such that its Cayley digraph with respect to that generating set is hyperbolic.

Gray and Kambites [11] asked whether every Cayley digraph of a hyperbolic monoid with respect to a finite generating set is hyperbolic. Proposition 7.2 allows us to give at least a partial answer.

Proposition 8.1. *For every hyperbolic right cancellative semigroup, each of its Cayley digraphs with respect to finite generating sets is hyperbolic.*

Proof. Let S be a hyperbolic right cancellative semigroup. Let A be a finite generating set and let D be the Cayley digraph of S with respect to A . Then the

out-degree of all vertices of D is $|A|$. Similarly, since S is right cancellative, the in-degree of all vertices is bounded by $|A|$. In particular, D satisfies (B1) and (B2).

Let B be a finite generating set of S such that the Cayley digraph D' of S with respect to B is hyperbolic. We also have the in- and out-degrees of all vertices bounded by $|B|$ in this situation. Thus, Lemma 3.2 implies that D' satisfies (B1) and (B2), too. Changing finite generating sets of semigroups lead to quasi-isometric Cayley digraphs, see e. g. [10, Proposition 4]. So Proposition 7.2 directly implies that D is hyperbolic. \square

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