THE CLASSIFICATION OF FINITE AND LOCALLY FINITE CONNECTED-HOMOGENEOUS DIGRAPHS

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Abstract. We classify the finite connected-homogeneous digraphs, as well as the infinite locally finite such digraphs with precisely one end. This completes the classification of all the locally finite connected-homogeneous digraphs.

1. Introduction

A graph is called *homogeneous* if every isomorphism between two finite induced subgraphs extends to an automorphism of the entire graph. The countable homogeneous graphs were classified in [5, 8, 19, 22]. Weakening the assumptions of homogeneity so that only isomorphisms between finite *connected* induced subgraphs have to extend to automorphisms leads to the notion of *connected-homogeneous* graphs, or simply *C-homogeneous* graphs. Countable C-homogeneous graphs were classified in [4, 6, 9, 12, 13].

For directed graphs, or digraphs, the same notions of homogeneity and C-homogeneity apply. The countable homogeneous digraphs were classified in [2, 17, 18]. Of the C-homogeneous digraphs only those that have more than one end have been classified [10, 11] (independent of their cardinality). This paper completes the classification of locally finite C-homogeneous digraphs, by describing those that are finite or have precisely one end (Theorem 7.1).

Undirected locally finite C-homogeneous graphs cannot have precisely one end (see [20]). Directed such graphs can; but they have a very restricted structure. We shall see in Section 6 that these digraphs are quotients of one particular locally finite C-homogeneous digraph with infinitely many ends, the digraph $T(2)$. This is the digraph in which every vertex is a cut vertex and lies on precisely two directed triangles and in no other block (for a picture of the digraph $T(2)$, see Figure 2). Some of the finite examples are also quotients of $T(2)$. It turns out that all the other finite connected C-homogeneous digraphs have their origin in the finite homogeneous digraphs; they are canonical generalizations of the homogeneous digraphs. See Section 4 and Section 5 for more details.

Recall that every connected locally finite transitive (di)graph has either none, one, two, or infinitely many ends, see [3]. Together with the classification by Gray and Möller [10] of the two-ended digraphs and the classification of the infinitely-ended digraphs [11], our results thus complete the classification of all the locally finite C-homogeneous digraphs (see Theorem 2.1 for the classification result of the locally finite C-homogeneous digraphs).

The paper is structured as follows: first, we define in Section 2 all necessary digraphs that we use in this paper and state the classification result of the locally finite C-homogeneous digraphs. After introducing more basic notation on digraphs in Section 3, we look at the out-neighborhood of any vertex in a locally finite C-homogeneous connected digraph. If this out-neighborhood is not independent, then we obtain in Section 4 a complete classification of this situation. In Section 5, we look at the case that the out-neighborhood is independent. Except for one subcase, this analysis will be completed in Section 5. We discuss the remaining situation in Section 6. In Section 7, we combine our previous results and prove the remaining direction of the classification result of all locally finite C-homogeneous connected digraphs with at most one end.

2. The classification

In this section, we will define all digraphs that occur in the classification of the locally finite C-homogeneous digraphs and we will state this classification in Theorem 2.1.

By C_m we usually denote directed cycles of length m. But if it is obvious from the context that we are considering a subdigraph of a bipartite reachability digraph (see Section 3 for the definition of reachability digraphs), then we also use C_m to denote a cycle in the reachability digraph. We also use C_m to denote a cycle of length m in an undirected graph. *Triangles* are cycles of length 3. We call a cycle in a (di-)graph *induced* if no two of its vertices are adjacent in the (di-)graph but not in the cycle.

A vertex set is *independent* if no two of its vertices are adjacent. The digraph $\overline{K}_n$ is the digraph on n vertices whose vertex set is independent.

For two digraphs D, D' we denote by $D[D']$ the *lexicographic product* of D and D', that is the digraph with vertex set $VD \times VD'$ and edge set

$$\{(x,x')(y,y') \mid xy \in ED \text{ or } (x = y \text{ and } x'y' \in ED')\}.$$

A *complete bipartite* digraph $K_{k,\ell}$ is a bipartite digraph which, for some bipartition $\{A, B\}$ with $|A| = k$ and $|B| = \ell$, contains all edges from A to B. The *(directed) complement of a perfect matching* CP_k is the (di-)graph obtained from the complete bipartite (di-)graph $K_{k,k}$ where a perfect matching between A and B is removed.

Let Y_k be the digraph with vertex set $V_1 \cup V_2 \cup V_3$ where the V_i denote pairwise disjoint independent sets of the same cardinality k such that the induced subdigraphs $D[V_i, V_{i+1}]$ with vertex sets $V_i \cup V_{i+1}$ (for $i = 1, 2, 3$ with $V_4 = V_1$) are complements of perfect matchings such that all edges are directed from V_i to V_{i+1} and such that the directed tripartite complement of D is the disjoint union of k copies of C_3, where the *directed tripartite complement* of D is the digraph

$$(VD, (\bigcup_{i=1,2,3} (V_i \times V_{i+1})) \setminus ED).$$

Let $\sim$ be an equivalence relation on the vertices of some digraph D. By $D_\sim$ we denote the digraph whose vertex set is the set of equivalence classes and with edges XY whenever there are representatives $x \in X$ and $y \in Y$ with $xy \in ED$. This is not a digraph in our restrictive meaning because it may have loops or for an edge xy there might also exist the edge yx. However, we just consider such equivalence relations that make $D_\sim$ into a digraph, that is, whose adjacency relation is irreflexive and anti-symmetric.

Given an edge-transitive bipartite digraph Δ with bipartition $\{A, B\}$ such that every edge is directed from A to B we define $DL(\Delta)$ to be the unique connected digraph with reachability digraph Δ such that each vertex separates the digraph and has both in- and out-neighbors (cf. [1, 10]). So $DL(\Delta)$ is the unique digraph of connectivity 1 such that each vertex v lies in precisely two blocks[1] each of which is isomorphic to Δ and such that one of these blocks contains all successors of v and the other contains all predecessors of v.

Let H be the digraph depicted in Figure 1.

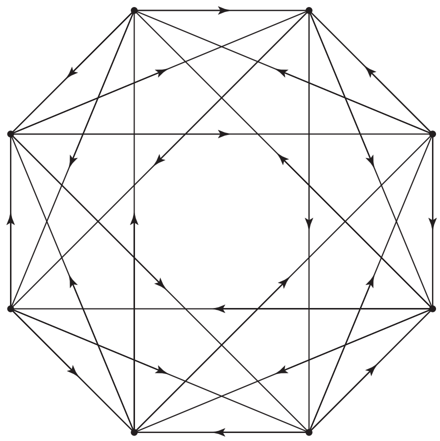

FIGURE 1. The digraph H

A tree (an undirected tree) is *semiregular* if vertices of even distance have the same degree. So there are at most two distinct degrees k and ℓ of a semiregular tree. We denote the tree by $T_{k,\ell}$. A digraph is a *tree* if its underlying undirected graph is a tree.

Let $\ell \geq 2$ be an integer. By $X_\ell(C_3)$ we denote the digraph with connectivity 1 such that each vertex is a cut vertex and lies in ℓ distinct blocks each of which is isomorphic to C_3. Then the digraph $T(2)$ mentioned in the introduction is the digraph $X_2(C_3)$. It is shown in Figure 2.

Let us define a class of digraphs with connectivity 2 and reachability digraph CP_k. Given integers $m \geq 2$ and $k \geq 3$ consider the tree $T_{k,m}$ and let $\{U, W\}$ be its natural bipartition such that the vertices in U have degree m. Now subdivide each edge once and endow the neighborhood of each $u \in U$ with a cyclic order. Then for each new vertex y let u_y be its unique neighbor in U and denote by $\sigma(y)$ the successor of y in the cyclic order of $N(u_y)$. For each $w \in W$ and each $x \in N(w)$ we add an edge directed from x to all $\sigma(y)$ with $y \in N(w) \setminus \{x\}$. Finally, we delete the vertices of the $T_{k,m}$ together with all edges incident with such a vertex to obtain the digraph $M(k,m)$. The left digraph in Figure 3 is the digraph $M(3,3)$ together with its construction tree.

The last class of digraphs that we define in preparation for the classification theorem is a class of digraphs with connectivity 2 and reachability digraph $K_{2,2}$. For an integer $m \geq 2$ consider the tree $T_{2,2m}$ and let $\{U, W\}$ be its natural bipartition such that the vertices in U have degree $2m$. Now subdivide every edge once and

[1]The *blocks* of a graph are its maximal 2-connected subgraphs and the *blocks* of a digraph are those of its underlying undirected graph.

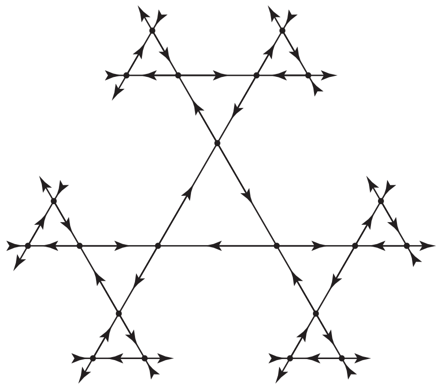

FIGURE 2. The digraph $T(2)$

enumerate the neighborhood of each $u \in U$ from 1 to $2m$ in a such way that the two neighbors of each $w \in W$ have distinct parity. For each new vertex x let u_x be its unique neighbor in U and define $\sigma(x)$ to be the successor of x in the cyclic order of $N(u_x)$. For any $w \in W$ we have a neighbor a_w with even index, and a neighbor b_w with odd index. Then we add edges from both a_w and $\sigma(a_w)$ to both b_w and $\sigma(b_w)$. Finally we delete the vertices of the $T_{2,2m}$ together with all edges incident with such a vertex. By $M'(2m)$ we denote the resulting digraph. The right digraph in Figure 3 is the digraph $M'(6)$ together with its construction tree.

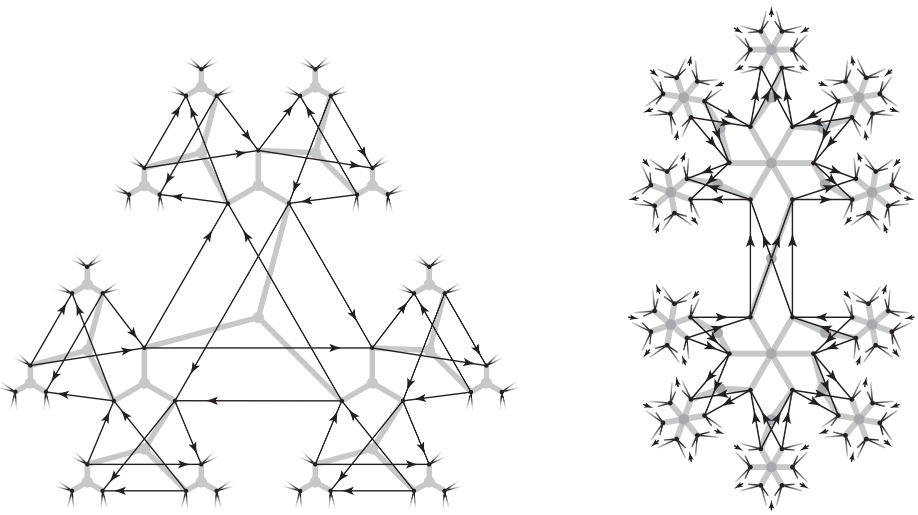

FIGURE 3. On the left side the digraph $M(3,3)$ and on the right side the digraph $M'(6)$. The grey tree underlying both digraphs, is the tree used for their construction.

The definition of all these digraphs enables us to state the classification result for the locally finite C-homogeneous digraphs that combines Theorem 7.1, [10, Theorem 6.2], and [11, Theorem 7.6].

Theorem 2.1. *Let D be a locally finite digraph. Then D is C-homogeneous if and only if all its components are isomorphic to the same one of the following digraphs:*

(i) K_1;
(ii) $C_m[\overline{K}_n]$ *for integers* $m \geq 3, n \geq 1$;
(iii) $H[\overline{K}_n]$ *for some integer* $n \geq 1$;
(iv) Y_k *for some integer* $k \geq 3$;
(v) $R[\overline{K}_n]$ *for some integer* $n \geq 1$ *where* R *is a directed double ray;*
(vi) *a tree with constant in- and out-degree;*
(vii) $X_\ell(C_3)$ *for some integer* $\ell \geq 2$
(viii) $DL(\Delta)$ *where* Δ *is*
  (a) CP_k *for some integer* $k \geq 3$,
  (b) C_{2m} *for some integer* $m \geq 2$, *or*
  (c) $K_{k,l}$ *for integers* $k, l \geq 2$;
(ix) $M(k,m)$ *for integers* $k \geq 3$ *and* $m \geq 2$;
(x) $M'(2m)$ *for some integer* $m \geq 2$;
(xi) $T(2)_\sim$, *where* $\sim$ *is a non-universal* $\mathrm{Aut}(T(2))$*-invariant equivalence relation on* $VT(2)$.

Note that the examples of (v)–(x) have more than one end, so they are covered by the classification of the connected C-homogeneous digraphs with more than one end, see [10, 11].

The only part that is not explicit in this classification result is (xi). We will discuss this situation in more detail in Section 6 and show that it is equivalent to give an explicit list here or to give an explicit list of those subgroups of the modular group that contain an involution.

3. Preliminaries

3.1. **Definitions.** A *digraph* $D = (VD, ED)$ consists of a non-empty set VD of *vertices* and an asymmetric, i.e. irreflexive and anti-symmetric, relation ED on VD, its *edges*. For $(x, y) \in ED$ we simply write $xy \in ED$ and say that the edge xy is *directed from x to y*. The vertices x and y are *adjacent* if either $xy \in ED$ or $yx \in ED$.

For $x \in VD$ we denote by $N^+(x)$ the *out-neighborhood* $\{y \in VD \mid xy \in ED\}$, by $N^-(x)$ the *in-neighborhood* $\{y \in VD \mid yx \in ED\}$, and by $N(x)$ the *neighborhood* $N^+(x) \cup N^-(x)$ of x. The *out-degree* $d^+(x)$, the *in-degree* $d^-(x)$, and the *degree* $d(x)$ of x are the cardinalities of $N^+(x)$, of $N^-(x)$, and of $N(x)$, respectively. If D is a transitive digraph, then we denote by d^+, d^- the value of $d^+(x), d^-(x)$, respectively, for any $x \in VD$. Every element of $N^+(x)$ is called a *successor* (or *out-neighbor*) of x and every element of $N^-(x)$ is called a *predecessor* (or *in-neighbor*) of x. By $D^+(x)$ we denote the induced subdigraph $D[N^+(x)]$ with vertex set $N^+(x)$ and by $D^-(x)$ we denote $D[N^-(x)]$.[2]

A vertex, vertex set, or subdigraph *separates* a digraph if its deletion leaves more than one component.

For a path P (not necessarily directed) and any two vertices x, y of P, let xPy denote the unique subpath of P that starts at x and ends at y. A *(k-)arc* is a

[2]Note that $D[X]$ has two different meanings depending on whether X is a digraph or a vertex set: if X is a digraph, it is the lexicographic product of D and X and, if $X \subseteq VD$ is a vertex set, it is a subgraph of D.

directed path (of length k). Notice that in general paths need not be directed paths. An *ancestor* (*descendant*) of a vertex x is any vertex y for which there exists an arc from y to x (from x to y). A *walk* is a sequence $x_0x_1\dots x_n$ of vertices such that either $x_ix_{i+1} \in ED$ or $x_{i+1}x_i \in ED$ for all $0 \leq i < n$ and it is an *alternating* walk if we have $x_{i-1} \in N^+(x_i) \Leftrightarrow x_{i+1} \in N^+(x_i)$ for all $1 \leq i \leq n-1$. If two edges lie on a common alternating walk then they are *reachable* from each other. This defines an equivalence relation, the *reachability relation* $\mathcal{A}$. By $\mathcal{A}(e)$ we denote the equivalence class of the edge e and by $\langle \mathcal{A}(e) \rangle$ the *reachability digraph* of D that contains e, that is, the digraph whose vertex set consists of those vertices incident with some edge in $\mathcal{A}(e)$ and whose edge set is $\mathcal{A}(e)$. If D is 1-*arc transitive*, that is, if $\mathrm{Aut}(D)$ is transitive on the 1-arcs of D, then all reachability digraphs of D are isomorphic and we denote by $\Delta(D)$ a digraph of their isomorphism class.

The reachability digraph of an edge e is a *bipartite reachability digraph* if it is bipartite, if one class of this bipartition has empty in-neighborhood in $\langle \mathcal{A}(e) \rangle$, and if the other class has empty out-neighborhood.

The following proposition is due to Cameron et al. [1, Proposition 1.1].

Proposition 3.1. *Let D be a connected 1-arc transitive digraph. Then $\Delta(D)$ is 1-arc transitive and connected. Furthermore, either*
(a) *$\mathcal{A}$ is the universal relation on ED and $\Delta(D) \cong D$, or*
(b) *$\Delta(D)$ is a bipartite reachability digraph.* □

We need some notations for infinite (di)graphs. Let G be a graph. A *ray* in G is a one-way infinite path. Two rays are *equivalent* if for every finite set S of vertices of G both rays lie eventually in the same component of $G - S$. This property is an equivalence relation whose equivalence classes are called the *ends* of G. The *rays* and the *ends* of a digraph are those of its underlying undirected graph, that we denote by $G(D)$.

3.2. **Group actions.** Let Γ be a group acting on a digraph D, let $U \subseteq VD$, let $e \in ED$, and let $x \in VD$. We denote by Γ_U the *(pointwise) stabilizer* of U, that is the subgroup of Γ that fixes each element of U. Similarly, we denote by Γ_e and Γ_x the stabilizer of e and of x, respectively. If Γ fixes the set U setwise, then we denote by Γ^U the group of all automorphisms of U that are obtained by restricting elements of Γ to U.

We will use the following theorem on subgroups of the symmetric group S_n.

Theorem 3.2. [14, Satz II.5.2] *Every proper subgroup of S_n with $n \neq 4$ is equal to A_n or has index at least n. If $n = 4$, then, except for A_n, the Sylow 2-subgroups are the only proper subgroups of index less than n.* □

3.3. **Homogeneous digraphs.** In this section we briefly recall the classification result of Lachlan [17] for homogeneous digraphs.

Theorem 3.3. [17, Theorem 1] *A finite digraph is homogeneous if and only if it is isomorphic to one of the following digraphs:*
(i) *C_4;*
(ii) *$\overline{K}_n$ for some $n \geq 1$;*
(iii) *$\overline{K}_n[C_3]$ for some $n \geq 1$;*
(iv) *$C_3[\overline{K}_n]$ for some $n \geq 1$;*
(v) *the digraph H.* □

4. The non-independent case

It is a straightforward argument that the out-neighborhood as well as the in-neighborhood of any vertex of a C-homogeneous digraph have to be homogeneous digraphs: extend any two finite isomorphic induced subdigraphs in $D^+(x)$ (in $D^-(x)$) for $x \in VD$ with the aid of x to connected such digraphs. As any of their isomorphisms extend to automorphisms of the whole digraph, so do the isomorphisms between the two original subdigraphs. Let us fix this as a lemma.

Lemma 4.1. *Let D be a C-homogeneous digraph and let $x \in VD$. Then $D^+(x)$ and $D^-(x)$ are homogeneous digraphs.* □

We investigate which of the homogeneous digraphs of Theorem 3.3 may occur as a subdigraph $D^+(x)$ or $D^-(x)$ for a vertex $x \in VD$. In this section we take a look at those cases that contain an edge and show that there is precisely one such case that may occur. This case is a generalization of the digraph H that occurs in the case (v) of Theorem 3.3. Our first aim is to show that neither $D^+(x)$ nor $D^-(x)$ is isomorphic to H.

Lemma 4.2. *Let D be a connected locally finite C-homogeneous digraph. Then $D^+(x) \not\cong H$ and $D^-(x) \not\cong H$ for all $x \in VD$.*

Proof. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may suppose that $D^+(x) \cong H$ for every $x \in VD$. Let $z \in N^+(x)$. As $D^+(x) \cong H$, the digraph $D^+(x) \cap D^+(z)$ consists of a directed triangle. Let v_1, v_2, v_3 be three vertices in $N^+(z) \smallsetminus N^+(x)$ such that v_1 has precisely two neighbors in $N^+(x) \cap N^+(z)$, such that $N^+(x) \cap N^+(z) \subseteq N^+(v_2)$, and such that $N^+(x) \cap N^+(z) \subseteq N^-(v_3)$. These vertices exist because $D^+(z) \cong H$. Then there are two vertices v_i, v_j $(i \neq j)$ such that they are both either in the in-neighborhood of x or not adjacent to x. This implies that $D[z, x, v_i] \cong D[z, x, v_j]$. As D is C-homogeneous, there is an automorphism of D mapping the first onto the second subdigraph that fixes x and z. But this is a contradiction to the choice of v_i and v_j as they behave differently to $N^+(x) \cap N^+(z)$. □

The next case that we exclude is that the out- or the in-neighborhood induces a subdigraph isomorphic to C_4.

Lemma 4.3. *Let D be a connected locally finite C-homogeneous digraph. Then $D^+(x) \not\cong C_4$ and $D^-(x) \not\cong C_4$ for all $x \in VD$.*

Proof. Analogously to the proof of Lemma 4.2, we may suppose that $D^+(x) \cong C_4$. Let us denote by $v_1, \ldots, v_4$ the four vertices in $N^+(x)$ such that $v_i v_{i+1} \in ED$ for $1 \leq i \leq 3$ and $v_4 v_1 \in ED$. According to Lemma 4.2, we know that $D^-(v_1) \not\cong H$.

Let us suppose that there is a vertex $y \in N^-(v_1) \cap N^-(v_2)$ distinct from x. An immediate consequence of C-homogeneity is $N^+(x) = N^+(y)$. Indeed, we can extend the isomorphism from $D[x, y, v_1]$ to $D[x, y, v_2]$ that fixes x and y to an automorphism of D, which implies that $v_3 \in N^+(y)$. Analogously, we have $v_4 \in N^+(y)$, too, so $N^+(x) = N^+(y)$. Hence, neither xy nor yx can be an edge of D. The subdigraph $D[x, y, v_4]$ is a subdigraph of $D^-(v_1)$ and thus, by Theorem 3.3, we have $D^-(v_1) \cong C_3[\overline{K}_n]$ for some $n > 1$. As$x \in N^-(v_1)$, there is a vertex in $N^+(x) \cap N^-(v_1)$ which is distinct from v_4. As this is impossible, we have proved

$$N^-(v_1) \cap N^-(v_2) = \{x\}. \tag{1}$$

Due to C-homogeneity, we know that (1) holds for every two adjacent vertices v_i and v_j in $N^+(x)$.

The next step in the proof is to show

$$N^-(v_1) \cap N^+(v_2) = \emptyset. \tag{2}$$

Let us suppose that there is a vertex $y \in N^-(v_1) \cap N^+(v_2)$. If y is neither adjacent to x nor to v_4, then by Theorem 3.3 $D^-(v_1)$ has to be isomorphic to $\overline{K}_n[C_3]$ for some $n > 1$. So there is a vertex $z \in N^-(v_1)$ that lies in $N^+(v_4) \cap N^-(x)$. As $xv_2 \in ED$ and as v_2 and v_4 are not adjacent, C-homogeneity implies that we must have $v_2 z \in ED$. Indeed, otherwise we could map z either to x or to v_4 and fix v_1 and v_2 by an automorphism of D. But both cases imply that then the whole directed triangle $D[x, v_4, z]$ in $D^-(v_1)$ must have the same adjacency to v_2 which is impossible. Both digraphs $D[z, v_1, v_2]$ and $D[y, v_1, v_2]$ are directed triangles. Hence, there is an automorphism α of D that maps z to y and fixes v_1 and v_2. But as x and y are not adjacent, we know that $x \neq x^\alpha$. Since also x^α lies in $N^-(v_1) \cap N^-(v_2)$, this contradicts (1). So y is adjacent to at least one of x and v_4.

If y is adjacent to x but not to v_4, then yx lies in ED as $y \notin \{v_1, \ldots, v_4\} = N^+(x)$. Since an induced 2-arc embeds into $N^-(v_1)$, we know that $D^-(v_1) \cong C_4$, as the only other possible case $D^-(v_1) \cong H$ is not possible due to Lemma 4.2. Hence, there is a vertex $z \in N^-(v_1)$ that lies in $N^+(v_4) \cap N^-(y)$ and that is not adjacent to x. As a consequence of (1) we know that zv_2 is not an edge in D. If z and v_2 are not adjacent, we also obtain a contradiction. Indeed, then there is an automorphism β of D that maps v_4 to z and fixes v_1 and v_2. So $x^\beta \neq x$ but both lie in $N^-(v_1) \cap N^-(v_2)$, which is impossible. Hence, we know that $v_2 z \in ED$. So there is an automorphism β of D that maps y to z and fixes v_1 and v_2. As x and y are adjacent but x and z are not, we have again two distinct vertices, x and x^β in $N^-(v_1) \cap N^-(v_2)$ which is impossible by (1).

If y is adjacent to v_4 but not to x, then we know by (1) applied to v_4 and v_1 that $yv_4 \notin ED$. So $v_4 y$ is an edge of D. This implies as above that $D^-(v_1) \cong C_4$. Hence, there is a vertex $z \in N^-(v_1) \setminus \{v_4, x, y\}$. If z is not adjacent to v_2, then there is an automorphism of D that maps z to v_4 and fixes v_1 and v_2. Since this automorphism cannot fix x, the image of x is a second vertex in $N^-(v_1) \cap N^-(v_2)$ contrary to (1). Hence, z and v_2 are adjacent. Due to (1), zv_2 is no edge of D, so we have $v_2 z \in ED$. Then there is an automorphism of D that maps y to z and fixes v_1 and v_2. Again, x and its image under that automorphism are distinct. But both lie in $N^-(v_1) \cap N^-(v_2)$ in contradiction to (1).

Thus, we conclude that both x and v_4 are adjacent to y. Due to (1), we have $v_4 y \in ED$ and not $yv_4 \in ED$, and because of $y \notin N^+(x)$ we have $yx \in ED$. By C-homogeneity, there is an automorphism γ of D that maps v_2 to v_4 and fixes y and x. Hence, we have $v_1^\gamma = v_3$ and $yv_3 \in ED$. But then $D[v_1, x, v_3]$ is a subdigraph of $N^+(y)$ that cannot be embedded into a C_4. This contradiction shows that (2) is true.

Let us suppose that there exists a vertex $y \in N^-(v_1) \cap N^+(v_4)$. Due to (2), we have $yv_3 \notin ED$. The existence of an edge $v_3 y$ in D implies that there is an automorphism α of D that maps v_3 to v_1 and fixes x and y. But then, we have $v_4^\alpha = v_2$ and hence $v_2 y \in ED$ contrary to (2). So we have $v_3 y \notin ED$. Thus, there is an automorphism β of D that maps v_1 to y and fixes v_3 and v_4. Since $y \notin N^+(x)$, we have $x \neq x^\beta \in N^-(v_3) \cap N^-(v_4)$ and thus a contradiction to (1). This shows

$$N^-(v_1) \cap N^+(v_4) = \emptyset. \tag{3}$$

Since there is a vertex in $N^-(v_1) \cap N^+(x)$, the same is true for $N^-(v_1) \cap N^+(v_4)$ due to C-homogeneity. This contradiction to (3) shows that $D^+(x)$ cannot be isomorphic to C_4. □

Lemma 4.4. *Let D be a connected locally finite C-homogeneous digraph such that $D^+(x) \cong \overline{K}_n[C_3]$ and $D^-(x) \cong \overline{K}_m[C_3]$ with $m, n \geq 1$ for all $x \in VD$. Then $m = n = 1$.*

Proof. Let $xy \in ED$. Then there exists $z \in N^-(y) \cap N^-(x)$. By considering $D^-(y)$, we obtain a vertex $a \in N^-(y) \cap N^+(x)$ with $az \in ED$. Let b be the third vertex of $N^+(x)$ in that isomorphic image of C_3 that contains y and a. If either zb or bz lies in ED, then we have either $by \in E(D^+(x) \cap D^+(z))$ or $ab \in E(D^+(x) \cap D^-(z))$. This is a contradiction as each of $N^+(x) \cap N^+(z)$ and $N^+(x) \cap N^-(z)$ consists of precisely one vertex by the assumption $D^+(x) \cong \overline{K}_n[C_3]$. Hence, z and b are not adjacent. So in the isomorphic copy $D[y, a, b]$ of C_3 in $D^+(x)$, there is an in- and an out-neighbor of z and one vertex not adjacent to z.

Let us suppose that $n > 1$. Then there exists a vertex $y' \in N^+(x)$ that is distinct from a, b, and y. So there is a vertex $v \in \{a, b, y\}$ and an automorphism of D that maps v to y' and fixes x and z. Hence, the isomorphic image of C_3 in $D^+(x)$ that contains y' contains a vertex of $N^+(z)$. We may suppose that this is y'. But then $D[y, x, y']$ is a digraph that cannot be embedded into $D^+(z)$. This contradiction shows $n = 1$. By a symmetric argument we also have $m = 1$. □

Lemma 4.5. *Let D be a connected locally finite C-homogeneous digraph. If for every $x \in VD$ either $D^+(x) \cong C_3[\overline{K}_n]$ or $D^-(x) \cong C_3[\overline{K}_n]$ for some $n \geq 1$, then $D \cong H[\overline{K}_n]$.*

Proof. Analogously to the proof of Lemma 4.2, we may suppose that $D^+(x) \cong C_3[\overline{K}_n]$ for some $n \geq 1$. Let $y \in N^+(x)$. Then x and n independent vertices of $N^+(x)$ lie in $N^-(y)$ and hence either $n = 1$ and $D^-(y) \cong \overline{K}_m[C_3]$ for some $m \geq 1$ or $D^-(y) \cong C_3[\overline{K}_m]$ for some $m \geq n$. In the first case, we have $m = 1$ according to Lemma 4.4. So in both cases, we have $D^-(y) \cong C_3[\overline{K}_m]$ for some $m \geq n$. With a symmetric argument we conclude $m = n$. Hence, there is a vertex $z \in N^-(x) \cap N^-(y)$. As $D^+(z) \cong C_3[\overline{K}_n]$ and $x \in N^+(z)$ and as $D^-(x) \cong C_3[\overline{K}_n]$ and $z \in N^-(x)$, we have that

(4) *$N^+(x) \cap N^+(z)$ and $N^-(x) \cap N^-(z)$ are independent sets of cardinality n.*

As D contains a directed triangle, an immediate consequence of the C-homogeneity of D is $N^+(x) \cap N^-(z) \neq \emptyset$. Our next aim is to show that

(5) *$N^+(x) \cap N^-(z)$ is an independent set of cardinality n.*

Let us suppose that there is an edge ab with its two incident vertices in $N^+(x) \cap N^-(z)$. Then the digraphs $D[x, z, a]$ and $D[x, z, b]$ are isomorphic and there is an automorphism α of D mapping a to b and fixing x and z. As a consequence of (4), both a and b have to be adjacent to all the vertices in $N^+(x) \cap N^+(z)$. Since $D^+(x) \cong C_3[\overline{K}_n]$ and $a, b \in N^+(x)$, we have $y'a \in ED$ and $by' \in ED$ for all $y' \in N^+(x) \cap N^+(z)$. Indeed, an edge ay' would imply that y' and b are not adjacent and the same would be true for an edge $y'b$. Thus, the automorphism α cannot exist and we conclude that no such edge ab exists. So $N^+(x) \cap N^-(z)$ is an independent set. Since every edge lies on at least n distinct directed triangles, there are at least n vertices in $N^+(x) \cap N^-(z)$ and, as a largest independent set in $N^+(x)$ consists of n vertices, we have proved (5).

As a further step in this proof, we prove the following:

(6) *Every two non-adjacent vertices in $N^+(x)$ have the same in-neighbors.*

Let $a, b \in N^+(x)$ be non-adjacent and $x' \in N^-(a)$ with $x' \neq x$. Let us first assume that x and x' are adjacent. In each of the two sets $N^+(x)$ and $N^+(x')$ there is precisely one maximal independent set that contains a as $D^+(x) \cong C_3[\overline{K}_n]$. Due to (4) applied to x and x' instead of x and z, these two maximal sets must be $N^+(x) \cap N^+(x')$. Hence, also b must lie in $N^+(x')$. So let us assume that x and x' are not adjacent. Then there is a third vertex x'' in $N^-(a)$ that is adjacent to both x and x'. Applying the previous case, we know that $x'' \in N^-(b)$ and hence also $x' \in N^-(b)$. This shows (6).

The remaining step in the proof is to show the following:

(7) *There is an equivalence relation $\sim$ on VD, each of whose equivalence classes has precisely n independent vertices, such that $D_\sim$ is isomorphic to H and $D_\sim[\overline{K}_n]$ is isomorphic to D.*

Let us define a relation $\sim$ on VD via

$$a \sim b \quad :\Longleftrightarrow \quad N^-(a) = N^-(b).$$

Obviously, $\sim$ is an equivalence relation. First, we note that every equivalence class must be an independent vertex set due to the definition of the relation $\sim$. Hence, there are more than one equivalence classes. Let A and B be two distinct equivalence classes, $a_1, a_2 \in A$, and $b_1, b_2 \in B$ such that $a_1b_1 \in ED$. According to the definition of $\sim$, we have $a_1b_2 \in ED$ and thus, $B \subseteq N^+(a_1)$. As B is an independent set and $D^+(a_1) \cong C_3[\overline{K}_n]$, there are at most n vertices in B. On the other side, (6) with x replaced by a_1 implies that there are n vertices in B, so B is the maximal independent set in $N^+(a_1)$ that contains b_1. The vertex b_1 has a successor c that is a predecessor of a_1. By definition of $\sim$, we have $ca_2 \in ED$. Since $|A| = n$, we conclude by (5) with x, z replaced by c, b_1 that $a_2b_1 \in ED$. So we also have $a_2b_2 \in ED$. Thus, $D_\sim$ is a digraph with $D \cong D_\sim[\overline{K}_n]$. The digraph $D_\sim$ is C-homogeneous, since D is C-homogeneous and since we can lift any connected induced subdigraph F of $D_\sim$ to a connected induced subdigraph of D that has as its vertices the union of the vertices of F – note that the vertices of F are equivalence classes of vertices of D. It remains to show that $D_\sim \cong H$. As $D_\sim$ is a C-homogeneous digraph with $D^+(v) \cong C_3$ for all $v \in VD_\sim$, it suffices to assume $n = 1$ and to show that $D \cong H$.

Let $x \in VD$. We know that $D^+(x) \cong C_3 \cong D^-(x)$. Let $N^+(x) = \{v_1, v_2, v_3\}$ and $N^-(x) = \{u_1, u_2, u_3\}$ with $v_iv_{i+1} \in ED$ and $v_iu_{i+1} \in ED$ (where $v_4 = v_1$ and $u_4 = u_1$). As xv_1 is an edge in $D[x, v_1, v_2]$, also u_1x must lie in the same position in some triangle. Thus, there is an edge from u_1 to one of the vertices v_i, say to v_1. Then $N^-(v_1) = \{u_1, x, v_3\}$ and hence, we have $v_3u_1 \in ED$. As $N^+(u_1) = \{u_2, x, v_1\}$, we have $v_1u_2 \in ED$. Now we can apply similar arguments and obtain that v_2u_3, u_2v_2, and u_3v_3 lie in ED. Let y be the third out-vertex of v_3 distinct from v_1 and u_1. Notice that y cannot be u_2. Because of $D^+(v_3) \cong C_3$, we have $yu_1 \in ED$ and $v_1y \in ED$. By $D^+(v_1) \cong C_3$ we conclude $v_2y \in ED$ and $yu_2 \in ED$ and $D^-(u_1) \cong C_3$ implies $yu_3 \in ED$. The constructed digraph has the correct out- and in-degree at every vertex and is isomorphic to H. This finishes the proof of Lemma 4.5. □

Let us combine the results of this section with Theorem 3.3:

Theorem 4.6. *Let D be a connected locally finite C-homogeneous digraph. Either $N^+(x)$ and $N^-(x)$ are independent vertex sets or there is an $n \geq 1$ such that $D^+(x) \cong C_3[\overline{K}_n] \cong D^-(x)$ and $D \cong H[\overline{K}_n]$.* □

5. The independent case

In this section, we consider the situation that every out-neighborhood – and hence due to Theorem 4.6 also every in-neighborhood – is independent. Let us briefly outline the content of this section. First, we show that if either the out-degree or the in-degree is 1, then the connected locally finite C-homomgeneous digraph is a tree (Lemma 5.1). Thereafter, we show in Lemmas 5.2 and 5.5 that the reachability relation, which we defined in Section 3, is not universal in our situation. So due to Proposition 3.1, the reachability digraphs are bipartite. That is why we turn our attention towards connected locally finite C-homogeneous bipartite graphs. Their classification (Theorem 5.7) is due to Gray and Möller [10] and we use it to obtain a complete classification in the case of connected locally finite C-homogeneous digraphs with at most one end if the digraphs contain no directed triangle (Lemma 5.10) and then a partial classification of such digraphs if they contain a directed triangle (Lemma 5.11). We continue the investigation of this situation in Section 6.

Lemma 5.1. *Let D be a connected vertex-transitive digraph and let $x \in VD$. If $N^+(x)$ or $N^-(x)$ consists of precisely one vertex, then D is either an infinite tree or a directed cycle.*

Proof. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $N^+(x)$ consists of precisely one vertex. Let us assume that D is not a tree. Then there is a cycle C in D. If C is not a directed cycle, then there is a vertex with out-degree at least 2 on that cycle. Hence, we may assume that C is a directed cycle. For every vertex on C, its descendants must lie on C, so they induce a subdigraph that is a cycle. If $D \neq C$, then there must be a vertex u outside C that is adjacent to some vertex v on C. The edge between u and v cannot be vu as we already mentioned, so it must be uv. So the descendants of u do not induce a directed cycle, as they contain u and all vertices of C. But as D is vertex-transitive, the descendants of u and those of v induce isomorphic digraphs. This contradiction shows that $D = C$ is a directed cycle. □

Notice that C-homogeneous digraphs are vertex-transitive and hence Lemma 5.1 holds for them. Let us now look at the reachability relation of C-homogeneous digraphs. The proof that this relation is not universal splits into two cases: whether a directed triangle embeds into D or not. We start with the latter case:

Lemma 5.2. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$ and such that D contains no directed triangle. Then the reachability relation of D is not universal.*

Proof. Let $x \in VD$. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $d^+(x) \geq d^-(x)$ and due to Lemma 5.1, we may also assume that $d^-(x) \geq 2$. Let $y \in N^+(x)$ and $\Omega = N^+(y)$. Since D is C-homogeneous and contains no directed triangle and since Ω and $N^+(x)$ are independent sets of vertices, the group $\Gamma := \mathrm{Aut}(D)_{xy}$ acts on Ω like S_Ω, the symmetric group on Ω, i.e. $\Gamma^\Omega \cong S_\Omega$.

By induction, we will show $(\Gamma_Q)^\Omega = \Gamma^\Omega$ for all alternating walks Q with initial edge xy. Let P be such an alternating walk. Let us assume that $(\Gamma_P)^\Omega = \Gamma^\Omega$ and let $e \in ED$ such that Pe is an alternating walk. Let z be the vertex incident with e but distinct from the end vertex of P. We will show that $(\Gamma_{Pe})^\Omega = \Gamma^\Omega$, and hence, $(\Gamma_z)^\Omega = \Gamma^\Omega$. There are at most $|\Omega| - 1$ vertices in $\{z^\alpha \mid \alpha \in \Gamma_P\}$, as this set is contained either in the out- or in the in-neighborhood of z', the other vertex that is incident with e, but it does not contain the neighbor of z' on P. So we have $|\Gamma_P : \Gamma_{Pe}| < |\Omega|$. Since $\Gamma^\Omega = (\Gamma_P)^\Omega$, we have either $|\Omega| = 2$ or

$$|(\Gamma_P)^\Omega : (\Gamma_{Pe})^\Omega| \leq |(\Gamma_P) : (\Gamma_{Pe})| < |\Omega|.$$

Let us first assume that $|\Omega| \neq 2$. Then, due to Theorem 3.2, either $(\Gamma_z)^\Omega$ is Γ^Ω or $(\Gamma_z)^\Omega$ is isomorphic to A_Ω, the alternating group on Ω, or $|\Omega| = 4$ and $(\Gamma_z)^\Omega$ is a Sylow 2-subgroup of Γ^Ω. In each of these three cases, the group $(\Gamma_z)^\Omega = (\Gamma_{Pe})^\Omega$ acts transitively on Ω. But then, C-homogeneity implies that $(\Gamma_z)^\Omega$ must be the full symmetric group S_Ω. Indeed, as Ω is an independent set, for any $A, B \subseteq \Omega$ with $|A| = |B|$, the digraph D_1 induced by Pe and A must be isomorphic to the subdigraph D_2 induced by Pe and B and any bijection from A to B extends to an isomorphism from D_1 to D_2 fixing Pe.

Let us now consider the case that $|\Omega| = 2$. Then we have $d^+(x) = d^-(x) = 2$. Hence, the orbit of z under Γ_P contains only z and we conclude $\Gamma = \Gamma_z$. As for $a \in \Omega$ the orbit of a under Γ contains both successors of y, the vertex z cannot lie in Ω.

In both cases, no vertex of Ω can lie on an alternating walk that contains the edge xy and thus, the reachability relation of D cannot be universal. □

Before we turn our attention to investigate the reachability relation if D contains directed triangles, we prove some lemmas.

Lemma 5.3. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If C_3 embeds into D, then $d^+(x) = d^-(x)$.*

Proof. Let n be the number of directed triangles that contain a fixed edge xy of D. As D is C-homogeneous, we conclude for the number of directed triangles that contain x:

$$|N^+(x)|n = |N^-(x)|n.$$

Hence, we have $d^+(x) = d^-(x)$. □

Lemma 5.4. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If D contains a directed triangle, then the number of directed triangles that contain a given edge $xy \in ED$ is either 1 or at least $(d^+ - 1)$.*

Proof. Let Ω_1 be the set of all vertices in $N^+(y)$ that lie on a common directed triangle with xy, let $\Omega_2 = N^+(y) \smallsetminus \Omega_1$, and let $\Omega_3 := N^+(x) \smallsetminus \{y\}$. Note that

$\Omega_3 \cap N^+(y) = \emptyset$ as $N^+(x)$ is an independent set. Let $d_1 = |\Omega_1|$ and $d_2 = |\Omega_2|$. Then we have $d = d_1 + d_2$ where $d := d^+$ which is the same as d^- by Lemma 5.3.

We may suppose that d_1 and d_2 are both at least 2, as otherwise the assertion follows immediately. Hence, we have $|\Omega_3| \geq 3$. We consider the action of $\Gamma := \mathrm{Aut}(D)_{xy}$ on Ω_3. Since $N^+(x)$ is an independent set and since D is C-homogeneous, Γ acts on Ω_3 like S_{Ω_3}, the symmetric group on Ω_3. For every $z \in \Omega_1$, we have $|\Gamma : \Gamma_z| = d_1 < d^+ - 1 = |\Omega_3|$. Thus and due to Theorem 3.2, we have either $(\Gamma_z)^{\Omega_3} \cong S_{\Omega_3}$, or $(\Gamma_z)^{\Omega_3} \cong A_{\Omega_3}$, or $|\Omega_3| = 4$ and $|\Gamma : \Gamma_z| = 3$. In each case, Γ_z acts transitively on Ω_3. As Ω_3 is an independent set, the subdigraph D_1 induced by x, y, z, and A is isomorphic to the subdigraph D_2 induced by x, y, z, and B for any two subsets A and B of Ω_3 with $|A| = |B|$ and, furthermore, any bijection from A to B extends to an isomorphism from D_1 to D_2 fixing x, y, and z. As D is C-homogeneous, each of these isomorphisms extends to an automorphism of D, so $(\Gamma_z)^{\Omega_3}$ cannot be a proper subgroup of S_{Ω_3} and Γ_z acts on Ω_3 like S_{Ω_3}. Thus, either none or all vertices of Ω_3 are predecessors of z. This implies that the edge zx and hence every edge lies either on precisely one or on d distinct directed triangles. This contradicts the assumptions that $d_1 \geq 2$ and $d_2 \geq 2$ and hence shows the assertion. □

Now we are able to prove also for connected locally finite C-homogeneous digraphs that contain directed triangles that their reachability relation is not universal.

Lemma 5.5. *Let D be a connected locally finite C-homogeneous digraph such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. If D contains a directed triangle, then the reachability relation of D is not universal.*

Proof. For this proof, we use two specific digraphs D_1 and D_2 depicted in Figure 4.

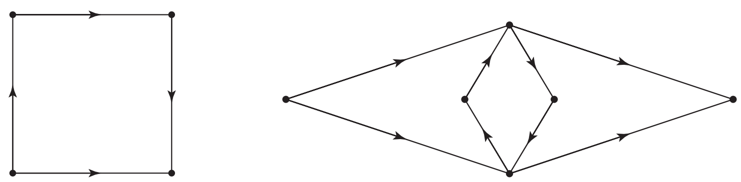

FIGURE 4. On the left side the digraph D_1 and on the right side the digraph D_2.

Let $d = d^+$. By Lemma 5.3 we have $d = d^-$. Let us suppose that the reachability relation $\mathcal{A}$ of D is universal. We say that a cycle C *witnesses that $\mathcal{A}$ is universal* if C contains a directed path of length 2 and if there is an edge xy on C such that C without the edge xy is an alternating walk. The digraph D_1 is an example of such a cycle (removing the uppermost edge leaves an alternating walk of length 3) and up to isomorphism D_1 is the only such cycle of length 4. As $\mathcal{A}$ is universal and as we find a directed (not necessarily induced) path xyz of length 2 in D, there must be a minimal alternating walk in D whose first edge is xy and whose last edge is yz. Either this walk is a cycle or there is a vertex incident with at least three edges of that walk. In the latter case, we find a shorter alternating path between the two edges of some 2-arc. In both cases

(8) *there is a cycle in D witnessing that $\mathcal{A}$ is universal.*

Let us show that

(9) *if D contains a cycle witnessing that $\mathcal{A}$ is universal, then it contains an induced such cycle of shorter or equal length.*

Let us suppose that none of the minimal such cycles is induced and let C be such a cycle of minimal length. Let x, y be vertices on C such that C without the edge xy is an alternating walk P. Note that P is a path. Since C is not induced, there is a chord uv in C. If both vertices u and v lie in the same set of the natural bipartition of P, then the subpath uPv of P together with the edge uv is a smaller cycle witnessing that $\mathcal{A}$ is universal. So let us assume that u and v lie in distinct sets of the natural bipartition of P. But then we also find a smaller cycle in C together with the edge uv depending on its direction: if the in-degree of v in P is 0, then we take uv together with uPv, the subpath of P between u and v, and otherwise we take uv together with that maximal path of C that has only the vertices u and v in common with uPv. This contradiction to the minimality of C shows (9).

The next step is to show:

(10) *If D contains an induced cycle C of even length witnessing that $\mathcal{A}$ is universal, then each edge lies on precisely one directed triangle.*

Let xyz be a directed path of length 2 on C. As C has even length, the path $C-y$ has an automorphism that interchanges x and z. This automorphism of $C-y$ extends to an automorphism α of D. As C is induced, the same holds for C^α. Thus and since y and y^α cannot be adjacent because $N^+(y)$ and $N^-(y)$ are independent, we obtain that y and y^α are not adjacent. Hence, y is the first vertex of at least two directed paths of length 2 that share the edge yz: one is yzy^α and the other is yzu where u is the second neighbor of z on C. Thus, the edge yz lies on at most $d^+(z)-2$ directed triangles which directly implies (10) due to Lemma 5.4 and as $\mathrm{Aut}(D)$ acts transitively on the edges of D.

Let us show:

(11) *If D contains an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal, then it contains an isomorphic copy of D_2.*

Let u, v, x, y be the vertices of D_1 such that $uv, vx, xy, uy \in ED$. Then there is an automorphism α of D that fixes u and interchanges v and y. As the out- and the in-neighborhood of x is independent, the vertices x and x^α are not adjacent and D_1 together with x^α forms all but the rightmost vertex of D_2 in that u is the left-most vertex and the inner cycle is $vxyx^\alpha$. Let $\beta \in \mathrm{Aut}(D)$ with $(u, v, x)^\beta = (v, x, y)$, and set $z = y^\beta$. An edge between z and u either contradicts (10) or leads to an out- or an in-neighborhood that is not independent – depending on its direction. Similarly, neither z and x nor z and x^α are not adjacent. This shows (11).

Now we exclude the existence of induced cycles witnessing that $\mathcal{A}$ is universal step by step: first we exclude such cycles if they have precisely four vertices, then we exclude odd such cycles of length at least 5 and last we exclude even such cycles of length at least 6. When we have shown that none of these cases occur, we have a contradiction to the assumption that $\mathcal{A}$ is universal.

(12) *No induced cycle of length 4 in D witnesses that $\mathcal{A}$ is universal.*

To show (12), let us suppose for a contradiction that there is an induced cycle of length 4 witnessing that $\mathcal{A}$ is universal. Due to (11), D contains an isomorphic copy D' of D_2. Let x be the leftmost and y the rightmost vertex and let a, b, u, v the vertices of the inner cycle such that x and y are adjacent to a and u and such that $uv \in ED$. Since D contains a directed triangle, there is a vertex $a' \in N^+(a) \cap N^-(x)$. Then a' is adjacent neither to b, nor to v, nor to y, since the only directed triangle that contains aa' is $D[x, a, a']$ and since the in- and the out-neighborhoods of every vertex are independent sets. Hence, there is an automorphism α of D that fixes a', x, and u, and maps v onto y. Then α also has to fix a, since it fixes together with x and a' the unique vertex in the directed triangle that contains the edge $a'x$. As $va \in ED$ but $ay \in ED$, this is a contradiction that shows (12).

(13) *No induced odd cycle of length at least* 5 *in D witnesses that $\mathcal{A}$ is universal.*

Let us suppose that D contains an induced odd cycle C of length at least 5 that witnesses that $\mathcal{A}$ is universal. Let xy be an edge on C such that either $d^+_C(x) = 2$ and $d^+_C(y) = 1$ or $d^+_C(x) = 1$ and $d^+_C(y) = 0$. Let z be the second neighbor of y on C. Then $C - x$ and $C - y$ are isomorphic and hence, there is an automorphism α of D that maps $C - x$ onto $C - y$. The digraph $D[x, y, z, x^\alpha]$ is isomorphic to D_1 because $N^-(z)$ and $N^+(z)$ are independent sets. This contradicts (12). So we proved (13).

The next claim will finish the proof of Lemma 5.5.

(14) *No induced even cycle in D witnesses that $\mathcal{A}$ is universal.*

Let us suppose that D contains an induced even cycle C of minimal length witnessing that $\mathcal{A}$ is universal. Due to (12), the length of C is at least 6. As its length is even, there is a directed path $xyzu$ on C. Due to C-homogeneity, D has an automorphism α that maps $C - y$ onto itself with $x^\alpha = z$. Hence, the path xyz lies on a directed cycle of length 4, the cycle induced by x, y, z, and y^α. Note that y and y^α cannot be adjacent as y has independent out- and independent in-neighborhood. Let a be the neighbor of u on C that is not z. As every edge lies on precisely one directed triangle due to (10), there are uniquely determined vertices a' and z' such that a, a', and u induce a directed triangle and the same holds for z, z', and u. Furthermore, the vertex a' is not adjacent to z or z' and z' is also not adjacent to a because of the independent out- and in-neighborhoods and due to (10). The induced 2-arc zua' lies on a directed cycle of length 4 as the same holds for xyz. Let y' be the fourth vertex on that cycle. Then y' cannot be adjacent to a as otherwise the in-neighborhood of a' is not independent. We shall show that $a'y \in ED$. This is true if $y' = y$, so let us assume that $y' \neq y$. Then the digraphs $D[a, u, z, y]$ and $D[a, u, z, y']$ are isomorphic. Hence, there is an automorphism β of D that fixes a, u, and z and maps y' to y. As a and u lie on precisely one common directed triangle, β must also fix a', so $y = y'^\beta$ must be adjacent to $a'^\beta = a'$. Then the digraph induced by a' and all the vertices of C but u and z contains a cycle C' witnessing that $\mathcal{A}$ is universal and this cycle C' has smaller length than C. Due to (9), there is also an induced such cycle C'' of at most the same length as C'. If the length of C'' is either 4 or odd, then we obtain the claim by (12) or (13), and if the length of C'' is even and at least 6, then we obtain a contradiction to the minimality of the length of C. This shows (14) and finishes the proof of the lemma. □

As the reachability relation is not universal for any locally finite C-homogeneous digraph D if $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$, we conclude with Proposition 3.1 that D has a bipartite reachability digraph. That is, why we are interested in the classification of the locally finite C-homogeneous bipartite graphs. A bipartite graph G (with bipartition $\{X, Y\}$) is *connected-homogeneous bipartite*, or simply *C-homogeneous bipartite*, if every isomorphism between two isomorphic connected induced finite subgraphs A and B of G that preserves the bipartition (that means $VA \cap X$ is mapped onto $VB \cap X$ and $VA \cap Y$ is mapped onto $VB \cap Y$) extends to an automorphism of G that preserves the bipartition.

The next lemma is due to Gray and Möller [10, Lemma 4.3], see also [11, Lemma 5.4], and it underlines our interest in the C-homogeneous bipartite graphs.

Lemma 5.6. *Let D be a connected C-homogeneous digraph. If $\Delta(D)$ is bipartite, then the underlying undirected graph of $\Delta(D)$ is a connected C-homogeneous bipartite graph.* □

The following result is the classification result of the C-homogeneous bipartite graphs. Its proof is due to Gray and Möller and uses the classification of the homogeneous bipartite graphs, see [7].

Theorem 5.7. [10, Theorem 4.6] *Let G be a locally finite connected graph. Then G is C-homogeneous bipartite if and only if it is isomorphic to one of the following graphs:*

(i) *a cycle C_{2m} with $m \geq 2$;*
(ii) *an infinite semiregular tree $T_{k,\ell}$ with $k, \ell \geq 2$;*
(iii) *a complete bipartite graph $K_{m,n}$ with $m, n \geq 1$;*
(iv) *a complement of a perfect matching CP_k with $k \geq 2$.* □

Now, we use the above classification result to continue our classification of the connected locally finite C-homogeneous digraphs. At this place the assumption that the digraphs have at most one end will be used for the first time in this paper and the remaining lemmas of this section will also build on it.

Lemma 5.8. *Let D be a locally finite connected C-homogeneous digraph with at most one end such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$. Then either $\Delta(D)$ is a finite digraph or C_3 embeds into D and $G(\Delta(D)) \cong T_{2,2}$.*

Proof. Due to Lemmas 5.2 and 5.5, we know that the reachability relation of D is not universal and hence that the reachability digraphs are bipartite by Proposition 3.1 and that we can apply Theorem 5.7. Let us suppose that $\Delta(D)$ is not finite. Since D is locally finite, we conclude from Theorem 5.7 that $G(\Delta(D)) \cong T_{k,\ell}$ for integers $k, \ell \geq 2$. Let us first assume that $k \geq 3$. By regarding the digraph whose edges are directed in the inverse way, if necessary, we may assume that $k = d^+(x)$.

Let $u \in VD$ and let x, y, z be distinct vertices of $N^+(u)$. As there is a ray in $G(\Delta(D))$ and as D has at most one end, it has precisely one end. Hence, removing the (finite) set S of all vertices with distance at most 3 to u separates D into components such that precisely one of them is infinite, because D is locally finite. Let C be this infinite component. Let Δ be the reachability digraph that contains u and x and let R_x, R_y be rays in Δ that start at u and contain x, y, respectively.

Since D is locally finite, there are vertices a, b on R_x, R_y, respectively, that lie in C. So we have $d(a, x) \geq 3$ and $d_\Delta(a, x) = d_\Delta(a, u) - 1$ as well as $d(b, y) \geq 3$

and $d_\Delta(b,y) = d_\Delta(b,u) - 1$, where d_Δ denotes the distance in $G(\Delta)$. Let P be a path (not necessarily directed) in C from a to b, and let Q be the path in Δ between a and x. Note that neither y nor z has a neighbor on P because of $P \subseteq D - S$. Let us suppose that y has a neighbor on Q. Due to the definition of a reachability digraph and as $uy \in E\Delta$ and $G(\Delta(D)) \cong T_{k,\ell}$, we know that y has no predecessor on Q. If y has a successor y^+ on Δ, the unique reachability digraph that contains all predecessors of y, then every successor of y lies on Δ by C-homogeneity. Since D contains no triangle, we can map the 2-arc uyy^+ onto any other 2-arc y^-yy^+ and obtain $d_\Delta(y^+,u) = d_\Delta(y^+,y^-)$, which contradicts the choice of y^+ on Q and $G(\Delta(D)) \cong T_{k,\ell}$. Similarly, z has no neighbor on Q. Hence, the digraph induced by P, Q, u, and y is isomorphic to the digraph induced by P, Q, u, and z, but there is no automorphism of D that maps one onto the other by fixing P, Q, and u and mapping y to z since $d_\Delta(b,y) = d_\Delta(b,z) - 2$, which follows from $d_\Delta(b,y) = d_\Delta(b,u) - 1$ as Δ is a tree. This shows $k = 2$. The case $\ell \geq 3$ is analogous, so we conclude $k = \ell = 2$ and $d^+ = d^- = 2$.

It remains to show that D contains a directed triangle. So let us suppose that there is no directed triangle in D. Let $z \in VD$, let x and y be the two predecessors of z and let z_1 be a successor of z. Due to the assumptions, $D[x,z,z_1]$ and $D[y,z,z_1]$ are induced 2-arcs and we conclude with C-homogeneity that $\Gamma := \mathrm{Aut}(D)_{zz_1}$ acts transitively on $\{x,y\}$.

Let $z_1z_2\ldots$ be the ray with $z_2 \neq z$ in that reachability digraph that contains z and z_1. The group Γ must fix z_2 as $d^- = 2$ and, inductively, it fixes every z_i as also $d^+ = 2$. Let z_i be a vertex on that ray that has distance at least 3 to z. As above, there is a path P from z_i to a vertex a that lies in the same reachability digraph as the edge xz and has distance at least 3 to z such that every vertex of P has distance at least 3 from z. So neither x nor y has a neighbor on P. Furthermore, $\Gamma = \Gamma_{zz_1\ldots z_i}$ acts transitively on $\{x,y\}$, so any successor or predecessor of x on $z_1\ldots z_i$ is also a successor or predecessor of y, respectively, and neither x nor y has a neighbor on P. Hence, the digraphs $D_1 := D[\{x,z,z_1,\ldots,z_i\} \cup VP]$ and $D_2 := D[\{y,z,z_1,\ldots,z_i\} \cup VP]$ are isomorphic. So the isomorphism that maps x to y and fixes all other vertices of D_1 extends to an automorphism α of D that fixes $\langle \mathcal{A}(xz)\rangle = \langle \mathcal{A}(yz)\rangle$. Hence, $a = a^\alpha$ has the same distance to x and to $x^\alpha = y$. But because of $d^+ = d^- = 2$ the unique path in $\langle \mathcal{A}(xz)\rangle$ from z to a contains either x or y but not both. Thus, a has distinct distance to x and to y. This contradiction shows that D contains a directed triangle if $G(\Delta(D)) \cong T_{2,2}$. □

Lemma 5.9. *Let D be a locally finite connected C-homogeneous digraph with at most one end such that $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$ and such that $\Delta(D)$ is finite. If some reachability digraph Δ separates D, then there is a second reachability digraph Δ' such that $\Delta' \smallsetminus \Delta$ has vertices from distinct components of $D \smallsetminus \Delta$.*

In particular, if some reachability digraph Δ separates D, then there is a second reachability digraph Δ' such that $\Delta \cap \Delta'$ separates Δ'.

Proof. As argued at the start of the proof of Lemma 5.8, the reachability digraph $\Delta(D)$ is bipartite. Let us suppose that some reachability digraph Δ_1 separates D and that every other reachability digraph of D has vertices from at most one component of $D \smallsetminus \Delta_1$. Let Δ_2 be a reachability digraph with $\Delta_1 \neq \Delta_2$ and $V\Delta_1 \cap V\Delta_2 \neq \emptyset$. Note that $V\Delta_1 \neq V\Delta_2$ as otherwise $VD = V\Delta_1$ and thus, Δ_1 does not separate D.

Let $x \in V\Delta_1 \cap V\Delta_2$ and let y be a neighbor of x in Δ_2. We may assume $xy \in E\Delta_2$. Let z be a successor of y. Then z lies outside of Δ_2 as otherwise every neighbor of y lies in Δ_2, which implies by C-homogeneity that every neighbor of x lies in a unique reachability digraph in contradiction to the assumption that Δ_1 separates D.

Let us show that

(15) *there are components D_i of $D - \Delta_i$, for $i = 1, 2$, with $D_1 \cong D_2$ and $D_2 \subsetneq D_1$.*

Let D_1 be the component of $D - \Delta_1$ that contains the vertices of $\Delta_2 - \Delta_1$. This is a unique component, since $V(\Delta_2 - \Delta_1)$ is a non-empty set of vertices and since Δ_1 does not separate Δ_2 by assumption. Analogously, there is a unique component of $D - \Delta_2$ that contains the successors of y. By C-homogeneity, we find for every vertex in Δ_2 whose successors lie outside of Δ_2 a unique component that contain its successors and all these components are isomorphic and they are isomorphic to D_1. Either one of those lies in D_1 and thus gives us (15) or all of them contain vertices of Δ_1. But then the edge xy in Δ_2 has the property that the component of $D - \Delta_2$ that contains the predecessors of x and the components that contains the successors of y are the same. By C-homogeneity, the same holds for every edge of Δ_2. As Δ_2 is connected, we conclude that $D - \Delta_2$, and thus also $D - \Delta_1$, has only one component in contradiction to the assumption. This shows (15).

By a symmetric argument, we obtain that

(16) *there are components D_i' of $D - \Delta_i$, for $i = 1, 2$, with $D_1' \cong D_2'$ and $D_1' \subsetneq D_2'$.*

Due to (15) and (16), the two components D_1 and D_1' are infinite. As D is locally finite, each of those two components contains an end of D. As D_1 and D_1' have empty intersection and Δ_1 is finite, D has at least two ends, contrary to our assumption. This shows the first part of the assertion and the second one follows from the first one immediately. □

The following lemma is the main lemma for the case that there is no isomorphic copy of C_3 in the C-homogeneous digraph.

Lemma 5.10. *Let D be a locally finite connected C-homogeneous digraph with at most one end that contains no directed triangle. If $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$, then D is isomorphic to $C_m[\overline{K_n}]$ for some $m \geq 4$, $n \geq 1$.*

Proof. As in the previous proofs, we know that $\Delta(D)$ is bipartite. So by Lemma 5.8, it is finite. Due to Lemma 5.1, we may assume that $d^+ \geq 2$ and $d^- \geq 2$. Define $x \sim y$ for $x, y \in VD$ if x and y *lie on the same side* of some reachability digraph, that is, both have the same out-degree and the same in-degree in that reachability digraph and one of these two values is 0. If x and y lie in a common reachability digraph but not on the same side they lie *on distinct sides* of a reachability digraph. Note that, a priori, $\sim$ is not an equivalence relation. But we shall show later that it is an equivalence relation in our situation.

Let Δ_1 and Δ_2 be two distinct reachability digraphs with non-empty intersection. If $\Delta_1 \cap \Delta_2$ does not lie on the same side of Δ_1, then $G(\Delta(D))$ cannot be a complete bipartite graph because Δ_2 contains vertices on distinct sides of Δ_1 which lie also on distinct sides of Δ_2 and thus are adjacent in Δ_1 and in Δ_2, which is impossible as the two subdigraphs share no edge and as D contains no edge yx for $xy \in ED$. Combining this with Theorem 5.7 and Lemma 5.8, we have just proved:

(17) *If $\Delta_1 \cap \Delta_2$ does not lie on the same side of Δ_1, then either $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$ or $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 4$.*

Let us also exclude the possibility of $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 4$ if $\Delta_1 \cap \Delta_2$ does not lie on the same side of Δ_1:

(18) *If $\Delta_1 \cap \Delta_2$ does not lie on the same side of Δ_1, then $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$.*

Let us suppose that $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 4$. Let $x, y \in \Delta_1 \cap \Delta_2$ be on distinct sides of Δ_1 with minimal distance in $G(\Delta_1)$. So x and y lie also on distinct sides of Δ_2. We may assume that x has only predecessors and y has only successors in Δ_1. Suppose that x and y are adjacent; so $yx \in ED$. Then some edge of Δ_1 has both its incident vertices in Δ_2 and as Δ_1 is connected, the same holds for any of its edges. By C-homogeneity, also every edge of Δ_2 has both its incident vertices in the same reachability digraph (besides Δ_2), which must be Δ_1. Thus, $D = \Delta_1 \cup \Delta_2$ and Δ_1 as well as Δ_2 contain all vertices of D. Let y' be the second predecessor of x and let z, z' be the two successors of x. Since D contains no triangles at all – neither directed nor the unique second kind of triangles, as $N^+(x)$ is an independent set –, the 2-arcs yxz and $y'xz$ as well as yxz' and $y'xz'$ are induced subdigraphs. By C-homogeneity, we find some $\alpha \in \mathrm{Aut}(D)$ with $(y, x, z)^\alpha = (y', x, z)$. As Δ_1 contains y, y', z, we have $d_{\Delta_1}(y, z) = d_{\Delta_1}(y', z)$, where d_{Δ_1} denotes the distance in Δ_1. Because of $G(\Delta(D)) \cong C_{2m}$ this implies $d_{\Delta_1}(x, z) = m$. Similarly, we obtain $d_{\Delta_1}(x, z') = m$. Thus, $z = z'$ in contradiction to their choice. This shows that x and y are not adjacent.

Thus, as x and y are not adjacent and as they do not lie on the same side of Δ_1, the distance between them in Δ_1 is at least 3. Let P be a minimal path in Δ_2 from x to y. Let x' be a neighbor of x in Δ_1, let y_1, y_2 be the two neighbors of y in Δ_1, and let y' be the neighbor of y on P. The subdigraphs induced by y', y, y_1 and by y', y, y_2 are isomorphic, as D contains no triangles. Thus, there is some $\alpha \in \mathrm{Aut}(D)$ with $(y', y, y_1)^\alpha = (y', y, y_2)$. This automorphism must fix the reachability digraph that contains the edge between y and y' setwise, which is Δ_2, and hence it fixes Δ_1 setwise, the only other reachability digraph that contains y, too. As $y^\alpha = y$ and $(y')^\alpha = y'$, the automorphism α fixes one edge of Δ_2 and hence the whole digraph Δ_2 pointwise because of $G(\Delta_2) \cong C_{2m}$. In particular, we have $x^\alpha = x$. Let P_i be the unique path in Δ_1 from y to x containing y_i, respectively. As α fixes x and y and maps y_1 to y_2, we conclude $P_1^\alpha = P_2$. Thus, they have the same length, which must be m. As $d_{\Delta_1}(x, y)$ is minimal with $x, y \in \Delta_1 \cap \Delta_2$ such that x and y are on distinct sides of Δ_2 and the maximum distance between any two vertices in Δ_1 is m, the vertices x and y are the only ones in $\Delta_1 \cap \Delta_2$. We conclude that the subdigraphs induced by $x'xPyy_1$ and $x'xPyy_2$ are isomorphic: if y_1 is adjacent to some vertex z on P, then $y_1^\alpha = y_2$ is adjacent to $z^\alpha = z$, and as $m \geq 4$, neither y_1 nor y_2 is adjacent to x'. As $x'xPyy_1$ and $x'xPyy_2$ are isomorphic via an isomorphism that fixes $x'xPy$, we conclude as before for y and x using the two paths Q_i in Δ_1 from y to x' such that y_i lies on Q_i that the distance between y and x' in Δ_1 is m. This contradiction shows (18).

In the situation $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$, we shall obtain some precise information about $\Delta_1 \cap \Delta_2$:

(19) *Either $\Delta_1 \cap \Delta_2$ lies on the same side of Δ_1 or $\Delta(D) \cong CP_k$ for some $k \geq 3$ and the intersection consists of precisely two vertices which are adjacent in the bipartite complement of CP_k.*

Let us assume that $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$. Let $x, y \in \Delta_1 \cap \Delta_2$ be on distinct sides of Δ_1 with minimal distance in $G(\Delta_1)$. So x and y lie also on distinct sides of Δ_2. We may assume that x has only predecessors and y has only successors in Δ_1. If x and y are adjacent, then some and hence every edge of Δ_1 has both of its incident vertices in the same two reachability digraphs of D. In particular, we have $D = \Delta_1 \cup \Delta_2$. The vertex x has $k-1$ predecessors in $\Delta_1 \cap \Delta_2$ and $k-1$ successors in $\Delta_1 \cap \Delta_2$ none of which lies on the same side of Δ_1 as x. Due to $k \geq 3$, we have

$$|N^+(x)| + |N^-(x)| = 2(k-1) > k \geq |N(x)|$$

and thus some vertex of $\Delta_1 \cap \Delta_2$ lies in $N^+(x) \cap N^-(x)$, which is impossible. Thus, $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are not adjacent in Δ_1, which shows (19).

For $x, y \in VD$, let $x \approx y$ if x and y lie on the same side of two reachability digraphs. As every vertex lies in precisely two reachability digraphs, $\approx$ is an equivalence relation. The next aim is to show that $\sim$ and $\approx$ are (despite their different definition) the same relation, that is:

(20) *For all $x, y \in VD$, we have $x \sim y$ if and only if $x \approx y$.*

As a first step we shall prove:

(21) *If $\sim$ and $\approx$ are different relations, then for every two successors x, y (predecessors x, y, respectively) of any vertex we have $x \sim y$ but $x \not\approx y$.*

If for each vertex every two of its successors are $\approx$-equivalent, then one whole side of some reachability digraph Δ lies in a second reachability digraph Δ' on the same side as Δ is connected. If $G(\Delta(D)) \not\cong K_{k,\ell}$ for any $k \neq \ell$, then its sides have the same size due to Theorem 5.7 as $\Delta(D)$ is finite by Lemma 5.8 and $d^+ \geq 2$ and $d^- \geq 2$. Thus, as $\Delta \cap \Delta'$ is one whole side of Δ, it is also one whole side of Δ'. Hence, $\sim$ and $\approx$ are the same relation. Thus, we may assume $G(\Delta(D)) \cong K_{k,\ell}$ for some $k \neq \ell$. But then some vertex in Δ' has two predecessors in $\Delta \cap \Delta'$ and by C-homogeneity every two of its predecessors, and hence one whole side of Δ', lie in $\Delta \cap \Delta'$. So $\Delta \cap \Delta'$ is one whole side of Δ and one of Δ'. As one of those has size ℓ and the other has size k, this contradicts $k \neq \ell$. Thus, for any vertex any two of its successors are not $\approx$-equivalent by C-homogeneity. By a symmetric argument for predecessors, we obtain (21).

The next step is to show that

(22) *if $\sim$ and $\approx$ are different relations, then no reachability digraph separates D.*

Let us suppose that $\sim$ and $\approx$ are distinct but some, and hence any, reachability digraph separates D. Due to Lemma 5.9, there are two reachability digraphs whose intersection separates one of them. As there is a 2-arc in these two reachability digraphs, we can map them onto any two reachability digraphs with non-trivial intersection. Due to Lemma 5.8 and because of $d^+ \geq 2$ and $d^- \geq 2$ the graph $G(\Delta(D))$ is not a tree. Thus, there is no separating vertex in any of the possible reachability digraphs given by Theorem 5.7 and we conclude that every two reachability digraphs with at least one common vertex have at least two common vertices. Thus and due to (19), either the intersection of every two reachability digraphs is contained on the same side of each of them or $\Delta(D) \cong CP_3$; for if $k \geq 4$, no two vertices in CP_k separate that digraph.

Let us first assume that the intersection of every two reachability digraphs is contained on the same side of each of them. Note that no two vertices with a common successor can lie in the intersection of two reachability digraphs due to (21). Thus, $G(\Delta(D))$ is neither a complete bipartite graph nor the directed complement of a perfect matching. So Theorem 5.7 implies that $G(\Delta(D))$ is a cycle of length $2m$ for some $m \in \mathbb{N}$, as $\Delta(D)$ is finite by Lemma 5.8. Since $C_4 \cong K_{2,2}$ and $C_6 \cong CP_3$, we may assume $m \geq 4$. Let a and b be two vertices in the the intersection of two distinct reachability digraphs Δ_1 and Δ_2 of minimal distance in Δ_1 to each other and let P be a minimal path between a and b in Δ_1. Due to (21), the length of P is at least 4. Let w_1, w_2 be the neighbors of b in Δ_2, let u_1 be the vertex on P that is adjacent to a, and let u_2 be a vertex in Δ_2 that is adjacent to a. Let v be the neighbor of b on P. Since D contains no triangles, $D[w_i, b, v]$ are induced 2-arcs. Thus, C-homogeneity implies the existence of some $\alpha \in \mathrm{Aut}(D)$ with $(w_1, b, v)^\alpha = (w_2, b, v)$. Then α fixes Δ_1 pointwise as it fixes the edge between b and v. So we have $a^\alpha = a$ and hence $d_{\Delta_2}(a, w_1) = d_{\Delta_2}(a, w_2)$ and $d_{\Delta_2}(a, b) = m$. Since α fixes Δ_1 pointwise, the digraphs induced by $aPbw_1$ and $aPbw_2$ are isomorphic. Because of $m \geq 4$, neither w_1 nor w_2 is adjacent to u_2. Thus, the digraphs induced by u_2aPbw_1 and by u_2aPbw_2 are isomorphic. So we also have $d_{\Delta_2}(u_2, w_1) = d_{\Delta_2}(u_2, w_2)$ and hence $d_{\Delta_2}(u_2, b) = m$. But this cannot be true since Δ_2 contains a unique vertex of distance m to b and since $a \neq u_2$.

Let us now assume that the intersection of every two reachability digraphs is not contained on the same side of each of them. In particular, we have $\Delta(D) \cong CP_3$ due to (19). Then the intersection of two reachability digraphs Δ_1, Δ_2 consists, if it is not empty, of precisely two vertices a, b which are adjacent in the bipartite complements of each of the two reachability digraphs due to (21). Let $uavw$ be a 3-arc in D. Let us assume that $ua \in E\Delta_1$ and $av \in E\Delta_2$. We cannot have $w \in V\Delta_2$, because Δ_2 contains no 2-arc. Since D contains no directed triangle, w cannot lie on the same side of Δ_1 as b since otherwise $wa \in E\Delta_1$. Since $v \notin V\Delta_1$, we have $vw \notin E\Delta_1$. As Δ_1 contains the edges from all predecessors of a to a but not the edge vw, the vertex w cannot lie on the same side of Δ_1 as a. This shows $w \notin V\Delta_1$. Let $\Delta_3 = \langle \mathcal{A}(vw) \rangle$ and let w' be a vertex in $\Delta_3 - \Delta_2$. If there is a 3-arc that has its first edge in Δ_1 and w' as its last vertex, then we just saw $w' \notin V\Delta_1$. If there is no such 3-arc, then vw' is no edge of Δ_3 and the structure of Δ_3 implies that w' is a predecessor of v', the neighbor of v in directed complement of Δ_2. Thus, there is a 3-arc whose first vertex is w' and whose last edge lies in Δ_1 where we may assume that this 3-arc contains b and v'. By reversing the direction of the edges in the argument of the case that $uavw$ is a 3-arc in D, we obtain that w' is no vertex of Δ_1. Thus, no vertex of Δ_3 lies in Δ_1 and $D[(V\Delta_2 \setminus V\Delta_1) \cup V\Delta_3]$ is connected. So Δ_2 has only vertices in a unique component of $D - \Delta_1$. Thus, Lemma 5.9 implies that no reachability digraph separates D. This contradiction shows (22).

Now we shall prove (20). Let us suppose that we find vertices x, y with $x \sim y$ but $x \not\approx y$. Due to (21), we may assume that x and y either have a common successor or a common predecessor. By considering the digraph whose edges are directed in the inverse way, if necessary, we may assume that x and y have a common successor v_1. Let $\Delta = \langle \mathcal{A}(xv_1)\rangle$. Due to (22), we find a second induced (aside from the edge yv_1) path from v_1 to y whose only vertices in Δ are v_1 and y and that does not use the edge yv_1. Let R be such a path of minimal length. Then the only vertices on R that are adjacent to x are v_1 or the neighbor of y on R. Indeed, by C-homogeneity, we find some $\alpha \in \mathrm{Aut}(D)$ with $(v_1, x, y)^\alpha = (v_1, y, x)$ and, if x had other neighbors on R, then y has some neighbor on R^α and $v_1 R^\alpha zy$ contradicts the minimality of R, as it also lies outside of Δ except for v_1 and x.

Let v_3, v_2, y be the last three vertices on R. So we have $v_2 y \in ED$, since $v_2 \notin V\Delta$. Because of $x \not\approx y$, the vertices x and v_2 are not adjacent. So v_1 is the only neighbor of x on R. Let us suppose that $v_3 \sim y$. Then we have $v_2 v_3 \in ED$. If $v_3 \sim x$, then as $v_2 \notin V\Delta$ their common reachability digraph must be the one that contains x and its predecessors. By definition of $\sim$, it must be $\langle \mathcal{A}(v_2 v_3)\rangle = \langle \mathcal{A}(v_2 y)\rangle$. So we have $x \approx y$ in contradiction to their choice. Thus, we have $v_3 \not\sim x$. By C-homogeneity and as neither x nor y have neighbors other than v_1 and v_3 on R, yv_1Rv_3 can be mapped onto xv_1Rv_3 by an automorphism of D that fixes v_1Rv_3 and thus, we obtain $v_3 \sim x$, a contradiction.

So we have $v_3 \not\sim y$ and hence $v_3 v_2 \in ED$. Again, we find an automorphism α of D that maps yv_1Rv_3 onto xv_1Rv_3 and fixes v_1Rv_3 by C-homogeneity. We conclude that there is a vertex $v_4 := v_2^\alpha$ in D with $v_3 v_4 \in ED$ and $v_4 x \in ED$. Let v_0 be the neighbor of v_1 on R. Since $v_0 \notin \Delta$, we have $v_1 v_0 \in ED$. As D contains no directed triangle and $N^+(x)$ is an independent set, D contains no triangle at all. If v_1 and v_3 are adjacent, then $v_1 v_3 \in ED$ and $v_0 = v_3$ as no inner vertex of R lies in Δ and, if v_1 and v_3 are not adjacent, then v_0 and v_2 are not adjacent by minimality of R. Thus, $D[v_3, v_2, y, v_1]$ and $D[v_2, y, v_1, v_0]$ are isomorphic and there is some automorphism $\beta \in \mathrm{Aut}(D)$ with $(v_3, v_2, y, v_1)^\beta = (v_2, y, v_1, v_0)$. Let $y' = v_4^\beta$ and $v_1' = x^\beta$. The vertices $v_1, v_0, v_1', y', v_2, y$ form a cycle. So if neither y' nor v_1' lies in Δ, then we could have chosen $R' = v_1 v_0 v_1' y' v_2 y$ instead of R and we are in the first case $v_3 \sim y$, which already led to a contradiction. Thus, either y' or v_1' lies in Δ. If y' lies in Δ, then we have that y and y' must lie on the same side of Δ since v_2 lies not in Δ. So we have $y \approx y'$. Since y and y' have a common predecessor, this contradicts (21). Thus, y' does not lie in Δ, but v_1' does. If v_1' lies on the same side of Δ as v_1, then we obtain again with $v_1 \approx v_1'$ a contradiction to (21). So v_1' lies on the same side as y and x. But then v_0 lies on the same side of Δ as v_1 and there is an edge between vertices of that side in contradiction to the assumption that $\Delta(D)$ is bipartite. This shows (20).

Since $\approx$ is an equivalence relation on VD, we conclude from (20) that the same is true for $\sim$. Let $\Gamma := D_\sim$. Let $X \in V\Gamma$, let $x_1, x_2 \in X$, and let $y_1 \in N^+(x_1)$ and $y_2 \in N^+(x_2)$. Since x_1 and x_2 lie on the same side of two reachability digraphs, y_1 and y_2 lie on the same side of one – and due to (20) of two – reachability digraphs. Thus, X has a unique successor X^+ in Γ: the $\approx$-equivalence class that contains y_1, which is not X. Symmetrically, X has a unique predecessor in Γ, which is neither X nor X^+. So $D_\sim$ is a digraph. Every equivalence class of $\sim$ is finite, since $\Delta(D)$ is finite by Lemma 5.8. If $G(\Gamma)$ is a double ray, then this implies that D has at least two ends. Since this is false,

(23) Γ *is a directed cycle* C_n *for some* $n \geq 3$.

An edge e of Γ corresponds to a reachability digraph Δ of D in that the two equivalence classes of $\sim$ in Δ are the two vertices that are incident with e. If $G(\Delta(D)) \cong K_{k,\ell}$ for some $k, \ell \in \mathbb{N}$, then $k = \ell$ due to (20). Thus,

(24) *if* $G(\Delta(D)) \cong K_{k,k}$, *then* $D \cong C_n[\overline{K}_k]$.

So it remains to show that $\Delta(D)$ is a complete bipartite digraph. Let $V_1, \ldots, V_n$ denote the equivalence classes of $\sim$ such that $V_iV_{i+1} \in E\Gamma$ for $i < n$ and $V_nV_1 \in E\Gamma$. Due to Theorem 5.7 and Lemma 5.8 and as $d^+ \geq 2$ and $d^- \geq 2$, we just have to show that $G(\Delta(D))$ is neither an undirected cycle C_{2m} nor the complement of a perfect matching CP_k.

Let us show

(25) $G(\Delta(D)) \not\cong C_{2m}$ *for any* $m \geq 4$.

We suppose that $G(\Delta(D)) \cong C_{2m}$ for some $m \geq 4$. Let $x \in V_1$ and let a, b be its successors. Let a_1 and a_2 be the successors of a. As D contains no directed triangle and as Γ is a directed cycle, x is adjacent neither to a_1 nor to a_2. Thus, there is an automorphism α of D that maps a_1 to a_2 and fixes a and x. Hence, also b must be fixed by α and the two a-b paths in $G(D[V_2 \cup V_3])$ must have the same length, which must be m. Let x' be the second predecessor of a and let b' be a successor of x' other than a. By C-homogeneity, we find some $\alpha \in \mathrm{Aut}(D)$ with $(a, x, b)^\alpha = (a, x', b')$. This automorphism fixes $\langle \mathcal{A}(xa) \rangle$ and thus also $\langle \mathcal{A}(aa_1) \rangle$ setwise. Thus, the distance between a and b' in $G(D[V_2 \cup v_3])$ is m, too. Thus, we have $b = b'$ and hence $m = 2$. This contradiction shows (25).

Now we show

(26) $G(\Delta(D)) \not\cong CP_k$ *for any* $k \geq 3$.

Let us suppose that $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$. Let $x \in V_1$. If $n = 3$, then there is a directed triangle in D, as $k \geq 3$, which is impossible. So we conclude $n \geq 4$. There exists a unique vertex in V_2 that is not adjacent to x and this vertex itself has a unique vertex $y \in V_3$ to which it is not adjacent. Let P be a path that consists of x, y, and of one vertex v_i from every V_i for $i \geq 4$. This path exists since $k \geq 3$. Let y' be a vertex of V_3 with $y' \neq y$ but that is adjacent to v_4. Then the path $xv_n \ldots v_4y'$ is isomorphic to P, but there is no automorphism of D that maps the first onto the second one, since there is a unique vertex in V_2 that is not adjacent to x and y, but for x and y' there is no such vertex. This shows (26).

So $\Delta(D)$ is a complete bipartite digraph. As D is C-homogeneous, it is transitive and thus, all equivalence classes have the same size, that is $\Delta(D) \cong K_{k,k}$ for some $k \geq 1$. As D contains no directed triangle, we also conclude that $n \geq 4$, which proves the assertion. □

Having completed the case that the locally finite connected C-homogeneous digraph with at most one end contains no directed triangle, we look at those that contain directed triangles. The following lemma is the main lemma for this situation. The case (iv) of the conclusions of Lemma 5.11 will be investigated in more detail in Section 6.

Lemma 5.11. *Let D be a locally finite connected C-homogeneous digraph that contains a directed triangle. If $N^+(x)$ and $N^-(x)$ are independent sets for all $x \in VD$, then one of the following cases holds.*

(i) *The digraph D has at least two ends.*

(ii) *The reachability digraph $\Delta(D)$ is isomorphic to a complete bipartite digraph $K_{k,k}$ for some $k \geq 3$ and D is isomorphic to $C_3[\overline{K}_k]$.*
(iii) *The reachability digraph $\Delta(D)$ is isomorphic to CP_k for some $k \geq 4$ and D is isomorphic to Y_k.*
(iv) *The underlying undirected graph of the reachability digraph $\Delta(D)$ is isomorphic either to C_{2m} for some $m \geq 2$ or to $T_{2,2}$.*

Proof. Due to Lemma 5.5 and Proposition 3.1, the reachability digraph $\Delta(D)$ is bipartite. Let us assume that D has at most one end and that $G(\Delta(D))$ is neither isomorphic to C_{2m} for some $m \geq 2$ nor isomorphic to $T_{2,2}$. Due to Lemma 5.1, we may assume $d^+ \geq 2$ and $d^- \geq 2$. According to Lemma 5.8 and Theorem 5.7, we know that $\Delta(D)$ is finite and either a complete bipartite digraph or the directed complement of a perfect matching.

Case (a): $G(\Delta(D)) \cong K_{k,\ell}$ for some $k, \ell \in \mathbb{N}$ but $G(\Delta(D)) \ncong K_{2,2}$ as that is a cycle. By Lemma 5.3, we know that $k = \ell$. If we have $|\Delta \cap \Delta'| \geq 2$ for two distinct reachability digraphs Δ and Δ', then $\Delta \cap \Delta'$ lies on one side of Δ and it is a direct consequence of C-homogeneity that $\Delta \cap \Delta'$ is a complete side of Δ and hence of Δ' since some two vertices in $\Delta \cap \Delta'$ have a common predecessor x in either Δ or Δ' and by C-homogeneity every two successors of x lie in $\Delta \cap \Delta'$. But then we consider – as in the proof of Lemma 5.10 – the following equivalence relation: $x \approx y$ if and only if they lie on the same side of two reachability digraphs. The equivalence classes of $\approx$ are the sides of the reachability digraphs since $\Delta \cap \Delta'$ is a whole side of Δ and of Δ'. Then the proof of (23) also holds with our situation here. So the quotient digraph $\Gamma := D_{\approx}$ is a directed cycle. Since D contains a directed triangle, we have $\Gamma \cong C_3$. Thus, (ii) holds.

So let us suppose that there are two distinct reachability digraphs Δ and Δ' with $|\Delta \cap \Delta'| = 1$. If an edge lies in more than one directed triangle, then it lies in at least $k-1$ distinct such triangles due to Lemma 5.4. So the intersection $\Delta \cap \Delta'$ has to contain at least $k-1$ elements which is a contradiction. Hence, every edge lies in a uniquely determined directed triangle.

To show that this situation cannot occur, let x and y be two vertices on the same side of Δ such that their out-degree in Δ is 0. Let u be a common predecessor of x and y. As every edge lies on a unique directed triangle, we find successors a, b of x, y, respectively, such that they are predecessors of u. Let c be a common successor of a and b distinct from u. Since every edge lies on precisely one directed triangle, neither $D[x, a, c]$ nor $D[y, b, c]$ are triangles. As $k \geq 3$, there is a second predecessor z of b such that z and c as well as z and u are not adjacent. If $za \in ED$, then y and x have to lie in two common reachability digraphs which we supposed to be false. If $az \in ED$, then z and c lie in a common reachability digraph and it is not a bipartite reachability digraph because zbc is a 2-arc in that reachability digraph. Thus, the vertices a and z cannot be adjacent. Furthermore, zx cannot be an edge of D, because then the edge yb would have its two incident vertices on the same side of a reachability digraph. Let us suppose that xz is an edge of D. Then there is an automorphism α of D that maps $D[x, a, c, b]$ onto $D[z, b, c, a]$. We conclude that there is a vertex $z' = z^\alpha \in N^-(a)$ with $zz' = x^\alpha z^\alpha \in ED$. But the edge zz' has the wrong direction: in a complete bipartite reachability digraph all edges are directed from one side to the other, but zz' is directed the other way round compared with the edges xa, xz, and $z'a$. This contradiction shows that x and z cannot be adjacent. Hence, we have shown that the subdigraphs $D[x, a, c, b, y]$ and

$D[x, a, c, b, z]$ are isomorphic. But there is no automorphism of D that maps one onto the other by fixing all of x, a, c, b, since x and y lie on the same side of a reachability digraph but x and z do not because of $uz \notin ED$. Thus, we showed that there are no two reachability digraphs whose intersection consists of precisely one vertex. This completes the case $G(\Delta(D)) \cong K_{k,l}$.

Case (b): $G(\Delta(D)) \cong CP_k$ for some $k \geq 3$. If $k = 3$, then $G(\Delta(D))$ is a cycle. So we may assume $k \geq 4$. Let Δ_1 and Δ_2 be two distinct reachability digraphs of D with non-trivial intersection. Let us suppose that $|\Delta_1 \cap \Delta_2| = 1$. Then this holds for any two distinct reachability digraphs with non-trivial intersection as each vertex lies in precisely two reachability digraphs and as we can map the unique vertex in $\Delta_1 \cap \Delta_2$ onto any vertex in the intersection of any two reachability digraphs by C-homogeneity. Let $a, b, c, v, w \in V\Delta_1$ such that $b, v, w \in N^+(a)$ and $b, w \in N^+(c)$ but $cv \notin ED$. Such vertices exist as $k \geq 4$. Since any edge lies in a directed triangle, there are $x, y \in N^-(a)$ with $x \in N^+(v)$ and $y \in N^+(w)$. Because of $|\Delta_1 \cap \Delta_2| = 1$, no other edges than the described ones lie in $D[a, b, c, v, w, x, y]$. Then the digraphs $D_1 := D[a, b, c, x]$ and $D_2 := D[a, b, c, y]$ are isomorphic but there is no automorphism of D that maps D_1 onto D_2 because such an automorphism has to map v, the unique predecessor of x in Δ_1, onto w, the unique predecessor of y in Δ_1, but w is adjacent to c and v is not. Thus, we have proved

$$|\Delta_1 \cap \Delta_2| \geq 2. \tag{27}$$

Let us suppose that $\Delta_1 \cap \Delta_2$ is not contained in any of the sides of Δ_1. Then $\Delta_1 \cap \Delta_2$ consists of precisely two vertices that are adjacent in the directed bipartite complement of Δ_1 and, furthermore, any edge lies in at most two directed triangles (because of $|\Delta_1 \cap \Delta_2| = 2$) and by Lemma 5.4 any edge lies in precisely one directed triangle (because of $k \geq 4$). Let us consider the subdigraph of Δ_1 with vertices a, b, c, d and edges ba, bc, dc such that $\{a, d\} = V(\Delta_1 \cap \Delta_2)$. Let z be the vertex on the unique directed triangle that contains ba and let x and y be two predecessors of d in Δ_2 such that x is the neighbor of z in the directed bipartite complement of Δ_2 and such that y is not adjacent to c. We can choose them in this way as $k \geq 4$ and as dc lies in precisely one directed triangle. In addition, we may replace c by some other vertex in $N^+(b) \cap N^+(d)$, if necessary, such that $D[x, d, c]$ is not a directed triangle, that is, such that x and c are not adjacent. Furthermore, neither x nor y can be adjacent to b, as – regardless of the direction of this edge – such an edge implies that b lies in $\Delta_1 \cap \Delta_2$, too, which is impossible due to $b \notin \{a, d\} = V(\Delta_1 \cap \Delta_2)$. Hence, the subdigraphs $D[b, c, d, x]$ and $D[b, c, d, y]$ are isomorphic to each other, so there is an automorphism α of D that fixes each of b, c, and d and maps x to y. Then also a must be fixed by α, as it is the unique neighbor of d in the directed bipartite complement of Δ_1, and hence, we also have $z^\alpha = z$ by the choice of z. But this is impossible because y and z are adjacent in contrast to x and z. Thus, we proved that $\Delta_1 \cap \Delta_2$ is contained in one side of Δ_1. C-homogeneity directly implies that $\Delta_1 \cap \Delta_2$ is a whole side of Δ_1, as we can map any two vertices of $\Delta_1 \cap \Delta_2$ with a common neighbor in Δ_1 onto any other two vertices on the same side as $\Delta_1 \cap \Delta_2$ of Δ_1 with a common neighbor in Δ_1. Thus, we have

$$|\Delta_1 \cap \Delta_2| = k. \tag{28}$$

Now, we are able to prove $D \cong Y_k$. Due to (28) and as every edge lies in a directed triangle, D consists of precisely three reachability digraphs Δ_1, Δ_2, and Δ_3. Let $V_i := V\Delta_i \cap V\Delta_{i+1}$ with $\Delta_4 = \Delta_1$ and let $\overline{D}$ denote the directed tripartite complement of D. Since $\Delta(D) \cong CP_k$, the digraph $\overline{D}$ is a union of directed cycles. We shall show that every component of $\overline{D}$ is a directed cycle of length 3. So let us suppose that this is not the case. Then there are $x, y \in V_1$ that lie in a common directed cycle of length at least 6 in $\overline{D}$ and have distance 3 on that cycle. Since $k \geq 4$, there is a vertex $a \in V_2$ that is adjacent in D to both x and y. We conclude by C-homogeneity that for every vertex $z \in V_1$, distinct from x, we have that x and z lie on a common directed cycle in $\overline{D}$ and have distance 3 on that cycle. It is a direct consequence that $k \leq 3$ in contrast to the assumption $k \geq 4$. Hence, we have shown $D \cong Y_k$. □

6. An imprimitive case

In this section, we investigate the situation from Lemma 5.11 (iv): we look at locally finite connected C-homogeneous digraphs that contain directed triangles, all whose vertices have independent out- and in-neighborhood and for whose reachability digraph the underlying undirected graph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$. In [10], Gray and Möller showed the existence of such a digraph, in that they showed that $T(2)$ has all these properties. It has infinitely many ends. But although we are interested only in digraphs with at most one end, this particular digraph turns out to be very important in our situation: we shall show that every digraph with the above described properties and with at most one end is a homomorphic image of $T(2)$. More precisely, we prove:

Theorem 6.1. *The following assertions are equivalent for any locally finite connected digraph D all whose vertices have independent out- and in-neighborhood.*

(i) *The digraph D is C-homogeneous and contains a directed triangle. If $D \ncong C_3$, then the underlying undirected graph of its reachability digraph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$.*
(ii) *There is a non-universal* $\mathrm{Aut}(T(2))$*-invariant equivalence relation $\sim$ on $VT(2)$ such that $T(2)_\sim$ is a digraph that is isomorphic to D.*

Furthermore, D has at most one end if and only if one, and hence every, equivalence class of $\sim$ consists of more than one element.

Proof. To see that (i) implies (ii), we may assume that D is not isomorphic to C_3: otherwise take any labeling of the vertices of $T(2)$ with labels 0, 1, 2 such that no two adjacent vertices have the same label and such that out-neighbors of vertices labeled by i are labeled by $i+1 \pmod 3$. This labeling induces an $\mathrm{Aut}(T(2))$-invariant equivalence relation $\sim$ on $VT(2)$ such that $T(2)_\sim$ is a directed triangle.

Therefore, every vertex of D has out-degree 2. So every edge lies in at most two directed triangles. Let us first assume that every edge of D lies in precisely two directed triangles. For an edge xy, the two successors of y are the two predecessors of x. So the other successor of x must have the same successors as y. The analogous statements hold for the second predecessor of y. It is a direct consequence that $G(\Delta(D)) \cong C_4 \cong K_{2,2}$ and that $D \cong C_3[\overline{K}_2]$. Let x_i, y_i, z_i for $i = 1, 2$ be the vertices of D such that $x_i y_j$, $y_i z_j$ and $z_i x_j$, for all $i, j \in \{1, 2\}$, are the edges of D. We label the vertices of $T(2)$ with labels from $V(D)$ so that for every vertex labeled by x_i its successors obtain different labels from $\{y_1, y_2\}$ and its predecessors obtain different labels from $\{z_1, z_2\}$ and so that the analogue statements hold for vertices labeled by y_i and by z_i. Starting with a triangle labeled by $x_1 y_1 z_1$, there is a unique way to extend its labelling to the whole digraph $T(2)$ such that the just described property holds. Two vertices are $\sim$-equivalent if they have the same label. Then by definition, $T(2)_\sim$ is a digraph and isomorphic to D. Furthermore, the $\mathrm{Aut}(T(2))$-invariance is a consequence of the unique extension property of the labeling by starting it at a directed triangle.

Let us now assume that every edge of D lies in precisely one directed triangle. As $d^+ = 2$, every vertex lies in precisely two. Let $xy \in ED$ and $ab \in ET(2)$. For every vertex u in $T(2)$ there exists a unique shortest path $P = a_1 \dots a_n$ from a to u. In D there are precisely two walks $x_1 \dots x_n$ and $y_1 \dots y_n$ starting at x (i.e. with $x_1 = x = y_1$) such that $D[x_i, x_{i+1}, x_{i+2}]$ and $D[y_i, y_{i+1}, y_{i+2}]$ are isomorphic to $D[a_i, a_{i+1}, a_{i+2}]$ for all $i \leq n-2$ in the canonical way (i.e. such that x_i and y_i are mapped to a_i and so on). That there are precisely two such walks in D follows from the fact that every vertex of D lies in precisely two directed triangles and in the middle of precisely two induced 2-arcs. In particular, no two end vertices of any subpath of length 2 of the walks in D are adjacent. If $a_2 = b$ or if a_2 is adjacent to b, then let Q be that one of the two above described walks in D whose second vertex is y or is adjacent to y, and in the other case for a_2 let Q be the other described walk in D. Let u_D denote the last vertex of Q. Thereby, we define for every vertex v of $T(2)$ a vertex v_D in D.

We are now able to define the equivalence relation $\sim$: let $u \sim v$ for two vertices $u, v \in VT(2)$ if $u_D = v_D$. Obviously, this is a non-universal equivalence relation. It remains to show that $T(2)_\sim$ is a digraph, that $D \cong T(2)_\sim$ and that $\sim$ is $\mathrm{Aut}(T(2))$-invariant. Let us first show that $\sim$ is $\mathrm{Aut}(T(2))$-invariant. Let π be the map from $T(2)$ to D that maps z to z_D, let $u, v \in VT(2)$ with $u \sim v$ and let ψ be an automorphism of $T(2)$. It suffices to show $u^\psi \sim v^\psi$. First, let us consider the case that the shortest path $P = u_1 \dots u_n$ from u to v does not contain any other vertex of the equivalence class that contains u. If we have shown this, then the assertion follows by an easy induction on the number of elements on P that are equivalent to u. We look at the images of P and P^ψ under π. These are walks due to the definition of π, because adjacent vertices in $T(2)$ are mapped to adjacent vertices of D. As $u \sim v$, the walk P^π starts and ends at the same vertex u_D. For every $i \leq n$, we can map $(u_1 \dots u_i)^\pi$ onto $(u_1 \dots u_i)^{\psi\pi}$ inductively, since D is C-homogeneous and since $(u_{i+1})_D$ is uniquely determined in D by the two walks $(u_1 \dots u_i)^\pi$ and $(u_1 \dots u_i)^{\psi\pi}$. We conclude that also the walk $(P^\psi)^\pi$ has the same end vertices. So we have $u^\psi \sim v^\psi$. Hence, $\sim$ is $\mathrm{Aut}(T(2))$-invariant.

Next, we show that $T(2)_\sim$ is a digraph. That there are no loops in $T(2)_\sim$ is a direct consequence of the definition of $\sim$, as we do not have $a'_D = a_D$ for any neighbor a' of a and as D is $\mathrm{Aut}(T(2))$-invariant. The only other obstacle for $T(2)_\sim$ being a digraph is that the edges are not asymmetric. Another consequence of the definition of $\sim$ is that no two neighbors of a are $\sim$-equivalent, as every vertex of D and every vertex of $T(2)$ lies in precisely two directed triangles. Let us suppose that there are vertices a_1, a_2, b_1, and b_2 in $T(2)$ with $a_1a_2,\ b_1b_2 \in ET(2)$ and $a_1 \sim b_2$ and $a_2 \sim b_1$. Due to transitivity of $T(2)$, there is an automorphism α of $T(2)$ that maps a_2 to b_1. Since $\sim$ is $\mathrm{Aut}(T(2))$-invariant, there is also an in-neighbor of b_1 in the same equivalence class as b_2, which is impossible as we already saw. Thus, we have shown that $T(2)_\sim$ is a digraph.

That D and $T(2)_\sim$ are isomorphic is a direct consequence of the definition of $\sim$, since they have the same in- and out-degree. This shows (ii).

Let us now assume that (ii) holds, more precisely, that $D = T(2)_\sim$. We shall prove (i). As $T(2)$ is vertex-transitive so is D. Let us assume that D is not a directed triangle. So every vertex of D has two successors and, as every edge lies in a directed triangle since they do so in $T(2)$, every vertex of D lies in at least two directed triangles and no two neighbors of a vertex of $T(2)$ are $\sim$-equivalent. Thus, for every $uv \in ED$ and every $x \in T(2)$ whose equivalence class is u, there is a vertex $y \in N^+(x)$ whose equivalence class is v, as $d^+(x) = 2 = d^-(x)$. We also obtain that $\Delta(D)$ is a homomorphic image of $\Delta(T(2))$, so its underlying undirected graph is either $T_{2,2}$ or C_{2m} for some $m \geq 2$. To show that D is C-homogeneous, let A and B be isomorphic induced connected subdigraphs of D and let $\varphi : A \to B$ be an isomorphism. Let T_A be a spanning tree of A. Then we can map T_A by an injective homomorphism π_A to $T(2)$ such that a is the equivalence class of $\pi_A(a)$ for all $a \in VA$. Notice that π_A is uniquely determined by the image of one vertex of A. Analogously, we define T_B and π_B such that $T_B = T_A^\varphi$. The subdigraphs of $T(2)$ induced by $A' := (T_A)^{\pi_A}$ and $B' := (T_B)^{\pi_B}$ are isomorphic by an isomorphism that induces on the equivalence classes of the vertices of A' and of B' the isomorphism φ. As $T(2)$ is C-homogeneous, this isomorphism extends to an automorphism ψ of $T(2)$. Since $\sim$ is $\mathrm{Aut}(T(2))$-invariant, this automorphism induces an automorphism ϕ of D that extends φ. So D is C-homogeneous.

The only remaining part to show is the additional claim on multi-ended digraphs which is a direct consequence of [10, Theorem 7.1], because $T(2)_\sim$ is not isomorphic to $T(2)$ as soon as each equivalence class contains at least two elements. □

Figure 5 shows two C-homogeneous digraphs that arise as quotient digraphs in Theorem 6.1 one of which is finite and the other being infinite and one-ended. In the finite digraph the edges of each reachability digraph, which is isomorphic to C_{10}, are drawn in different styles. The reachability digraphs of the infinite digraph are the cycles of length 6.

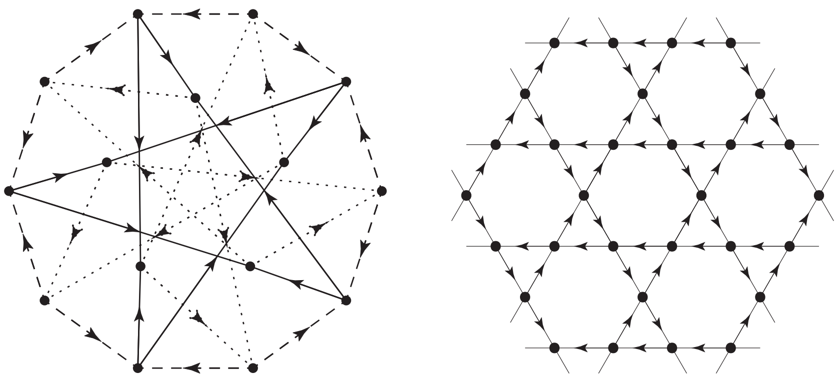

FIGURE 5. A finite and an infinite one-ended C-homogeneous digraph

As the automorphism group of $T(2)$ is a free product of the cyclic groups C_2 and C_3, it is isomorphic to the modular group. Let us consider the Cayley digraph Λ of $\Gamma := C_2 * C_3 = \langle x \rangle * \langle y \rangle$ with respect to the two canonical generators x and y. If we contract the edges in Λ that correspond to the involution x, then we obtain the digraph $T(2)$. Let $\sim$ be an $\mathrm{Aut}(T(2))$-invariant equivalence relation on $VT(2)$ and let X be the equivalence class that contains the vertex that arose from 1 and x in Λ by contracting the edges labeled by x. It is straight-forward to show that X corresponds to vertices of Λ that coincide with a subgroup of Γ that contains x. Conversely, the cosets of any subgroup of Γ that contains x induce in a canonical way a partition of $VT(2)$ and hence an equivalence relation of $VT(2)$ that is $\mathrm{Aut}(T(2))$-invariant. Therefore, instead of giving a precise list of the digraphs that may occur as quotients in Theorem 6.1, it is equivalent to describe all those subgroups of $C_2 * C_3$ that contain x. By Kurosh's Subgroup Theorem [16], every subgroup of the modular group is a free product of cyclic groups of orders 2, 3, or ∞ and the involutions form a conjugacy class in Γ. Thus, any subgroup of Γ that contains an involution is – up to conjugation – an example of a subgroup that corresponds to a C-homogeneous digraph in Theorem 6.1. As the number of cosets of a subgroup of Γ coincides with the number of vertices in the C-homogeneous digraph to which it corresponds in the above sense, the subgroups of finite index correspond to the finite and the subgroups of infinite index correspond to the infinite C-homogeneous digraphs in Theorem 6.1. There are numerous papers written on the subgroups of the modular group. Some of them deal with those of finite index, see [15, 21], and some with those of infinite index, see [23, 24, 25].

7. The main theorem

Let us now state our main result. We shall prove it by applying the results of the previous sections to show that no other than the described locally finite connected digraphs with at most one end are C-homogeneous. Additionally, we have to show that all those digraphs are C-homogeneous.

Theorem 7.1. *Let D be a locally finite connected digraph with at most one end. Then D is C-homogeneous if and only if one of the following cases holds:*

(i) $|VD| = 1$;
(ii) $D \cong C_m[\overline{K}_n]$ *for integers* $m \geq 3, n \geq 1$;
(iii) $D \cong H[\overline{K}_n]$ *for some integer* $n \geq 1$;
(iv) $D \cong Y_k$ *for some integer* $k \geq 3$;
(v) *there is a non-trivial and non-universal* $\mathrm{Aut}(T(2))$*-invariant equivalence relation* $\sim$ *on* $VT(2)$ *such that* $D \cong T(2)_\sim$.

Proof. First, let us assume that D is C-homogeneous and that D has at least one edge. If the out-neighborhood (or symmetrically the in-neighborhood) of any vertex of D is not independent, then we conclude from Theorem 4.6 that D is finite and isomorphic to $H[\overline{K}_n]$ for some $n \geq 1$. So we may assume that the out-neighborhood of each vertex is independent. Then, it is a direct consequence of Lemma 5.10, Lemma 5.11, and Theorem 6.1 that either (ii), (iv), or (v) holds.

Let us now show that all digraphs described in (i) to (v) are C-homogeneous. This is obvious in the situation (i). For those described in (v), it holds due to Theorem 6.1. That $H[\overline{K}_n]$ is C-homogeneous, follows from the fact that H is homogeneous. Obviously, C_m is C-homogeneous for all $m \geq 3$, so the same is true for $C_m[\overline{K}_n]$ as its reachability digraph is a complete bipartite digraph.

It remains to prove that the digraphs Y_k with $k \geq 3$ are C-homogeneous. Let A and B be two isomorphic connected induced subdigraphs of $D := Y_k$. Let V_1, V_2, V_3 be the three vertex sets as in the proof of Lemma 5.10 and let $\Delta_1, \Delta_2, \Delta_3$ be the corresponding reachability digraphs such that $\Delta_i = D[V_i \cup V_{i+1}]$ with $V_4 = V_1$. Let

α be an isomorphism from A to B. It is straightforward to see that $(VA \cap V_i)^\alpha$ is precisely the intersection of VB with some V_j: consider an undirected path between two vertices of VA and subtract from the number of forward directed edges on that path the number of backward directed edges. The resulting number is divisible by 3 if and only if the end vertices of the path lie in the same V_i. Hence, we may assume that $(VA \cap V_i)^\alpha = VB \cap V_i$ for all $i \leq 3$. Let us first assume that $\Delta_i \cap A$ is connected for some $i \leq 3$, say for $i = 1$. Let Δ_1' be a minimal subdigraph of Δ_1 isomorphic to some CP_ℓ with $\ell \leq k$ such that $A \cap \Delta_1 = A \cap \Delta_1'$. By replacing B by B^γ, for an automorphism γ of D, we may assume that also $B \cap \Delta_1 = B \cap \Delta_1'$ holds. Since $G(CP_\ell)$ is a C-homogeneous bipartite graph, we can extend every isomorphism from $\Delta_1' \cap A$ to $\Delta_1' \cap B$, in particular the restriction of α, to an automorphism of Δ_1'. Let α' be the automorphism of Δ_1' that extends the above restriction of α. Let $V_3' \subseteq V_3$ be the set of those vertices that are non-adjacent to at least one vertex of Δ_1'. As each vertex in V_3' is uniquely determined by two non-adjacent vertices one of which lies in $V_1 \cap V\Delta_1'$ and the other in $V_2 \cap V\Delta_1'$, the isomorphism α' has precisely one extension β on $D' := D[V\Delta_1' \cup V_3']$. By the construction of β it is easy to see that the restriction of α to $A \cap D'$ is again an isomorphism from $A \cap D'$ to $B \cap D'$ and is equal to the restriction of β to $A \cap D'$. Since all vertices of $A \cap (V_3 \setminus V_3')$ are adjacent to all vertices of $A \cap (V_1 \cup V_2)$ and since the same holds for B instead of A, the isomorphism β can be extended to an automorphism of D whose restriction to A is α.

If no $\Delta_i \cap A$ is connected, then we have $|V_i \cap VA| \leq 2$ for all $i \leq 3$. In particular, we have $|VA| \leq 6$. As $|VA| \leq 4$ also leads to some connected $\Delta_i \cap A$, we have $5 \leq |VA| \leq 6$. Hence, we may assume that $|VA \cap V_1| = 2 = |VA \cap V_2|$ and $|VA \cap V_3| \in \{1, 2\}$. As $\Delta_1 \cap A$ is not connected, it is a perfect matching. Either the same holds for $\Delta_2 \cap A$ and $\Delta_3 \cap A$ and we conclude that $A \cong C_6$, or $|V_3 \cap VA| = 1$ and A is a directed path of length 4. In both cases it is easy to verify that α extends to an automorphism of D. □

Acknowledgement

I would like to thank R.G. Möller for pointing out the connections to the modular group in Section 6 and I would like to thank the referees for their valuable comments.

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