ACCESSIBILITY IN TRANSITIVE GRAPHS

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Abstract. We prove that the cut space of any transitive graph G is a finitely generated $\mathrm{Aut}(G)$-module if the same is true for its cycle space. This confirms a conjecture of Diestel which says that every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible. In addition, it implies Dunwoody's conjecture that locally finite hyperbolic transitive graphs are accessible. As a further application, we obtain a combinatorial proof of Dunwoody's accessibility theorem of finitely presented groups.

1. Introduction

A locally finite transitive graph is *accessible* if there exists some $k \in \mathbb{N}$ such that any two ends can be separated by at most k edges. Relating this notion to the accessibility of finitely generated groups, Thomassen and Woess [18] proved that a finitely generated group is accessible if and only if some (and hence every) of its locally finite Cayley graphs is accessible.

Dunwoody [9] proved that the finitely presented groups are accessible. In this paper, we obtain as a corollary of our main theorem a result for the larger class of all locally finite transitive graphs that is similar to the accessibility theorem for finitely generated groups. Note that we have to make additional assumptions on the graph, as Dunwoody gave examples of locally finite inaccessible transitive graphs, see [10, 12].

Dunwoody [12] wrote that it seemed likely that all hyperbolic graphs are accessible. More generally (see Section 4.2), Diestel [6] conjectured in 2010 that locally finite transitive graphs are accessible as soon as their cycle spaces are generated by cycles of bounded length. We shall confirm both conjectures and prove the following theorem (for further definitions, see Section 2):

Theorem 1.1. *Let G be a 2-edge-connected graph. If its cycle space is a finitely generated $\mathrm{Aut}(G)$-module, then so is its cut space.*

The following special case of Theorem 1.1 also follows from a result of Mosher et al. [16, Theorem 15], see also Cornulier [1, Theorem 4.C.3], who independently confirmed Diestel's conjecture using simplicial 2-complexes.

Theorem 1.2. *Every locally finite transitive graph whose cycle space is generated by cycles of bounded length is accessible.*

Our proof includes a combinatorial proof of Dunwoody's accessibility theorem for finitely presented groups (Section 4.1), see [9]. In Section 4.2, we will deduce Dunwoody's conjecture on hyperbolic graphs from our main theorem. Using 2-manifolds, Dunwoody [11] proved that locally finite transitive planar graphs are accessible. In a forthcomming paper [15] we will obtain as a further corollary of Theorem 1.2 a combinatorial proof for the accessibility of such graphs. In the last section (Section 4.4), we discuss the connections of our main theorem with compactly presented groups. Note that Cornulier [1, Theorem 4.H.1] proved that such groups are accessible.

We might ask for an 'if and only if' in Theorem 1.1. Note that there are one-ended finitely generated groups that are not finitely presentable, e. g. the lamplighter groups. The cut space of each of their locally finite Cayley graphs G is generated by the cuts $E(\{v\}, V(G) \smallsetminus \{v\})$ for all $v \in V(G)$, the set of edges incident with v. These cuts form a single orbit under the automorphisms as G is transitive. However, as the group has no finite presentation, it seems unlikely, that the cycle space is a finitely generated $\mathrm{Aut}(G)$-module. Thus, we ask the following.

Question. *Does there exist a transitive 2-edge-connected graph G whose cut space is a finitely generated $\mathrm{Aut}(G)$-module but whose cycle space is not?*

2. Preliminaries

Let G be a graph. A *ray* is a one-way infinite path and a *tail* of a ray is a subgraph that is a ray. Two rays are *equivalent* if they lie eventually in the same component of $G - F$ for every finite edge set $F \subseteq E(G)$. It is easy to show that this is an equivalence relation whose classes are the *edge ends* of G. An edge set $F \subseteq E(G)$ *separates* two ends if the rays of these ends lie eventually in different components of $G - F$.

Let $\mathcal{B}(G)$ be the set of all ordered bipartitions (A, B) with $A \cap B = \emptyset$ and $A \cup B = V(G)$ such that the set $E(A, B)$ of all edges between A and B is finite. The set $\mathcal{B}(G)$ is a vector space over $\mathbb{F}_2$ where the addition of two ordered bipartitions has as its first component the symmetric difference between the two first components of the summands (and the set of all other vertices as its second component). For $n \in \mathbb{N}$ let $\mathcal{B}_n(G)$ be the subspace induced by the bipartitions (A, B) with *order* at most n, i.e., with $|E(A, B)| \leq n$. So we have $\mathcal{B}(G) = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n(G)$. Note that the action of $\mathrm{Aut}(G)$ on G induces an action of $\mathrm{Aut}(G)$ on $\mathcal{B}(G)$. We equip the set $\mathcal{B}(G)$ with a relation

$$(A, B) \leq (C, D) \;:\Longleftrightarrow\; A \subseteq C \text{ and } D \subseteq B.$$

It is easy to verify that this is a partial order. We call a subset $\mathcal{E}$ of $\mathcal{B}(G)$ *nested* if for any two $(A, B), (C, D) \in \mathcal{E}$ either (A, B) and (C, D) or (A, B) and (D, C) are $\leq$-comparable. We call $(A, B) \in \mathcal{B}(G)$ *tight* if the two subgraphs of G induced by A and by B are connected graphs. The *cut space* of G is the set of all edge sets $E(A, B)$ with $(A, B) \in \mathcal{B}(G)$ seen as a vector space over $\mathbb{F}_2$, where an edge lies in the sum of two *cuts* $E(A, B)$ and $E(A', B')$ if and only if it lies in precisely one of them. There is a canonical epimorphism between $\mathcal{B}(G)$ and the cut space of G that maps an ordered bipartition to its cut. So two distinct ordered bipartitions (A, B) and (C, D) of positive order are mapped to the same cut if and only if $(A, B) = (D, C)$.

The following theorem is due to Dicks and Dunwoody [3].

Theorem 2.1. [3, Theorem 2.20 and Remark 2.21 (ii)] *If G is a connected graph, then there is a sequence $\mathcal{E}_1 \subseteq \mathcal{E}_2 \subseteq \ldots$ of subsets of $\mathcal{B}(G)$ such that each $\mathcal{E}_n$ is an $\mathrm{Aut}(G)$-invariant nested set of tight elements of order at most n that generates $\mathcal{B}_n(G)$.* □

A useful lemma on tight elements of $\mathcal{B}(G)$ is the following – see e. g. [18, Proposition 4.1] for the first part, the second one follows by using the first one for the edges on any fixed path between the two vertices:

Lemma 2.2. *Let G be a connected graph and $n \in \mathbb{N}$.*

(1) [18, Proposition 4.1] *For every $e \in E(G)$, there are only finitely many tight $(A, B) \in \mathcal{B}(G)$ with $e \in E(A, B)$ and order at most n.*
(2) *For every $x, y \in V(G)$, there are only finitely many tight $(A, B) \in \mathcal{B}(G)$ of order at most n with $x \in A$ and $y \in B$.* □

Lemma 2.3. *Let G be a graph, $X, Y \subseteq V(G)$, and $\mathcal{E}$ be a nested subset of $\mathcal{B}(G)$. Then the following holds:*

(1) *The partial order $\leq$ is a total order on the set*

$$\mathcal{E}_{(X,Y)} := \{(A, B) \in \mathcal{E} \mid X \subseteq A, Y \subseteq B\}.$$

(2) *If $\mathcal{E} \subseteq \mathcal{B}_n(G)$ for some $n \in \mathbb{N}$ and if every element of $\mathcal{E}$ is tight, then $\mathcal{E}_{(X,Y)}$ is a finite chain.*

Proof. Let $(A, B), (C, D) \in \mathcal{E}_{(X,Y)}$. If (A, B) and (D, C) are $\leq$-comparable, then either $A \cap C$ or $B \cap D$ is empty; the first case contradicts $X \subseteq A \cap C$ and the second contradicts $Y \subseteq B \cap D$. Thus, (A, B) and (C, D) are $\leq$-comparable and (1) follows immediately.

Then (2) is just a corollary of (1) and Lemma 2.2 (1) applied to any vertices $x \in X$ and $y \in Y$. □

The *sum* of finitely many cycles $(C_i)_{i \in I}$ (over $\mathbb{F}_2$) in G is the subgraph induced by those edges that occur in an odd number of C_i. The set of all these sums of cycles forms a vector space over $\mathbb{F}_2$, the *cycle space* of G.

Lemma 2.4. *Let G be a graph, $C \subseteq G$ a cycle, and F a finite cut with precisely two edges e, f of C. Let $\mathcal{C}$ be any finite set of cycles in G such that $C = \sum_{D \in \mathcal{C}} D$. Then there is an alternating sequence $e_1 C_1 e_2 C_2 \ldots e_n$ of edges $e_i \in F$ and cycles $C_i \in \mathcal{C}$ with $e = e_1$ and $f = e_n$ such that e_i and e_{i+1} are edges of C_i.*

Proof. Let $\mathcal{D} \subseteq \mathcal{C}$ consist of precisely those cycles that lie on alternating sequences $e_1 C_1 \ldots e_n$ with $e = e_1$ and $e_i, e_{i+1} \in E(C_i) \cap F$ and $C_i \in \mathcal{C}$. Then $\sum_{D \in \mathcal{D}} |E(D) \cap F|$ is even as every cycle intersects with the finite cut F in an even number of edges. Note that for each edge $b \in F$ in $\bigcup_{D \in \mathcal{D}} E(D)$ except for e and f an even number of cycles in $\mathcal{C}$ contains b because of $\sum_{D \in \mathcal{C}} D = C$ and because the only edges of C in F are e and f and note that, if some cycle $D \in \mathcal{D}$ contains an edge $b \in F$, then every cycle $D' \in \mathcal{C}$ containing b lies in $\mathcal{D}$. Thus,

$$\sum_{D \in \mathcal{D}} |E(D) \cap (F \setminus \{e, f\})|$$

is even. As e lies in $\bigcup_{D \in \mathcal{D}} (E(D) \cap F)$ but an odd number of times because it lies on $C = \sum_{D \in \mathcal{C}} D$, we conclude by parity that also f lies in that set. This proves the assertion. □

The cut space and the cycle space of G form a natural $\mathrm{Aut}(G)$-module, as G acts canonically on these spaces and these actions respect the vector space properties. We call such an $\mathrm{Aut}(G)$-module *finitely generated* if there are finitely many elements that, together with all its images under $\mathrm{Aut}(G)$, are a generating set for the vector space.

3. Proof of the main theorem

In this section, we shall prove Theorem 1.1. Let us give a brief outline of its proof. By the theorem of Dicks and Dunwoody, Theorem 2.1, we obtain a nested generating set of the cut space invariant under the automorphisms. We then show that the number of orbits in this generating set is bounded in terms of a finite generating set of the cycle space, more precisely, in terms of its size and the length of a largest of its cycles. We will do this by showing that, if there are too many generators for the cut space all of which are nested, then there is one that lies so closely to one of the other generators that they do not differ.

Proof of Theorem 1.1. Instead of dealing with the cuts directly, we consider the space $\mathcal{B}(G)$. Because the canonical correspondence between the elements of the cuts space and those of $\mathcal{B}(G)$ commutes with the automorphisms of G, it suffices to prove that $\mathcal{B}(G)$ is a finitely generated $\mathrm{Aut}(G)$-module.

Let $\mathcal{C}$ be a finite generating set of the cycle space of G as an $\mathrm{Aut}(G)$-module and set

$$\mathcal{D} := \bigcup_{\psi\in\mathrm{Aut}(G)} \mathcal{C}\psi.$$

Let k be the length of a largest cycle in $\mathcal{C}$. Due to Theorem 2.1, we find for every $n \in \mathbb{N}$ some $\mathrm{Aut}(G)$-invariant nested $\mathcal{E}_n$ all of whose elements are tight and such that $\mathcal{E}_n$ generates $\mathcal{B}_n(G)$ as a vector space. Furthermore, we may assume $\mathcal{E}_n \subseteq \mathcal{E}_{n+1}$ for all $n \in \mathbb{N}$. Since 2-edge-connectivity implies that every edge lies on some cycle, we may assume $\mathcal{E}_1 = \emptyset$.

For every non-trivial ordered bipartition (X, Y) of the vertex set of any cycle C, those $(A, B) \in \mathcal{E}_n$ with $X \subseteq A$ and $Y \subseteq B$ form a finite chain by Lemma 2.3 (2). If this chain is not empty, it has a smallest and a largest element. Thus, among the elements of $\mathcal{E}_n$ that induce a non-trivial ordered bipartition on $V(C)$, there are at most $2^{|V(C)|}$ many smallest and at most $2^{|V(C)|}$ many largest such elements.

Hence and by definition of k, if $\mathcal{E}_n$ contains more than $2^{1+k}|\mathcal{C}|$ orbits, then there must be one orbit such that none of its elements is smallest or largest for any bipartition of any $C \in \mathcal{D}$. Let (A, B) be an element of such an orbit and let $C \in \mathcal{D}$ such that (A, B) induces a non-trivial bipartition of $V(C)$. Lemma 2.3 (2) implies that the set of all $(A', B') \in \mathcal{E}_n$ that induce the same bipartition on $V(C)$ as (A, B), i. e. with $A \cap V(C) = A' \cap V(C)$, forms a finite chain. So we find a unique $(A', B') < (A, B)$ among them such that no other element of $\mathcal{E}_n$ lies between (A', B') and (A', B'), that is, (A', B') is the predecessor of (A, B) in this finite chain. Note that $E(A, B)$ and $E(A', B')$ coincide on C.

We shall show

$(*)\quad A = A'$ *and* $B = B'$

and thus obtain a contradiction to the choice of (A', B'), as it is strictly smaller than (A, B).

The first step to prove $(*)$ is to show that (A, B) and (A', B') coincide on every cycle in $\mathcal{D}$ that contains some edge of $E(C) \cap E(A, B)$. So let $C' \in \mathcal{D}$ such that C and C' share an edge $xy \in E(A, B)$. As $E(A, B)$ and $E(A', B')$ coincide on C, the edge xy lies also in $E(A', B')$ and hence both ordered bipartitions (A, B) and (A', B') induce non-trivial bipartitions on $V(C')$. Since any $(E, F) \in \mathcal{E}_n$ with $(A', B') < (E, F) < (A, B)$ induces the same bipartition on C like (A, B), we conclude by the choice of (A', B') that no such element of $\mathcal{E}_n$ exists. Once more, Lemma 2.3 (2) implies that the bipartitions (E, F) with $x \in E$ and $y \in F$ form a finite chain. Hence, the bipartition (A', B') is the unique predecessor of (A, B) in $\mathcal{E}_n$ in this chain. Since, by its choice, (A, B) is neither minimal nor maximal with respect to its induced bipartition on $V(C')$ and since the elements of $\mathcal{E}_n$ inducing the same bipartition on C' as (A, B) form a finite chain by Lemma 2.3 (2), which is a subset of the finite chain of bipartitions (E, F) with $x \in E$ and $y \in F$, the unique predecessor of this chain must be (A', B'). In particular, (A', B') induces the same bipartition of $V(C')$ as (A, B).

We have shown

(†) *If $C_1, C_2 \in \mathcal{D}$ share an edge of $E(A, B)$, if (A, B) and (A', B') coincide on C_1, and if (A', B') is maximal in $\mathcal{E}_n$ among those that are smaller than (A, B) and that coincide with (A, B) on C_1, then (A, B) and (A', B') coincide on C_2.*

The strategy to prove $(*)$ is to use (†) inductively along some suitable sequence of cycles obtained by Lemma 2.4. As (A, B) is tight, we find for any two edges in $E(A, B)$ some cycle that meets $E(A, B)$ in precisely these two edges, just by joining the end vertices of these edges in A and in B, respectively. Let $e, f \in E(A, B)$ such that e is an edge of C. So e also lies in $E(A', B')$. We shall show

(‡) *f lies in $E(A', B')$.*

As e and f lie on a cycle containing only these two edges of $E(A, B)$ and as $\mathcal{D}$ generates the cycles space of G, Lemma 2.4 implies the existence of a sequence $e_1 C_1 e_2 \dots e_n$ with $e_1 = e$ and $e_n = f$ such that every C_i lies in $\mathcal{D}$ and such that e_i and e_{i+1} lie on C_i. By adding eC at the beginning of this sequence, if necessary, we may assume that $C_1 = C$. Applying (†), we conclude that (A, B) and (A', B') coincide on C_2. Inductively, they coincide on every cycle C_i, in particular, they coincide on C_n, which contains f. Thus, we have shown (‡).

As f was an arbitrary edge of $E(A, B)$, every edge of $E(A, B)$ lies in $E(A', B')$. So we have $E(A, B) \subseteq E(A', B')$ and hence $(A, B) = (A', B')$ as (A, B) and (A', B') are tight. This proves $(*)$ and contradicts the choice of (A', B') being strictly smaller than (A, B). This contradiction shows that there are at most $2^{1+k}|\mathcal{C}|$ orbits in $\mathcal{E}_n$ and that $\mathcal{B}_n(G)$ is a finitely generated $\mathrm{Aut}(G)$-module.

As $\mathcal{E}_n$, for every $n \in \mathbb{N}$, has at most $2^{1+k}|\mathcal{C}|$ orbits, some $\mathcal{E}_i$ contains maximally many orbits. Since $\mathcal{E}_i \subseteq \mathcal{E}_j$ for every $j \geq i$, the orbits of $\mathcal{E}_i$ are also orbits of $\mathcal{E}_j$ and hence we have $\mathcal{E}_i = \mathcal{E}_j$ for all $j \geq i$. So $\bigcup_{k \in \mathbb{N}} \mathcal{E}_k = \mathcal{E}_i$ generates $\mathcal{B}(G)$ and thus $\mathcal{B}(G)$ is a finitely generated $\mathrm{Aut}(G)$-module. □

We extend the notion of accessibility to arbitrary graphs: a graph is *accessible* if there exists some $k \in \mathbb{N}$ such that any two edge ends can be separated by at most k edges.

Theorem 3.1. *Every graph G whose cycle space is a finitely generated* $\mathrm{Aut}(G)$*-module is accessible.*

Proof. Let H be a maximal 2-edge-connected subgraph of G. As no two cycles of H within the same $\mathrm{Aut}(G)$-orbit lie in different $\mathrm{Aut}(H)$-orbits, the cycle space of H is a finitely generated $\mathrm{Aut}(H)$-module. By Theorem 1.1, the space $\mathcal{B}(G)$ is finitely generated as $\mathrm{Aut}(G)$-module. As the cycle space of G is a finitely generated $\mathrm{Aut}(G)$-module, there are only finitely many $\mathrm{Aut}(G)$-orbits of maximal 2-edge-connected subgraphs of G. Let n be the largest size of any cut in some finite generating set $\mathcal{E}$ of all the maximal 2-edge-connected subgraphs of G. Then we have $\mathcal{B}(G) = \mathcal{B}_n(G)$.

Let R, R' be two rays of G that are not equivalent. If these rays do not have tails in the same maximal 2-edge-connected subgraph, then they are separated eventually by some edge. If they have tails in the same 2-edge-connected subgraph, then they are separated eventually by at most n edges by the choice of n. □

We call a graph *quasi-transitive* if its automorphism group has only finitely many orbits on the vertex set. As every locally finite quasi-transitive graph G has only finitely many $\mathrm{Aut}(G)$-orbits of cycles of length at most k, we obtain as a corollary of Theorem 3.1:

Corollary 3.2. *Every locally finite quasi-transitive graph whose cycle space is generated by cycles of bounded length is accessible.* □

We note that the assumption of local finiteness is necessary in Corollary 3.2, as we shall see in Example 3.3.

Example 3.3. We construct an infinite transitive graph all of whose vertices have infinite degree and whose cycle space is generated by triangles but that is not accessible, that is, we show that the assumption of local finiteness in Corollary 3.2 is necessary. For $i = 2, 3, \ldots$, let V_i be a set of i vertices. The graph H' has vertex set $\bigcup_{i\geq 2} V_i$ and two vertices $x \in V_i$ and $y \in V_j$ are adjacent if and only if $|i-j| \leq 1$. So V_i and V_{i+1} form a complete graph on $2i+1$ vertices.

Consider H' with infinitely many copies $H_2, H_3, \ldots$ of H'. For $i = 2, 3, \ldots$, we identify each vertex in V_i with its copy in H_i. The resulting graph H has its cycle space generated by its triangles and, for any $n \in \mathbb{N}$, there are two ends that cannot be separated by less than n edges, for example take the end given by H' and the end given by H_n (if $n > 1$). Note that H is 2-connected.

So all that remains to show is that there is a transitive graph that contains H such that no two ends of H belong to the same end of G. In order to do that, we glue together copies of H in a treelike way. To make this precise, take for every two vertices $x \neq y \in V(H)$ a copy H_{xy} of H and identify x with the copy of y. We continue with this in the new copies of H, recursively, so that in the end the following hold.

(i) Every block of G is isomorphic to H.
(ii) Every vertex separates G.
(iii) If $\mathcal{X}$ is the set of blocks that contain a vertex x, then there is a bijection $\varphi\colon V(H) \to \mathcal{X}$ such that, for every $y \in V(H)$, the vertex x is a copy of y in the block $\varphi(y)$.

The resulting graph G is transitive and, for every $n \in \mathbb{N}$, every block of G has two ends that are not separable by less than n edges. This finishes the example.

It is an easy exercise (cp. [5] for finite graphs) to show that the cycle space of a graph G is generated by all its *geodesic* cycles, i. e. cycles C with $d_C(x, y) = d_G(x, y)$ for all vertices x, y on C.

Corollary 3.4. *Every locally finite quasi-transitive inaccessible graph has geodesic cycles of unbounded length.* □

4. Applications

4.1. **Finitely presented groups.** Stallings [17] proved that every finitely generated group with more than one end splits as a non-trivial free product with amalgamation over a finite subgroup or as an HNN-extension over a finite subgroup. We can continue this splitting process if one of its factors also has more than one end. We call a group *accessible* if this splitting stops after finitely many steps. Wall [19] conjectured that every finitely generated group is accessible. Dunwoody proved that this is true if the group is also finitely presented [9] but that it is false without the additional assumption [10]. Here, we use our result to give a combinatorial proof of Dunwoody's accessibility theorem.

Let $G = \langle S \mid R \rangle$ be a finitely presented group. Then any word over S in G that represents 1, is a finite product of relators in R or its conjugates. Thus, in its Cayley graph Γ, every cycle and thus every element of the cycle space is the sum of the elements of the cycle space that are given by elements of R and its G-images. Hence, we conclude by Theorem 1.2 that Γ is accessible, which in turn is equivalent to G being accessible due to Thomassen and Woess [18, Theorem 1.1]. Note that Diekert and Weiß [4] offered a combinatorial proof of this equivalence (its original proof applied a result due to Dunwoody [8] that used some algebraic topology). So we have just proved combinatorially:

Theorem 4.1. [9, Theorem 5.1] *Every finitely presented group is accessible.* □

4.2. **Hyperbolic graphs.** For $\delta \in \mathbb{N}$, we call a graph G *δ-hyperbolic* if it is connected and if for any three vertices and any three shortest paths P_1, P_2, P_3, one between any two of the three vertices, every vertex on P_1 has distance at most δ to some vertex on P_2 or P_3. We call G *hyperbolic* if it is δ-hyperbolic for some $\delta \in \mathbb{N}$. A finitely generated group Γ is *hyperbolic* if one of its locally finite Cayley graphs is hyperbolic. As hyperbolic groups are finitely presented (see [14]), they are accessible due to Theorem 4.1. In this section we will prove the analogue result for quasi-transitive hyperbolic graphs.

It is not hard to show that accessibility of locally finite graphs is preserved under quasi-isometries. The same holds for hyperbolicity (see e. g. [14]). But quasi-transitive locally finite graphs need not be quasi-isometric to some locally finite Cayley graphs due to Eskin et al. [13]. Thus, we cannot obtain the accessibility of locally finite quasi-transitive hyperbolic graphs directly from the accessibility of finitely generated hyperbolic groups.

Lemma 4.2. *Let G be a δ-hyperbolic graph. Then the cycles of length less than $4\delta + 4$ generate its cycles space.*

Proof. Let us suppose that this is not the case. Then we take some cycle C that cannot be written as a sum of shorter cycles and whose length is at least $4\delta + 4$. The distance between any two vertices of C is realized on C as any shortcut leads to two cycles that are shorter than C but whose sum is C. We pick $x, y, z \in V(C)$ such that $d(x, y) = 2\delta + 2$ and such that z lies in the middle of the longer path between x and y on C, i. e. $|d(x, z) - d(y, z)| \leq 1$. We pick three shortest paths, one between any two of the three vertices, such that their union is C. So z does not lie on the chosen shortest path between x and y. Let u be the vertex on the shortest path between x and y that has distance $\delta + 1$ to x and to y. Then its distance to any vertex on the other two shortest paths is at least $\delta + 1$ as C has no shortcuts. This contradiction shows the assertion. □

Dunwoody [12] thought it likely that every transitive locally finite hyperbolic graph is accessible. As a direct consequence of Corollary 3.2 and Lemma 4.2, we can confirm this:

Theorem 4.3. *Every locally finite quasi-transitive hyperbolic graph is accessible.* □

We note that the graph given in Example 3.3 is hyperbolic. Thus, the assumption of local finiteness is necessary in Theorem 4.3.

4.3. **Planar graphs.** In [7], Droms showed that finitely generated planar groups are accessible and finitely presented. This is a hint that the same might be true for locally finite transitive planar graphs. Indeed, Dunwoody [11] showed that these graphs are accessible.

In [15], it is shown that the cycle space of a locally finite transitive planar graph G is a finitely generate $\mathrm{Aut}(G)$-module. Together with Theorem 3.1, we obtain a new proof of Dunwoody's result that locally finite transitive planar graphs are accessible.

4.4. **Compactly presented groups.** Another application of our main theorem lies in the area of locally compact groups, in particular in the area of compactly presented groups. A locally compact group G is *compactly presented* if it has a presentation $\langle S \mid R \rangle$ such that S is a compact generating set and all relators in R have bounded length. We briefly sketch the proof of Cornulier [1, Section 4.H] that compactly presented groups are accessible and show where our main theorem can be used.

Let $G = \langle S \mid R \rangle$ be a finitely presented group. If the component G_0 of G that contains the neutral element of G is not compact, then G has at most two ends and is accessible, see [1, Lemma 4.B.1 and Corollary 4.D.2]. If G_0 is compact, then G admits a continuous proper cocompact combinatorial action on a locally finite simply connected simplicial 2-complex X, see [1, Proposition 4.H.3], and thus also on its 1-skeleton, which is a graph Γ whose cycle space is generated by boundaries of 2-simplices, i. e. whose cycle space is generated by triangles. Since G is accessible if and only if Γ is accessible [1, Theorem 4.F.1], Theorem 1.2 implies the following theorem of Cornulier.

Theorem 4.4. [1, Theorem 4.H.1] *Every compactly presented locally compact group is accessible.* □

A locally compact group is *hyperbolic* if it is compactly generated and any Cayley graph with respect to a compact generating set is hyperbolic. As hyperbolic locally compact groups are compactly presented, see [2, Proposition 8.A.25], they are accessible [1, Corollary 4.H.2].

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References

[1] Y. Cornulier, *On the quasi-isometric classification of locally compact groups*, arXiv:1212.2229, 2015.

[2] Y. de Cornulier and P. de la Harpe, *Metric geometry of locally compact groups*, arXiv:1403.3796, 2015.
[3] W. Dicks and M.J. Dunwoody, *Groups Acting on Graphs*, Cambridge Stud. Adv. Math., vol. 17, Cambridge Univ. Press, 1989.
[4] V. Diekert and A. Weiß, *Context-free groups and their structure trees*, Internat. J. Algebra Comput. **23** (2013), no. 3, 611–642.
[5] R. Diestel, *Graph Theory* (4th edition), Springer-Verlag, 2010.
[6] ______, *Personal communication*, 2010.
[7] C. Droms, *Infinite-ended groups with planar Cayley graphs*, J. Group Theory **9** (2006), 487–496.
[8] M.J. Dunwoody, *Accessibility and groups of cohomological dimension one*, Proc. London Math. Soc. (3) **38** (1979), no. 2, 193–215.
[9] ______, *The accessibility of finitely presented groups*, Invent. Math. **81** (1985), no. 3, 449–457.
[10] ______, *An inaccessible group*, Geometric Group Theory (G.A. Niblo and M.A. Roller, eds.), L.M.S. Lecture Note Ser., vol. 181, Cambridge University Press, 1992, pp. 75–78.
[11] ______, *Planar graphs and covers*, preprint, 2007.
[12] ______, *An Inaccessible Graph*, Random walks, boundaries and spectra, Progr. Probab., vol. 64, Birkhäuser/Springer Basel AG, 2011, pp. 1–14.
[13] A. Eskin, D. Fisher, and K. Whyte, *Coarse differentiation of quasi-isometries I: Spaces not quasi-isometric to Cayley graphs*, Ann. of Math. (2) **176** (2012), no. 1, 221–260.
[14] E. Ghys and P. de la Harpe, *Sur les groupes hyperboliques, d'après M. Gromov*, Progress in Math., vol. 83, Birkhäuser, Boston, 1990.
[15] M. Hamann, *Planar transitive graphs*, submitted, arXiv:1511.08777.
[16] L. Mosher, M. Sageev, and K. Whyte, *Quasi-actions on trees I. Bounded valence*, Ann. of Math. **158** (2003), 115–164.
[17] J.R. Stallings, *Group theory and three dimensional manifolds*, Yale Math. Monographs, vol. 4, Yale Univ. Press, New Haven, CN, 1971.
[18] C. Thomassen and W. Woess, *Vertex-transitive graphs and accessibility*, J. Combin. Theory (Series B) **58** (1993), no. 2, 248–268.
[19] C.T.C. Wall, *Pairs of relative cohomological dimension one*, J. Pure Appl. Algebra **1** (1971), 141–154.

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