Lecture notes

Geometric Group Theory

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These are the notes of my class *Geometric Group Theory*, taught in Winter 2022/23 at Hamburg University. I am greatful for letting me know of any typos or errors in these notes. You can send them via email: `matthias.hamann@math.uni-hamburg.de`.

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Chapter 0

Introduction

Groups play a major role in many (if not all) mathematical subjects. Mostly, they occur as automorphisms groups but sometimes, e. g. in Galois theory they occur directly. The intention of this class is to understand the groups themselves better. But beforehand let us discuss the following question.

What is geometric group theory?

Generally speaking, geometric group theory considers groups as geometric objects and tries to relate their geometric and algebraic properties. Sometimes, instead of looking at the geometric properties of groups, we use their actions on other geometric objects to obtain results for the groups.

For example the statement 'Subgroups of free groups are free.' is purely algebraic while an elegant proof uses a geometric characterisation of free groups via their action on trees.

Our most important objects will be *Cayley graphs*: for every group and each of its generating sets we can construct a Cayley graph. It will be important for us that the structure of different Cayley graphs for the same finitely generated group but for different finite generating sets will change the geometry of the Cayley graphs only locally: they are quasi-isometric to each other. This implies that every geometric property that is invariant under quasi-isometries is true for one of these Cayley graphs if and only if it is true for all of them. Thereby, we can view this property as a property of the group.

This way we can talk about *ends* or *growth* of groups. As an example between the geometric and algebraic properties of groups we will prove Stallings' theorem. It says that a finitely generately generated group has more than one end if and only if it is one of two well described group products.

Chapter 1

Basics

Remark. A **group** is a pair $(G, \cdot)$ consisting of a set G and a binary function $\cdot\colon G \times G \to G$ satisfying the ollowing properties.

- associative: $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ for all $f, g, h \in G$;
- neutral element: there exists $e \in G$ with $e \cdot g = g = g \cdot e$ for all $g \in G$;
- inverse elements: for every $g \in G$ there exists $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$.

Usually, we omit the function $\cdot$ and write gh instead of $g \cdot h$.

1.1 Group actions

In this section, we will make the following sentence precise from the groups theoretic point of view: 'A group acts on a mathematical object like automorphisms.'

Definition. A group G **acts (from the right)** on a set X if there is a function $\bullet\colon X \times G \to X$ such that

(1) $x \bullet 1 = x$ for all $x \in X$ and

(2) $(x \bullet g) \bullet h = x \bullet (gh)$ for all $g, h \in G$ and $x \in X$.

We call the function the **(right) action** of G on X.

Analogously, G **acts (from the left)** on X if there is a function $\bullet\colon G \times X \to X$ such that

(1′) $1 \bullet x = x$ for all $x \in X$ and

(2′) $g \bullet (h \bullet x) = (gh) \bullet x$ for all $g, h \in G$ and $x \in X$.

We call the function the **(left) action** of G on X.

Comment. Usually, we will omit the function $\bullet$ for group actions.

Let us look at some examples for group actions.

Example 1.1.1. Let G be a group and $U \leq G$ a subgroup.

(1) G acts from the right (left) via multiplication from the right (left) on itself.

(2) G acts on itself via **conjugation**, i. e. $x \bullet g := x^g := g^{-1}xg$.

(3) G acts from the right via multiplication from the right at the set of right cosets of U, i.e. at the set $\{Ug \mid g \in G\}$ where $Ug := \{ug \mid u \in U\}$.

(3′) G acts from the left via multiplication from the left at the set of left cosets of U, i.e. at the set $\{gU \mid g \in G\}$ where $gU := \{gu \mid u \in U\}$.

(4) Let F be a field and V a F-vector space. Then the multiplicative group $(F^*, \cdot)$ of F acts on V via Scalar multiplication.

Definition. Let a group G act on a set X. It acts **faithfully** if for all $g \in G$ with $g \neq 1$ there exists $x \in X$ such that $xg \neq x$.

Example 1.1.2 (continuation of Example 1.1.1)**.**

(1) Multiplication (from the left and from the right) are faithful actions.

(2) Conjugation is a faithful action if and only if the *center* $C(G) := \{g \in G \mid gh = hg \ \forall h \in G\}$ of G is trivial.

(3) Multiplication on the sets of cosets is not faithful. (Example?)

(4) Scalar multiplication on non-trivial vector spaces is a faithful action.

Remark. Usually, we consider left actions and then omit ‘left’. Contrary, when we use a right action, we shall explicitly state that.

Lemma 1.1.3. *Let G be a group and X be a set. Then G acts (non-trivially) on X if and only if there is a (non-trivial) group homomorphism $G \to S_X$.*[1]

Additionally, G acts faithfully on X if and only if this group homomorphism is injective.

Proof. First, let G act non-trivially on X. For every $g \in G$, set $\varphi_g \colon X \to X$, $x \mapsto gx$. Let $g \in G$. Because of $x = 1x = gg^{-1}x$ for every $x \in X$, we have $\varphi_g\varphi_{g^{-1}} = id_X$ and hence $\varphi_g \in S_X$. This Permutation must be non-trivial as the Operation is non-trivial. Furthermore, since $(\varphi_g\varphi_h)(x) = \varphi_g(\varphi_h(x)) = ghx = \varphi_{gh}(x)$ holds for all $g, h \in G$ and $x \in X$, we obtain the homomorphism property of the map $\varphi\colon G \to S_X,\ g \mapsto \varphi_g$. If the action is faithful, then there exists for every $g \in G$ an $x \in X$ with $gx \neq x$ and thus we have $\varphi_g(x) \neq \varphi_1(x)$. So φ is injective.

[1] Reminder: S_X is the symmetric group on X.

Now let $\varphi\colon G \to S_X$ be a non-trivial group homomorphism. For every $g \in G$ we set $gx := \varphi(g)(x)$. Then we have $1x = \varphi(1)(x) = id(x) = x$ and

$$(gh)x = \varphi(gh)(x) = (\varphi(g)\varphi(h))(x) = \varphi(g)(\varphi(h)(x)) = g(hx)$$

for all $g, h \in G$ and $x \in X$. Hence, this defines an action of G on X that is non-trivial since there exists $g \in G$ with $\varphi(g) \neq id$, so there exists $x \in X$ with $\varphi(g)(x) \neq id(x) = x$. If φ is injective, then there is no $g \in G$ such that $\varphi(g) = id$. Hence, there exists for every $g \in G$ some $x \in X$ with $gx = \varphi(g)(x) \neq x$ and thus the action is faithful. □

We can already obtain as corollary from our results above an important theorem (of Cayley). If states that – in order to understand all groups, it suffices to understand the subgroups of all symmetric groups. Unfortunately, it is false if we believe that this makes everything easier.

Theorem 1.1.4 (Theorem of Cayley)**.** *Every group is isomorphic to a subgroup of some symmetric group.*

Proof. According to Example 1.1.2 (1), the group G acts faithfully on itself via multiplication. So Lemma 1.1.3 implies the existence of an injective group homomorphism $\varphi\colon G \to S_G$. We directly obtain $G \cong \varphi(G) \leq S_G$. □

In our next section (Section 1.2), we shall prove an even stronger version of Cayley's theorem, which says that the group G can be found as subgroup of the automorphism group of some connected directed graph.

Lemma 1.1.3 is a reason for us to look at actions on other mathematical objects, not only sets.

Definition. A group G **acts** on a mathematical object X (a graph, a vector space, etc.), if it acts on the underlying set of X and if every $g \in G$ does not only define an element of S_X according to Lemma 1.1.3 but also an automorphism of X.

Analogously to the definition of *faithful actions on sets* we call the action of G on X **faithful** if G acts faithfully on the underlying set of X.

Remark. According to Lemma 1.1.3, a group G acts (faithfully) on a mathematical object X if there exists a (injective) group homomorphism $G \to \mathrm{Aut}(X)$.

Example 1.1.5. The action in Example 1.1.1 (2) is a faithful action of G on the *group* G and in Example 1.1.1 (4) it is an action of F^* on the *vector space* V.

In the following we will use the sentence 'A group G acts on X.' interchangeably for 'A group G acts on a mathematical object X.'.

Definition. Let G be a group acting on X and let $x \in X$.

(1) The **stabiliser** of x in G is the set

$$G_x := \{g \in G \mid gx = x\}.$$

(2) The **orbit** of x under G is the set

$$Gx := \{gx \mid g \in G\}.$$

Remark 1.1.6. Let G be a group acting on X. Then all stabilisers of elements $x \in X$ are subgroups of G.

We obtain the following relation between stabilisers and orbits.

Theorem 1.1.7. *Let G be a group acting on X. Then for every $x \in X$ the map from Gx into the set of left cosets of G_x defined by $gx \mapsto gG_x$ is bijective.*

Proof. Let $g, h \in G$. Then the following equivalences hold.

$$\begin{aligned} & gx = hx \\ \Leftrightarrow\ & h^{-1}gx = x \\ \Leftrightarrow\ & h^{-1}g \in G_x \\ \Leftrightarrow\ & h^{-1}gG_x = G_x \\ \Leftrightarrow\ & gG_x = hG_x \end{aligned}$$

□

Definition and Remark 1.1.8. *Let G be a group and $U \leq G$ be a subgroup. The* **index** *of U in G is the number of left cosets of U in G (or equivalently the number of right cosets of U in G) and we denote it by $|G : U|$. It is easy to see that $|G| = |U| \cdot |G : U|$.*

Corollary 1.1.9. *Let G be a finite group acting on X. The we have for every $x \in X$:*

$$|G| = |G_x| \cdot |Gx|$$

Proof. We obtain

$$|G| = |G_x| \cdot |G : G_x| = |G_x| \cdot |Gx|$$

directly by Remark 1.1.8 and Theorem 1.1.7. □

Let us discuss another relation between stabilisers and orbits.

Lemma 1.1.10. *Let G be a group acting on X. Let $x, y \in X$ such that $gx = y$ for some $g \in G$. Then we have $(G_x)^g = G_y$ and $G_x = (G_y)^{g^{-1}}$.*

Proof. Let $g \in G$ such that $gx = y$ and let $h \in G_x$. Then we have

$$h^g y = g^{-1}hgy = g^{-1}hx = g^{-1}x = y.$$

So we get $h^g \in G_y$ and thus $G_x^g \subseteq G_y$. Using and analogue argument, we obtain $G_y^{g^{-1}} \subseteq G_x$ and hence $(G_x)^g = G_y$ and $G_x = (G_y)^{g^{-1}}$. □

Definition. Let G be a group acting on X. G **moves** $x \in X$ **freely** if $G_x = 1$. The action of G on X is **free** if G moves every $x \in X$ freely.

Comment. In this course we do not consider graphs as topological objects, in particular we do not consider them as CW-complexes. That is why we have to strengthen the previous definitions for graphs a bit.

Definition. Let G be a group acting on a graph $\Gamma = (V, E)$. The action is **free** on X if not only the action induced on V but also the action induced on E is free.

1.2 Cayley graphs

In this section, we introduce an important object on which groups acts in a canonical way and which we will use extensively: their Cayley graphs. Before we introduce them, we need some more definitions.

Definition. Let G be a group. A subset $S \subseteq G$ **generates** G if every elements of G can be written as a (finite!) product of elements in S or their inverses. The set S is called a **generating set** of G. If S is a generating set of G, then we write $G = \langle S \rangle$.

The group G is **finitely generated** if there is a finite subset of G generating G.

Example 1.2.1. Every symmetric group S_n for $n \in \mathbb{N}$ is generated by its transpositions.

Comment. Example 1.2.1 is wrong if we look at symmetric groups on infinitely many elements. (Why?)

Definition. A **directed graph** or **digraph** is a pair (V, E) with $E \subseteq V \times V$.

If we speak of paths, walks etc. in a digraph (V, E), then we always consider those in the **underlying undirected (multi)graphs** of (V, E) via the map $f \colon E \to [V]^2, (x, y) \mapsto \{x, y\}$.

Definition. Let G be a group that is generated by $S \subseteq G$. Then

$$\Gamma_{G,S} = (G, \{(g, gs) \mid g \in G, s \in S\})$$

defines a digraph, the **Cayley digraph** of G and S. Der underlying undirected graph without multiple edges and without loops is the **Cayley graph** of G and S. We also denote the Cayley graph by $\Gamma_{G,S}$. It will be clear from the context whether $\Gamma_{G,S}$ is directed or not.

Remark 1.2.2.

(i) The digraph $\Gamma_{G,S}$ has no loops if and only if $1 \notin S$.

(ii) The underlying undirected graphs of $\Gamma_{G,S}$ has at most double edges. It has them if and only if S contains s^{-1} for some $s \in S$. That latter holds in particular, if S contains an **involution**, i. e. an element of order 2.

Example 1.2.3. Let C_n be the cyclic group on n elements and let S be a generating set of C_n.

(1) If $S = C_n$, then the Cayley digraph is complete: for all $g \neq h \in C_n$ there is an edge (g, h) and an edge (h, g). Additionally, every loop (g, g) exists.

(2) If $|S| = 1$, the the Cayley digraph is a directed cycle on n vertices.

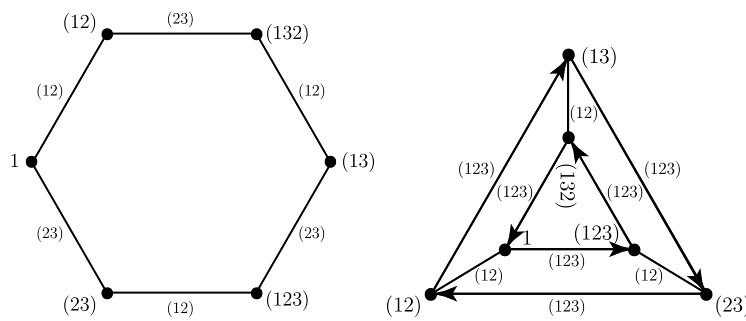

Figure 1.1: Two Cayley digraphs for the symmetric group S_3

Example 1.2.4. Let us consider two Cayley digraphs for the group S_3. Let $S = \{(12), (23)\}$ and $S' = \{(12), (123)\}$. Both Cayley digraphs can be found in Figure 1.1, where edges without directions are used to replace double edges with one possible orientation each. The edges are labelled with the elements of S or S' they originate from.

Theorem 1.2.5 (Cayley, strong version)**.** *For every group G there exists a connected graph on which G acts faithfully.*

If G is finitely generated, then we may choose the graph to be locally finite.[2]

Proof. Let S be a generating set of G and let $\Gamma_{G,S}$ be their Cayley graph. Then G acts faithfully on Γ via $g\colon G \to G, h \mapsto gh$. Note that an edge (h_1, h_2) is mapped onto an edge (gh_1, gh_2) and has $(g^{-1}h_1, g^{-1}h_2)$ as its preimage. The action is faithful by Example 1.1.2 (1).

If G is finitely generated, then we may choose S to be finite. Since every vertex g is adjacent to only the edges gs and gs^{-1} for all $s \in S$, every vertex of Γ has finite degree. □

Comment. To obtain a faithful right action, we can use in the definition of a Cayley graph the edge set $\{(g, sg) \mid g \in G, s \in S\}$. The reason, why we prefer edges (g, gs) is implied by the following remark.

Remark 1.2.6. For every walk $v_0e_0v_1 \ldots e_{k-1}v_k$ in a Cayley graph $\Gamma_{G,S}$ there is a sequence $s_0 \ldots s_{k-1}$ of elements elements of $S \cup S^{-1}$ (with $S^{-1} := \{s^{-1} \mid$

[2]A graph is **locally finite** if every vertex has finite degree.

$s \in S\}$) in the following way: $s_i = v_i^{-1} v_{i+1}$. That means that the edge e_i lies in the Cayley graph because of the generator s_i or s_i^{-1}. For the product of the s_i, we obtain $s_0 \cdots s_{k-1} = v_0^{-1} v_k$.

Using Lemma 1.1.3, we can formulate Theorem 1.2.5 analogously to Theorem 1.1.4 in the following way:

Theorem 1.2.7. *Every (finitely generated) group is a subgroup of the automorphism group of some connected (locally finite) graph.* □

For finitely generated groups, we can even strengthen this:

Theorem 1.2.8. *Every finitely generated group is isomorphic to the automorphism group of some graph.*

We may choose this graph to be connected and locally finite.

Proof. Let $S = \{s_1, \ldots, s_n\}$ be a finite generating set of the group G. Let Γ_G, S be the Cayley digraph of G and S. For every $i \in \{1, \ldots, n\}$ let T_i be a tree consisting of a path P_i of length 3 such that a path of length 1 starts at an inner vertex x_i of P_i and such that a path of length $i + 1$ starts at the other inner vertex y_i. Obviously, every automorphism of T_i that fixes the end vertices of P_i setwise must fix the whole tree pointwise. Note that all trees T_i are distinct. In $\Gamma_{G,S}$ we replace every directed edge from g to gs_i by T_i, where g is the end vertex of P_i that is adjacent to x_i and gs_i is the end vertex of P_i that is adjacent to y_i. Let Γ be the resulting graph. Obviously, Γ is connected and locally finite.

Let φ be an automorphism of Γ. Then φ must fix all vertices of Γ setwise that were not already in $\Gamma_{G,S}$ and it must fix setwise the vertices that were in $\Gamma_{G,S}$. Thus, φ induces a bijective map of the vertex set of $\Gamma_{G,S}$. Since the trees T_i are distinct, the tree that replaced a directed edge e of $\Gamma_{G,S}$ must be mapped onto a tree of the same kind. Thus, φ induces an automorphism $\overline{\varphi}$ of $\Gamma_{G,S}$. Let $g \in G$ such that $\overline{\varphi}(1) = g$. Since $\overline{\varphi}$ maps edges that belong to a generator s_i to edges that belong to s_i, too, and since it must keep their orientations, the neighbour s_i of 1 is mapped by $\overline{\varphi}$ to the neighbour gs_i of g. Inductively, g and $\overline{\varphi}$ coincide on $\Gamma_{G,S}$ and every other automorphism ψ of Γ that maps 1 to g must coincide with φ, too. We obtain an injective map Φ from the automorphism groups of Γ to G. Every $g \in G$ induces an automorphism φ_g of Γ with $\Phi(\varphi_g) = g$. So Φ is surjective. It is easy to verify that Φ is a group homomorphism. Thus, the automorphism group of Γ is isomorphic to G. □

Definition. Let G be a group acting on X. It acts **transitively** on X if for all $x, y \in X$ there exists $g \in G$ such that $gx = y$.

If $X = (V, E)$ is a (di-)graph, then G acts **(vertex-)transitively** on X (or **edge-transitively** on X), if the action induced on V (or on E) is transitive.

Remark 1.2.9. Every group acts transitively and free on each of its Cayley digraphs, since the left multiplication of a group on itself is transitive and free.

Proposition 1.2.10. *Let G be a group and S a generating set of G. The left multiplication on G induces a free action on the Cayley graph of G and S if and only if S contains no involution.*

Proof. Every $s \in S$ with $s^2 = 1$ fixes the edge $1s = ss^2$. Thus, the action cannot be free.

Conversely, let us assume that the action is not free. Obviously, the action induced on the vertices is free. Hence, the action induced on the edges is not free. So there exist $g \in G$ with $g \neq 1$ and an edge uv such that $g(uv) = uv$. Wlog let $s \in S$ such that $u = vs$. If $gu = u$, then we directly obtain $g = 1$. Hence, we have

$$v = gu = g(vs) = (gv)s = us = vs^2$$

and thus $s^2 = 1$. Since $s \neq 1$, it must be an involution. □

1.3 Sabidussi's theorem

In this section, we shall obtain a first result how to deduce information about a group by using its action on a graph. We will obtain a result that can be seen as a reverse to Cayley's theorem.

Definition. Let G be a group acting on a connected graph Γ. A **fundamental domain** of this action is a connected subgraph that contains exactly one element of each orbits on the vertices.

A priori it is not obvious that every action on a graph admits a fundamental domain. This is the content of the following theorem.

Theorem 1.3.1. *For every action of a group on some connected graph there exists a fundamental domain.*

Proof. Let G be a group acting on a connected graph $\Gamma = (V, E)$. We may assume that Γ has at least one vertex. Let $\mathcal{F}_G$ be the set of all connected subgraphs of Γ that contain at most one vertex of each orbit. Obviously, $\mathcal{F}_G$ is not empty (it contains the empty graphs and every subgraph on exactly one vertex) and every chain in $\mathcal{F}_G$ has an upper bound (the union of its elements). Zorn's lemma implies the existence of a maximal element F in $\mathcal{F}_G$. Let us show that F is a fundamental domain.

Let ussuppose that this is false. Then there exists a vertex $x \in V$ such that the orbit Gx contains no vertex from F. Let P be a path in Γ starting at x and ending at a vertex of F. Thenthere are two adjacent vertices u, v on P such that Gu contains no vertex of F but $Gv \cap V(F) \neq \emptyset$. Let $g \in G$ such that $gv \in V(F)$. Then gu lies in the same orbit as u; in particular is lies outside of F and $G(gu)$ contains no vertex of F. But gu has a neighbour gv in F. Thus, $F' = (V(F) \cup \{gu\}, E(F) \cup \{\{gv, gu\}\})$ is a connected subgraph of Γ that must lie in $\mathcal{F}_G$ by its construction. This contradicts the maximality of F. So F is a fundamental domain. □

Theorem 1.3.2. *Let F be a fundamental domain of the action of a group G on a connected graph Γ. Let S be the set of those $g \in G$ that satisfy*

$$V(gF) \cap (V(F) \cup N(V(F))) \neq \emptyset,$$

i. e. such that gF contain a vertex or a neighbour of a vertex of F. Then S is a generating set of G.

Proof. Let $g \in G$. We shall write g as a finite product of elements of $S \cup S^{-1}$. For this, let $v \in V(F)$ and let P be a v-gv path. Let $(F = g_0F, g_1F, \ldots, g_nF = gF)$ be a finite sequence of images of F under elements of G with the following properties.

(1) Every vertex of P lies in some g_iF.

(2) Either g_iF and $g_{i+1}F$ have a common vertex or g_iF has a vertex that has a neighbour in $g_{i+1}F$.

The existence of such a sequence can be seen as follows: For every vertex x_i on $P = x_1 \ldots x_m$ we choose some g_i such that $x_i \in V(g_iF)$. Then the claim follows for the sequence $(g_0F, g_1F, \ldots, g_mF, g_{m+1}F)$ with $g_0 = 1$ and $g_{m+1} = g$.

Let us show inductively that every g_i can be written a a product of elements of S. By the choice of the g_i, this holds trivially for g_0 and g_1. By (2), either the subgraphs $F = g_i^{-1}g_iF$ and $g_i^{-1}g_{i+1}F$ have a common vertex or some vertex in F has a neighbour in $g_i^{-1}g_{i+1}F$. In both cases we obtain by the definition of S that $g_i^{-1}g_{i+1}$ is an element of S. By induction g_{i+1} is an element of $\langle S \rangle$. We conclude $g = g_n \in \langle S \rangle$ and $G = \langle S \rangle$. □

Remark. The generating set obtained in Theorem 1.3.2 is usually not a minimal one (even if we ignore the neutral element) as the following example shows. Let Γ be the complete graph on three vertices and let G be its automorphism group. Then the fundamental domain is a single vertex and every automorphism of Γ has to be put into the generating set S. Thus, S contains the whole automorphism group G, which is isomorphic to the symmetric group S_3. Since there are minimal generating sets on two elements, S cannot be one of them.

As an application, we shall prove Sabidussi's theorem, which characterises Cayley graphs.

Theorem 1.3.3 (Sabidussi)**.** *A connected graph on which some group acts transitively and free is a Cayley graph.*

Proof. Let Γ be a connected graph and let G be a group that acts transitively and freely on Γ. Let $v \in V(\Gamma)$. Since G acts transitively on Γ, the graph $(\{v\}, \emptyset)$ is a fundamental domain.

Let $S \subseteq G$ be a minimal subset of G such that $S \cup S^{-1}$ is the generating set of Theorem 1.3.2. We want to show that Γ is the Cayley graph $\Gamma_{G,S}$ of G and S. For this, we define a map

$$\varphi \colon \Gamma_{G,S} \to \Gamma, \; g \mapsto gv.$$

Since the action on Γ is transitive, φ must be surjective and, since G acts freely, φ is injective, so it is bijective. It remains to show that φ preserves the adjacency relation. Let $\{u, w\} \subseteq V(\Gamma)$ a vertex set consisting of two distinct elements. As

G acts transitively on Γ, there exists $g \in G$ with $gu = v$. By considering $\{v, gw\}$ instead of $\{u, w\}$, we may assume that $u = v$. There exists $h \in G$ with $w = hv$. If $vw \in E(\Gamma)$, then $h \in S \cup S^{-1}$ by the choice of S and hence $\varphi(v)$ and $\varphi(w)$ are adjacent. If $vw \notin E(\Gamma)$, then $h \notin S \cup S^{-1}$ and $\varphi(v)$ and $\varphi(w)$ cannot be adjacent. Thus, φ is a graph isomorphism. $\square$

Chapter 2

Free groups

2.1 Free groups and trees

Definition. Let S be a set. A finite sequence of the form $w = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ with $s_i \in S$ and $\varepsilon_i \in \{\pm 1\}$ is a **word** over $S \cup S^{-1}$. We call $|w| := n$ the **lenth** of w. The word is **reduced** if there is no $i \leq n-1$ with $s_i^{\varepsilon_i} = s_{i+1}^{-\varepsilon_{i+1}}$. For $s, s_i \in S$, $\varepsilon, \varepsilon_i \in \{\pm 1\}$, we call a word $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ an **elementary reduction** of the word $s_1^{\varepsilon_1} \dots s_i^{\varepsilon_i} s^{\varepsilon} s^{-\varepsilon} s_{i+1}^{\varepsilon_{i+1}} \dots s_n^{\varepsilon_n}$. A word v is a **free reduction** of a word u if there is a finite sequence $u = w_1, \dots, w_n = v$ of words such that w_{i+1} is an elementary reduction of w_i and if v is reduced.

A group G is **free** with **free generating set** $S \subseteq G$ if $\langle S \rangle = G$ and there is no non-trivial reduced word w over $S \cup S^{-1}$ such that $w = 1$ in G. We call $|S|$ the **rank** of G.

If $w = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ and $v = t_1^{\varepsilon_1} \dots t_m^{\varepsilon_m}$ are words over $S \cup S^{-1}$, then we the word $wv = s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n} t_1^{\varepsilon_1} \dots t_m^{\varepsilon_m}$ is the **concatenation** of w and v.

Comment. In particular, no free generating set S contains 1 since the word 1 is distinct from the trivial word over S, which is the empty word.

Example 2.1.1. The additive group $\mathbb{Z}$ is a free group of rank 1.

Comment. A priori it is not obvious that the rank of a free group is well-defined. We shall prove that in Section 2.2.

First, we want to ensure that free groups exist.

Theorem 2.1.2. *Let S be a set. Then there exists a free group with S as free generating set.*

We will sketch the standard proof of Theorem 2.1.2 before proving a slightly stronger result that contains Theorem 2.1.2.

Sketch of the proof of Theorem 2.1.2. We will define a relation $\sim$ on the set of words over $S \cup S^{-1}$ via $v \sim w$ if and only if there is a sequence $v = w_1, \dots, w_n = w$ such that either w_i is an elementary reduction of w_{i+1} or vice versa. Obviously, this is equivalence relation. It can be proved tha every equivalence class of this relation contains exactly one reduced word. Then we can define a multiplication on this set in the following way: $[\alpha][\beta] := [\alpha\beta]$ for any two words α, β over $S \cup S^{-1}$, where $\alpha\beta$ is their concatenation. It can be shown that the set of equivalence classes with this multiplication forms a free group.

Strictly speaking, S is not a free generating set for F, since S is no subset of F. But since every $s \in S$ is a reduced wird, we can identify every s and $[s]$ to satisfy this formality. □

Comment. Since the equivalence classes of the equivalence relation in the sketch of the proof of Theorem 2.1.2 contain a unique recued word, it is possible (and also reasonable) to think of the elements of free groups as reduced words. Of course, one has to keep in mind that the product of two such elements is not simply their concatenation but the free reduction of that. Note that this concatenation is uniquely determined since every equivalence class of $\sim$ contains a unique reduced word.

Theorem 2.1.3. *Let S be a set. There exists a free group G with S as free generating set that acts transitively and free on a tree T.*

Proof. We define a graph T. Its vertex set V is the set of reduced words over $S \cup S^{-1}$ (including the empty word) and its edge set E is defined as follows: we add for every reduced word $s_1 \dots s_n$ with $s_i \in S \cup S^{-1}$ the edge $\{s_1 \dots s_n, s_1 \dots s_n s\}$ for all $s \in S$ with $s \neq s_n^{-1}$ and the edge $\{s_1 \dots s_n, s_1 \dots s_{n-1}\}$ (without multi-edges). To show that T is connected, it suffices to verify that every reduced word lies in the same component as the empty word[1]. Since the sequence $\emptyset, s_1, s_1 s_2, \dots, s_1 \dots s_n$ of vertices defines a path from the empty word to the word $s_1 \dots s_n$, the graph T is connected.

Let us suppose that T contains a cycle C. This cycle contains a vertex $u = u_1 \dots u_n$ whose word has maximum length for all vertices on C. By definition of the edges, the neighbours of u on C must have length $|u| - 1$ and both must be the word $u_1 \dots u_{n-1}$. But then, C was not a cycle. Thus, T is a tree.

For every $s \in S \cup S^{-1}$ we define a map $\varphi_s \colon V \to V$ such that

$$\varphi_s(s_1 \dots s_n) = \begin{cases} s_2 \dots s_n, & \text{if } s = s_1^{-1}, \\ s s_1 \dots s_n, & \text{if } s \neq s_1^{-1}. \end{cases}$$

Obviously, φ_s maps edges to edges and non-adjacent vertices to non-adjacent vertices, that is, it is an automorphism of T. Also, the equality $\varphi_s^{-1} = \varphi_{s^{-1}}$ is easily verifiable.

Let $\Phi_S = \{\varphi_s \mid s \in S\}$ and let G be the subgroup of $\mathrm{Aut}(T)$ that is generated by Φ_S. We will show that G is a free group that acts transitively and freely on T and that Φ_S generates G freely.

Let $\varphi_{s_1} \dots \varphi_{s_n}$ be a reduced word over $\Phi_S \cup \Phi_S^{-1}$. Then we have $s_i \neq s_{i+1}^{-1}$, since $\varphi_s^{-1} = \varphi_{s^{-1}}$ and since the word is reduced. So $s_1 \dots s_n$ is a reduced

[1] The empty word will be denoted by $\emptyset$.

word over $S \cup S^{-1}$ and we have $\varphi_{s_1} \dots \varphi_{s_n}(\emptyset) = s_1 \dots s_n \neq \emptyset$. We obtain $\varphi_{s_1} \dots \varphi_{s_n} \neq id$ and hence G is a free group freely generated by Φ_S.

Since the $s_1 \dots s_n$ is the image of the empty word under $\varphi_{s_1} \dots \varphi_{s_n}$, the action of G must be transitive. Let $v \in V$ and let $\varphi \in F$ such that $\varphi(v) = v$. Since G acts transitively on T, we may assume by Lemma 1.1.10 that v is the empty word. Let $\varphi_{s_1} \dots \varphi_{s_n}$ be the shortest word over $\Phi_S \cup \Phi_S^{-1}$ such that $\varphi_{s_1} \dots \varphi_{s_n} = \varphi$. In particular, we have $\varphi_{s_i}^{-1} \neq \varphi_{s_{i+1}}$ and $s_i^{-1} \neq s_{i+1}$ for all $i < n$. Thus, $s_1 \dots s_n$ is a reduced word. Hence, $\emptyset = \varphi(\emptyset) = s_1 \dots s_n$. Since $s_1 \dots s_n$ is reduced, we obtain $n = 0$ and $\varphi = id$. This implies that G acts freely on the vertices of T. It remains to show that G also acts freely on the edges of T. Let $e \in E$. Since G acts transitively on T, we may apply Lemma 1.1.10 once more to assume that $e = \{\emptyset, s\}$ for some $s \in S \cup S^{-1}$. Let us suppose that there exists $\varphi = \varphi_{s_1} \dots \varphi_{s_n}$ such that $\varphi(e) = e$ and $\varphi \neq id$, where the n is shortest possible. Since G acts freely on the vertices of T, we know that $\varphi(\emptyset) \neq \emptyset$. So we have $\varphi(\emptyset) = s$ and $\varphi(s) = \emptyset$. We also get $\varphi_s(\emptyset) = s$ and, since the action of G on T is free on the vertices of T, we conclude $\varphi = \varphi_s$. But we have $\varphi_s(s) = ss \neq \emptyset = \varphi(s)$. This contradiction shows that G acts freely on T.

Just like in the sketch of the proof of Theorem 2.1.2, we can use a formal trick to guarantee that G is generated by S instead of Φ_S. □

Before we take a closer look at the relation between trees and free groups, let us show an important characterisation of free groups.

Theorem 2.1.4 (Universal property)**.** *The following two statements are equivalent for every group F with subset $S \subseteq F$.*

(i) *F is a free group with free generating set S.*

(ii) *for every group G and every map $\varphi : S \to G$ there exists a uniquely determined group homomorphism $\overline{\varphi}\colon F \to G$ that extends φ.*

In the proof of this theorem, we consider the elements of the free group as being equivalence classes of words just as in the sketch of the proof of Theorem 2.1.2.

Proof of Theorem 2.1.4. First, let us assume that F is a free group and S a free generating set of F. Let G be another group and let $\varphi\colon S \to G$ be a map. We set $\overline{\varphi}(s) := \varphi(s)$ and $\overline{\varphi}(s^{-1}) := (\varphi(s))^{-1}$ and for every word $w = s_1 \dots s_n$ over $S \cup S^{-1}$ we set $\overline{\varphi}(w) := \overline{\varphi}(s_1) \dots \overline{\varphi}(s_n)$. By definition, $\overline{\varphi}$ is a group homomorphism as soon as we make sure that it is well-defined. If the word $s_1 \dots s_n$ is an elementary reduction of the word $s_1 \dots s_i t t^{-1} s_{i+1} \dots s_n$, the we have the following:

$$\begin{aligned}\overline{\varphi}(s_1 \dots s_n) &= \overline{\varphi}(s_1) \dots \overline{\varphi}(s_n) \\ &= \overline{\varphi}(s_1) \dots \overline{\varphi}(s_i)\overline{\varphi}(t)\overline{\varphi}(t^{-1})\overline{\varphi}(s_{i+1}) \dots \overline{\varphi}(s_n) \\ &= \overline{\varphi}(s_1 \dots s_i t t^{-1} s_{i+1} \dots s_n).\end{aligned}$$

Thus, $\overline{\varphi}$ is well-defined on the equivalence classes of words and hence induces a group homomorphism $F \to G$ that extends φ. Furthermore, every group homomorphism must have the two properties $\overline{\varphi}(s^{-1}) = (\varphi(s))^{-1}$ and $\overline{\varphi}(w) = \overline{\varphi}(s_1)\dots\overline{\varphi}(s_n)$. That is, why $\overline{\varphi}$ is uniquely determined.

Let us assume that (ii) holds. Let G be a free group with free generating set S and let $\varphi\colon F \to G$ a group homomorphism with $\varphi(s) = s$ for all $s \in S$. Let us suppose that F is not free. Then there exists a non-trivial reduced word $s_1 \dots s_n$ over $S \cup S^{-1}$ such that $s_1 \dots s_n = 1$. We obtain

$$1 = \varphi(s_1 \dots s_n) = \varphi(s_1)\dots\varphi(s_n) = s_1 \dots s_n.$$

Hence there is a non-trivial reduced word $w = \varphi(s_1)\dots\varphi(s_n)$ in G with $w = 1$, a contradiction to the definition of a free group. Thus, F is a free group with free generating set S. □

A direct consequence of the universal property of free groups is the following (even though we still do not know whether the rank of free groups is well-defined).

Corollary 2.1.5. *Every two free groups of the same rank are isomorphic.*

Proof. Let F, G be two free groups of the same rank. We may assume that both groups are freely generated by the same set S. By Theorem 2.1.4 there are two group homomorphisms $\varphi\colon F \to G$ and $\psi\colon G \to F$ such that $\varphi|_S = id$ and $\psi|_S = id$. Then we must have $\varphi(\psi(s)) = s$. Since F is generated by S, we have $\varphi\psi = id$ and thus φ and ψ are group homomorphisms that are inverse to each other. □

Free groups and trees have more connections that the one obtained in Theorem 2.1.3.

Lemma 2.1.6. *Every Cayley graph of a free group and one of its free generating sets is a tree.*

Proof. Let G be a free group with free generating set S and let Γ be the der Cayley graph of G and S. If Γ contains a cycle, the it also contains a cycle that contains the vertex 1 since G acts transitively on Γ by Remark 1.2.9. This cycle belongs to a closed walk starting and ending at 1. According to Remark 1.2.6, this walk corresponds to a word over $S \cup S^{-1}$. Since this word must be reduced, G cannot be free. Since every Cayley graph is connected, this contradiction shows that Γ is a tree. □

In general, the reverse statement of Lemma 2.1.6 does not hold as the following examples show.

Example 2.1.7. 1. The Cayley graph of the cyclic group $C_2 = \langle a \rangle$ with $\{a\}$ as generating set if a tree on two vertices.

2. The Cayley graph of the group $\mathbb{Z}$ with $S = \{1, -1\}$ as generating set is a tree, too.

Essentially, the problems highlighted in Example 2.1.7 are the only ones preventing a successful reverse statement of Lemma 2.1.6 as we will see in the following lemma.

Lemma 2.1.8. *Let G be a group and let S be a generating set of G that satisfies $st \neq 1$ for all $s, t \in S$. If the Cayley graph $\Gamma_{G,S}$ is a tree, then G is a free group and S a free generating set of G.*

Proof. Let F be a free group with free generating set S. We will show that F and G are isomorphic. According to Theorem 2.1.4, there is a group homomorphism $\varphi\colon F \to G$, whose restriction to S is the identity. This homomorphism is surjective since S generates G. In order to show that F and G are isomorphic, it suffices to verify that φ is injective. Let us suppose that there is a reduced word over $S \cup S^{-1}$ in $\ker(\varphi)$ that is not the empty word. Let $s_1 \dots s_n$ be such a word of minimum length. Because of $\varphi(s) = s \neq 1$ for all $s \in S$, we must have $n \geq 2$. If $n = 2$, then we have $1 = \varphi(s_1 s_2) = \varphi(s_1)\varphi(s_2) = s_1 s_2$. Since $s_1 s_2$ is reduced, this contradicts the assumption $st \neq 1$ for all $s, t \in S$. So we may assume $n \geq 3$. Due to minimality of n, the group elements $\varphi(s_1 \dots s_i)$ for all $0 \leq i \leq n$ are distinct since if there are $i < j < n$ with $s_1 \dots s_i = s_1 \dots s_j$, then $s_{i+1} \dots s_j$ is a word of shorter length over $S \cup S^{-1}$ with $\varphi(s_{i+1} \dots s_j) = 1$. Since the group elements $\varphi(s_1 \dots s_i)$ for all $0 \leq i \leq n$ are distinct, they induce a cycle in $\Gamma_{G,S}$. This contradiction to the assumption on $\Gamma_{G,S}$ shows that φ is injective. □

We can even use the action on trees to characterise free groups.

Theorem 2.1.9. *A group is free if and only if it acts freely on a tree.*

Proof. By Proposition 1.2.10, every free group acts free on any of its Cayley graphs. So Lemma 2.1.6 shows the first implication.

Let the group G act freely on the tree T. Let T' be a fundamental domain of this action, which exists by Theorem 1.3.1. Since G acts freely on T, there exists a unique $g \in G$ with $T' \cap gT' \neq \emptyset$, which is $g = 1$; this is true, since $gv \neq v$ for all $v \in V(T')$ and all $g \neq 1$ and since by definition of fundamental domains, $gv \in V(T')$ implies $gv = v$.

An edge is **essential** if exactly one of its incident vertices lies in T'. Since T' is a fundamental domain, there is for every essential edge w some $g_e \in G$ such that the vertex of e that does not lie in T' is contained in $g_e T'$. Set

$$\widetilde{S} := \{g_e \in G \mid e \text{ is essential}\}.$$

We shall prove that the set $\widetilde{S}$ has the following properties:

(i) $1 \notin \widetilde{S}$;

(ii) $\widetilde{S}$ contains no involution;

(iii) if e, e' are essential edges with $g_e = g_{e'}$, then $e = e'$;

(iv) for every $g \in \widetilde{S}$ we have $g^{-1} \in \widetilde{S}$;

(v) for every $g \in G$ with $V(gT') \cap (V(T') \cup N(V(T'))) \neq \emptyset$ we have $g = 1$ or $g \in \widetilde{S}$.

While (i) immediately follows from the definition of $\widetilde{S}$, we need small proofs for the remaining claims. Property (ii) is true, since every involution g_e maps the subtree g_eT' to T' and thus must fix the uniquely determined edge e in T between T' and g_eT', which contradicts the fact the action is free. Since T is a tree and thus contains a unique edge connecting the subtrees T' and g_eT' we obtain (iii). Since e connects the subtrees T' and g_eT', the edge $g_e^{-1}e$ connects the subtrees $g_e^{-1}T'$ and T' and we obtain (iv). Let $g \in G$ with $V(gF) \cap (V(F) \cup N(V(F))) \neq \emptyset$. If $g \neq 1$, then we already verified $gF \cap F = \emptyset$. Thus, gT' contains a vertex incident with an essential edge e, which must not be the in T' but in g_eT'. So we have $g_e^{-1}gT' \cap T' \neq \emptyset$. As we already verified above, we obtain $g_e^{-1}g = 1$ and thus $g = g_e$. This shows (v).

Let $S \subseteq \widetilde{S}$ be a minimal subset such that $S \cup S^{-1} = \widetilde{S}$. This is possible by (iv). By (ii) S and S^{-1} are disjoint. Theorem 1.3.2, and (v) imply that $\widetilde{S} \cup \{1\}$, and hence also S, generates G. It remains to show that S is a free generating set. For this, according to Lemma 2.1.8, it suffices to show that the Cayley graph $\Gamma_{G,S}$ is a tree. Let us suppose that $\Gamma_{G,S}$ is not a tree. Since it is connected, it contains a cycle $h_0 \dots h_n h_0$ for some $n \geq 2$. The edge $h_n h_0$ and all edges $h_i h_{i+1}$ correspond to some element $s_n := h_0 h_n^{-1}$ or $s_i := h_{i+1} h_i^{-1}$ from $S \cup S^{-1}$ and these in turn belong to unique essential edges e_i for all $1 \leq i \leq n$ by (iii).

For every $j < n$ the subtree s_jT' contains a vertex v_j incident with the edge e_j and a vertex w_j incident with the edge $s_j e_{j+1}$. Since T' is connected, there exists a path P_j from $h_j v_j$ to $h_j w_j$ in $h_j s_j T' = h_{j+1} T'$. Analogously, T' contains a path P_0 from v_0, the vertex in T' incident with e_n, to the vertex w_0 in T' incident with the edge e_1. Then

$$v_0 P_0 w_0 v_1 P_1 w_1 \dots v_n P_n w_n v_0$$

is a cycle in T. This contradiction together with Lemma 2.1.8 shows that G is a free group. □

We obtain the following corollary from the proof of Theorem 2.1.9.

Corollary 2.1.10. *Let T' be a fundamental domain of a free action of a free group G on a tree T. Then there is a free generating set X of G such that the set S defined in Theorem 1.3.2 satisfies the following:*

$$S = X \cup X^{-1} \cup \{1\}. \qquad \square$$

As corollary of Theorem 2.1.9, we obtain a central result on free groups, more specifially on their subgroups.

Corollary 2.1.11 (Nielsen-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. Let H be a subgroup of a free group G. Then G acts free on a tree T by Theorem 2.1.9. As a subgroup of G, also H acts freely on T and thus is a free group by Theorem 2.1.9. □

We want to finish this section with a lemma that guarantees the existence of free subgroups in arbitrary groups under certain conditions.

Lemma 2.1.12 (Ping-Pong-Lemma)**.** *Let G be a group acting on X. Let $(A_i)_{i\in I}, (B_i)_{i\in I}$ with $|I| \geq 2$ be two families of non-empty subsets of X such that all A_i and B_j are pairwise disjoint. If there are $g_i \in G$ such that $X \setminus B_i \subseteq g_i A_i$ for all $i \in I$, then $\langle g_i \mid i \in I\rangle$ is a free subgroup of G.*

Proof. From $X \setminus B_i \subseteq g_i A_i$ we obtain $X \setminus g_i^{-1} B_i \subseteq A_i$ and hence $X \setminus A_i \subseteq g_i^{-1} B_i$ and $g_i(X \setminus A_i) \subseteq B_i$.

Let $s_n \dots s_1$ be a word over $S \cup S^{-1}$ for $S := \{g_i \mid i \in I\}$. Let $i, j \in I$ with $s_1 \in \{g_j, g_j^{-1}\}$ and $s_n \in \{g_i, g_i^{-1}\}$. If $i = j$, then let $k \in I \setminus \{i\}$; otherwise set $k := j$. Let $\varepsilon \in \{1, -1\}$ with $s_1 = g_j^{\varepsilon}$. If $k \neq j$, then pick $x \in A_k \cup B_k$. If $k = j$ and $\varepsilon = 1$, then pick $x \in B_j$. If $k = j$ and $\varepsilon = -1$, then pick $x \in A_j$. Using induction on ℓ, we obtain $s_\ell \dots s_1 x \in B_m$, if $s_\ell = g_m$ for some $m \in I$, or $s_\ell \dots s_1 x \in A_m$, if $s_\ell = g_m^{-1}$ for some $m \in I$. Thus, $s_n \dots s_1 x$ lies in either A_i or B_i and in particular it does not lie in the set $A_k \cup B_k$, which contains x. Thus, the element $s_n \dots s_1$ of G is distinct from 1. So $\langle S\rangle$ is free and freely generated by S. □

2.2 The rank of free groups

In this section, we will show that the rank of free groups is well-defined.

Theorem 2.2.1. *Every two generating sets of a free group have the same cardinality.*

Proof. Let G be a free group. If S and S' are infinite free generating sets of G, then we must have $|S| = |G| = |S'|$.

Let S be a finite free generating set of G. Every homomorphism $\varphi\colon G \to C_2$ is uniquely determined by the restriction of φ to the set S. Also, every map $S \to C_2$ can be extended to a homomorphism. Thus, there are $2^{|S|}$ homomorphisms from G to C_2. Since this number only depends on G and not on the particular generating set, we have $2^{|S|} = 2^{|S'|}$ for every generating set S' of G. So S and S' have the same number of elements. □

Theorem 2.2.1 implies that the rank of free groups is well-defined. One might assume that the ranks of subgroups of a free group G (which are free groups themselves by Corollary 2.1.11) are bounded by the rank of G. This however is far from being true as our next result shows.

Proposition 2.2.2. *Let G be a free group of rank $n \in \mathbb{N}$ and let H be a subgroup of G of index $k \in \mathbb{N}$. Then H is a free group of rank $k(n-1)+1$.*

Proof. Let T be the Cayley graph of G and a free finite generating set S. By Lemma 2.1.6, we know that T is a tree. Since G acts freely on its Cayley graph, H acts freely on T, too. Since H has finite index in G and since G acts transitively on T, there are at most $|G : H|$ orbits of the action of H on T. Thus, every fundamental domain T' of the action of H on T, which exists by Theorem 1.3.1, is finite and has the size $k = |G : H|$. Since S is finite, T is locally finite and hence the free generating set X of H defined in Corollary 2.1.10 is finite. It remains to show that the size of X is $k(n-1)+1$.

The sum of all degrees in T of all vertices T' is $2n|T'| = 2nk$, since T is a $2n$-regular tree. The subtree T' contains $|T'|-1 = k-1$ edges, so there are $2kn - 2(k-1) = 2(k(n-1)+1)$ edges with on of its incident vertices in T' and the other outside of T'. □

We can apply the previous result to arbitrary finitely generated groups to get informations about some of their subgroups.

Corollary 2.2.3. *Let G be a finitely generated group. Every subgroup of G of finite index is finitely generated.*

Proof. Let $H \leq G$ be a subgroup of G such that $|G : H| \in \mathbb{N}$. Let S be a finite generating set of G and let F be a free group with free generating set S. Then there is a surjective homomorphism $\varphi\colon F \to G$ such that $\varphi|_S = id$. Let H' be the preimage of H under φ. We shall show that $|F : H'| \leq |G : H|$ for the index of H' in F. For this, let $g, h \in F$ with $gH' \neq hH'$. Then we have $h^{-1}g \notin H'$. So we have $\varphi(h^{-1})\varphi(g) = \varphi(h^{-1}g) \notin H$ und hence $\varphi(g)H \neq \varphi(h)H$. Thus, distinct cosets of H' will be mapped by φ to distinct cosets of H. Hence, we have $|F : H'| \leq |G : H|$. By Proposition 2.2.2, the group H' is finitely generated. Since $\varphi|_{H'}$ maps H' to H surjectively and since this map is defined by its definition on a generating set of H' by Theorem 2.1.4, the image of this generating set must generate H. Thus, H is finitely generated. □

2.3 Group presentations

A corollary of Theorems 2.1.2 and 2.1.4 is the following.

Corollary 2.3.1. *Every group is the image of some free group.* □

This is the reason for us to define presentations of groups.

Definition. Let G be a group that is the image of a free group F under some homomorphism φ. Let S be a free generating set of F. A word w over $S \cup S^{-1}$ with $\varphi(w) = 1$ is a **relator**. A subset $R \subseteq \ker(\varphi)$ is a **set of defining relators** if $\langle R\rangle^{\trianglelefteq} = \ker(\varphi)$, where $\langle R\rangle^{\trianglelefteq}$ is the smallest normal subgroup of F that contains R.[2] If $uv \in \ker(\varphi)$, then we call $\varphi(uv) = 1$ a **relation**. A set of relations is a **set of defining relations** if the corresponding relators form a set of defining relators.

[2]Reminder: (1) A subgroup U is **normal** if $U^g = U$ for all $g \in G$. (2) Kernels of homomorphisms are normal subgroups.

Remark. The smallest normal subgroup that contains the set R in a group G must contain R^{-1} and all $r^g = g^{-1}rg$ for all $r \in R$ and $g \in G$. The finite products of elements of $R \cup R^{-1}$ and $R^g \cup (R^{-1})^g$ already form a normal subgroup. This must be $\langle R \rangle^{\trianglelefteq}$.

Definition. Let S be a set and let R be a subset of the free group F that is freely generated by S. Then we call $\langle S \mid R \rangle$ a **presentation** of a group G if $G \cong F/\langle R \rangle^{\trianglelefteq}$ and we write $G = \langle S \mid R \rangle$. Alternatively, R could be a set of defining relations, as well. Then we call $\langle S \mid R \rangle$ a **presentation** of G if $\langle S \mid R' \rangle$ is a presentation of G, where R' is the set of those relators that belong to R.

We call $\langle D \mid R \rangle$ a **finite presentation** if S and R are finite or, if we emphasis the group, we call it **finitely presented** if S and R are finite.

Example 2.3.2. (1) A free group F with free generating set S has the presentation $\langle S \mid \emptyset \rangle$.

(2) Finite cyclic groups C_n have a presentation $\langle g \mid g^n \rangle$.

Theorem 2.3.3. *Let S be a set and let R a set of words over $S \cup S^{-1}$. Then there is a group with presentation $\langle S \mid R \rangle$.*

Proof. Let F be a group with free generating set S. Then the group $F/\langle R \rangle^{\trianglelefteq}$ has $\langle S \mid R \rangle$ as a presentation. □

Similar to free groups, also groups with presentations have a universal property.

Theorem 2.3.4 (Universal property)**.** *Let $G = \langle S \mid R \rangle$ and let F be a free group with free generating set S. Let H be a group and let $\varphi\colon F \to H$ be a group homomorphism. If $\varphi(r) = 1$ for all $r \in R$, then there is a unique group homomorphism $\psi\colon G \to H$ with $\varphi(s) = \psi(s)$ for all $s \in S$.*

Proof. Let us define a map $\psi\colon G \to H$ in that we set $\psi(s) := \varphi(s)$ and $\psi(s^{-1}) := (\varphi(s))^{-1}$ for all $s \in S$ and $\psi(s_1 \ldots s_n) := \varphi(s_1) \ldots \varphi(s_n)$ for all $s_1, \ldots, s_n \in S \cup S^{-1}$. Then ψ is uniquely determined by the equalities $\varphi(s) = \psi(s)$ and it remains to show that ψ is a group homomorphism. The homomorphism properties directly follow from the definition of ψ. So we just have to show that ψ is well-defined. By assumption, we have $\langle R \rangle \leq \ker(\varphi)$. Since $\ker(\varphi)$ is a normal subgroup, we also obtain $\langle R \rangle^{\trianglelefteq} \leq \ker(\varphi)$. Thus, ψ is well-defined. □

The proof showing the free group of fixed rank are uniquely determined up to isomorphisms (Corollary 2.1.5) carries over almost verbatim to our situation here and we obtain the following.

Corollary 2.3.5. *Every two groups with the same presentation are isomorphic.* □

2.4 Tietze transformations

In this section, we are interested in the relations between different presentations of the same group. For this, we define four ways how to obtain new presentations out of old ones without changing the group.

Definition. Let $G = \langle S \mid R \rangle$. **Tietze transformations** are the following four possible modifications of the presentation $\langle S \mid R \rangle$:

(1) For $R' \subseteq \langle R \rangle^{\trianglelefteq}$, we can **add redundant relators**

$$\langle S \mid R \rangle \longrightarrow \langle S \mid R \cup R' \rangle.$$

(2) For $R' \subseteq R$ with $\langle R \rangle^{\trianglelefteq} = \langle R' \rangle^{\trianglelefteq}$, we can **remove redundant relators**

$$\langle S \mid R \rangle \longrightarrow \langle S \mid R' \rangle.$$

(3) For a set S' with $S \cap S' = \emptyset$ and a set $\{w_s \mid s \in S'\}$ of words over $S \cup S^{-1}$, we can **add redundant generators**

$$\langle S \mid R \rangle \longrightarrow \langle S \cup S' \mid R \cup \{s^{-1} w_s \mid s \in S'\} \rangle.$$

(4) If $S = S_1 \dot\cup S_2$ and $R = R' \dot\cup \{s^{-1} w_s \mid s \in S_2\}$, where R' is a set of relators over S_1 and $\{w_s \mid s \in S_2\}$ is a set of words over $S_1 \cup S_1^{-1}$, we can **remove redundant generators**

$$\langle S \mid R \rangle \longrightarrow \langle S_1 \mid R' \rangle.$$

We obtain the following directly from the definition.

Remark 2.4.1. If $\langle S' \mid R' \rangle$ can be obtained from $\langle S \mid R \rangle$ using Tietze transformations, then the two groups are isomorphic.

If we consider the reverse direction of Remark 2.4.1, then it is not immediately obvious that distinct presentation of the same group can be transformed into each other using Tietze transformations. But that this holds nonetheless, we will prove in the next theorem.

Theorem 2.4.2. *Two presentation define isomorphic groups if and only if there is a finite sequence of Tietze transformations that transforms one into the other.*

Comment. In the literature, sometimes Tietze transformations are defined by adding or removing only one generator or relator. Then the finiteness condition in Theorem 2.4.2 has to be dropped. Instead, you will find the following additional statement: *If both presentations are finite, then the sequence can be chosen to be finite, too.*

Proof of Theorem 2.4.2. If a presentation is obtained from another presentation by finitely many Tietze transformations, the Remark 2.4.1 implies that both groups are isomorphic.

For the reverse direction, let $G_1 := \langle S_1 \mid R_2\rangle$ and $G_2 := \langle S_2 \mid R_2\rangle$ be presentations of isomorphic groups and let $\varphi\colon G_1 \to G_2$ be an isomorphism. We may assume that S_1 and S_2 are disjoint. For $s \in S_1$ let w_s be a word over $S_2 \cup S_2^{-1}$ such that $\varphi(s) = w_s$ and for $s \in S_2$ let w_s be a wordover $S_1 \cup S_1^{-1}$ such that $\varphi^{-1}(s) = w_s$. Let $i \neq j \in \{1, 2\}$. Wie consider the following Tietze transformations:

$$\begin{aligned}\langle S_i \mid R_i\rangle &\longrightarrow \langle S_1 \cup S_2 \mid R_i \cup \{s^{-1}w_s \mid s \in S_j\}\rangle \\ &\longrightarrow \langle S_1 \cup S_2 \mid R_i \cup \{s^{-1}w_s \mid s \in S_j\} \cup \{s^{-1}w_s \mid s \in S_i\} \cup R_j\rangle\end{aligned}$$

Thus, we can transform both groups using Tietze transformations into a third group. Since Tietze transformations are closed under reverting a transformation, we can transform $\langle S_1 \mid R_1\rangle$ into $\langle S_2 \mid R_2\rangle$ by a finite sequence of Tietze transformations. □

We are interested in if we can transform an arbitrary presentation of a finitely presented group into a finite presentation and if so how we can do it. First, we deal with the generating set.

Theorem 2.4.3. *Let $G = \langle S \mid R\rangle$ be a finitely generated group. Then there is a finite subset S' of S and a set R' of relators over $S' \cup S'^{-1}$ such that $G \cong \langle S' \mid R'\rangle$.*

Proof. Let X be a finite generating set of G. Then there exists for every $x \in X$ a word w over $S \cup S^{-1}$ such that $w = x$. Thus, it suffices to take some finite subset S' of S to write every $x \in X$ as word over $S' \cup S'^{-1}$. For every $s \in S \setminus S'$ we may choose words v_s, w_s over $S' \cup S'^{-1}$ such that $s = v_s$ and $s^{-1} = w_s$ and such that the free reduction of $v_s w_s$ is the empty word. We replace in every word in R each subword s by v_s and each subword s^{-1} by w_s, then we obtain a set R' of relators such that $\langle S \mid R\rangle = \langle S' \mid R'\rangle$. □

Theorem 2.4.4. *Let $G = \langle S \mid R\rangle$ be a finitely presented group and let S be finite. Then there exists a finite subset R' of R such that G is isomorphic to $\langle S \mid R'\rangle$.*

Proof. Let $\langle X \mid Q\rangle$ be a finite presentation of G. For $s \in S$ let w_s be a word over $X \cup X^{-1}$ such that $s = w_s$ and for $x \in X \cup X^{-1}$ let v_x be a word over $S \cup S^{-1}$ such that $x = v_x$. Using Tietze transformations, we can modify the presentations as follows.

$$\begin{aligned}\langle X \mid Q\rangle &\longrightarrow \langle S \cup X \mid Q \cup \{s^{-1}w_s \mid s \in S\}\rangle \\ &\longrightarrow \langle S \cup X \mid Q \cup \{s^{-1}w_s \mid s \in S\} \cup \{x^{-1}v_x \mid x \in X\}\rangle.\end{aligned}$$

Additionally, we can apply two Tietze transformations to replace the set Q by a set $Q[S]$ that was obtained as follows: for every $q \in Q$ we replace every

$x \in X \cup X^{-1}$ in q by v_x. Analogously, let w'_s be obtained from w_s by replacing every $x \in X \cup X^{-1}$ by v_x for every $s \in S$.

$$\begin{aligned} &\langle S \cup X \mid Q \cup \{s^{-1}w_s \mid s \in S\} \cup \{x^{-1}v_x \mid x \in X\}\rangle \\ \longrightarrow\ &\langle S \cup X \mid Q[S] \cup \{s^{-1}w'_s \mid s \in S\} \cup \{x^{-1}v_x \mid x \in X\}\rangle. \end{aligned}$$

We remark that $Q[S]$ is a finite set since Q is finite. Now, some generators are obsolete and we remove them.

$$\begin{aligned} &\langle S \cup X \mid Q[S] \cup \{s^{-1}w'_s \mid s \in S\} \cup \{x^{-1}v_x \mid x \in X\}\rangle \\ \longrightarrow\ &\langle S \mid Q[S] \cup \{s^{-1}w'_s \mid s \in S\}\rangle. \end{aligned}$$

Since $Q[S]$ and S are finite sets, the presentation $\langle S \mid Q[S] \cup \{s^{-1}w'_s \mid s \in S\}\rangle$ is a finite presentation of G. Since each of those finitely many relators in the set

$$Q[S] \cup \{s^{-1}w'_s \mid s \in S\}$$

lies in $\langle R\rangle^{\trianglelefteq}$, we find a finite subset R' of R such that $G = \langle S \mid R'\rangle$. □

Remark 2.4.5. In an exercise we shall see that, generally, for presentations $\langle S \mid R\rangle$ of some finitely presentable group G it is not possible to find finite subsets $S' \subseteq S$ and $R' \subseteq R$ such that $G = \langle S' \mid R'\rangle$.

2.5 Group products

In this section, we will discuss several possibilities how to obtain new groups from old ones. Most of the time, these will be products; just the 'HNN extension' has a different role.

Definition. Let $(G_i)_{i\in I}$ be a family of groups. The **direct product** $\prod_{i\in I} G_i$ of the G_i is defined on the cartesian product of the G_i where multiplication is given componentwise $(g_i)_{i\in I} \cdot (h_i)_{i\in I} := (g_ih_i)_{i\in I}$.

Example 2.5.1. (1) $\mathbb{Z}^n$ with componentwise addition is the direct product of n copies of $\mathbb{Z}$.

(2) If $m, n \in \mathbb{N}$ are coprime, then $C_m \times C_n = C_{mn}$.

2.5.1 Free products (with amalgamation)

Definition. Let $(G_i)_{i\in I}$ be a family of disjoint groups with $G_i = \langle S_i \mid R_i\rangle$. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. Then the group

$$\left\langle \bigcup_{i\in I} S_i \mid \bigcup_{i\in I} R_i \cup \bigcup_{i\neq j\in I} \left\{(\iota_i(a))^{-1}(\iota_j(a)) \mid a \in A\right\} \right\rangle$$

is the **free product of the $(G_i)_{i\in I}$ with amalgamation over A** and we write $G = *_{A,i\in I}G_i$. If $A = 1$, then we call the product simply the **free product** and write $G = *_{i\in I}G_i$.

If the groups G_i are not disjoint, we can make them disjoint artificially, e. g. by identifying every $g \in G_i$ with (g, i). Thereby, we can define free product of groups that need not be disjoint families $(G_i)_{i\in I}$.

Theorem 2.3.3 implies the existence of free products with amalgamation immediately.

Theorem 2.5.2. *Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. Then the free product of amalgamation $*_{A,i\in I}G_i$ exists.* □

Example 2.5.3. Let F be a free group with free generating set S. Let $\mathcal{S}$ be a partition of S. For every $X \in \mathcal{S}$ let F_X be a free group with free generating set X. Then $F \cong *_{X\in\mathcal{S}}F_X$.

Definition. Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. For every $i \in I$ let X_i be a **transversal** of $\iota_i(A)$ in G_i, i. e. a subset of G_i that contain exactly one element of each right coset of $\iota_i(A)$ in G_i, where 1 is the element in X for the left coset $\iota_i(A)$. A **reduced form** is a finite sequence $g_1 \dots g_n$ with $g_j \in \bigcup_{i\in I} G_i \setminus \{1\}$ such that $g_j \in G_i$ implies $g_{j+1} \notin G_i$. A **normal form** over $(G_i)_{i\in I}$ and A is a finite sequence $ag_1 \dots g_n$ with $a \in A$ and $g_j \in \bigcup_{i\in I} X_i \setminus \{1\}$ such that $g_j \in X_i$ implies $g_{j+1} \notin X_i$. We call n the **length** of the reduced form or the normal form. A (reduced form or) normal form is trivial if $n = 0$ and $a = 1$.

Remark 2.5.4. Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. If $A = 1$, then G_i is a transversal of $\iota_i(A)$ in G_i and thus, for free products, a reduced form is always a normal form. That is why we will use both notions interchangeably.

Theorem 2.5.5. *Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. Let X_i be transversals of $\iota_i(A)$ in G_i. Then every $g \in *_{A,i\in I}G_i$ has a unique normal form over $(G_i)_{i\in I}$ and A.*

In particular, there exists no non-trivial normal form for 1.

Proof. First, we show the existence of a normal form and then its uniqueness. Let $g = s_1 \dots s_n$ with $s_j \in \bigcup_{i\in I} S_i$ for all $1 \leq j \leq n$. If there exists $i \in I$ with $s_1, \dots, s_n \in S_i$, then there exists $x \in X_i$ such that the coset $\iota_i(A)x$ in G_i contains g. There exists $a \in A$ with $g = \iota_i(a)x$ and then ax is a normal form over g. For general g, we apply induction on the number of subwords of $s_1 \dots s_n$ that lie in some common S_i. Let $s_j \dots s_n$ be such that all $s_j, \dots, s_n$ lie in a common S_i but such that s_{j-1} does not lie in S_i. As we already saw, there exists $a \in A$ and $x_1 \in X_i$ such that $\iota_i(a)x_1 = s_j \dots s_n$. Let i' the index such that $s_{j-1} \in S_{i'}$. Because of $\iota_i(a) = \iota_{i'}(a)$, there exists $s'_j, \dots, s'_k \in S_{i'}$ such that $\iota_{i'}(a) = s'_j \dots s'_k$. By induction, $s_1 \dots s_{j-1}s'_j \dots s'_k$ has a normal form $bx_\ell \dots x_2$.

If $x_2 \notin S_i$, then $bx_\ell \dots x_1$ is a normal form of g. Otherwise, $\iota_m(b)x_\ell \dots x_1$, where $m \in I$ with $x_\ell \in X_m$, has fewer maximal subwords in some common S_p for some $p \in I$ and we can apply induction directly to obtain a normal form $b'y_{\ell'} \dots y_1$ of $\iota_m(b)x_\ell \dots x_1$, which is also a normal form of g.

To show uniqueness of the normal form, we will apply an argument that is similar to the on we used for the existence of free groups in Theorem 2.1.3. Let Ω be the set of normal form over $(G_i)_{i\in I}$ and A. For $g \in \bigcup_{i\in I} G_i$, let $\varphi_g \colon \Omega \to \Omega$ such that

$$ag_1 \dots g_n \mapsto \begin{cases} bxg_n \dots g_1, & \text{if } g_n \notin G_i, \\ b'g_n'g_{n-1} \dots g_1, & \text{if } g_n \in G_i \text{ and } g_n \neq x^{-1}, \\ bg_{n-1} \dots g_1, & \text{if } g_n \in G_i \text{ and } g_n = x^{-1}, \end{cases}$$

where $g \in G_i$ and $\iota_i(b)x = g\iota_i(a)$ such that $b \in A$ and $x \in X_i$ or in the second case $b' \in A$ and $g_n' \in G_i$ such that $\iota_i(b')g_n' = g\iota_i(a)g_n$. It is easy to see that φ_g and $\varphi_{g^{-1}}$ are inverse functions. So both of them lie in S_Ω. We consider the subgroup $H = \langle \varphi_g \mid g \in \bigcup_{i\in I} G_i \rangle$ of S_Ω. Note that each G_i acts on Ω and for every $i \neq j$ the maps $\varphi_{\iota_i(a)}$ and $\varphi_{\iota_j(a)}$ coincide. So we can extend the canonical map $\bigcup_{i\in I} S_i \to H,\ g \mapsto \varphi_g$ to a homomorphism $*_{A,i\in I} G_i \to H$ by Theorem 2.3.4 (universal property for group presentations). This implies that for every $g \in G$ its image φ_g is unique determined. If $cx_1 \dots x_k$ is a normal form of g, then $\varphi_g(1) = \varphi_c\varphi_{x_1} \dots \varphi_{x_k}(1) = cx_1 \dots x_k$. If $c'y_1 \dots y_\ell$ is a different normal form of g, then we have

$$c'y_1 \dots y_\ell = \varphi_{c'}\varphi_{y_1} \dots \varphi_{y_\ell}(1) = \varphi_g(1) = \varphi_c\varphi_{x_1} \dots \varphi_{x_k}(1) = cx_1 \dots x_k.$$

Since $c'y_1 \dots y_\ell$ and $cx_1 \dots x_k$ are the same element in Ω, we must have $c = c'$, $k = \ell$ and $x_i = y_i$ for all $1 \leq i \leq k$. This shows the uniqueness of the normal form. □

For free products with amalgamation over a non-trivial group, the reduced forms need not be unique. But for free products, this still holds, as we mentioned in Remark 2.5.4. Thus, we obtain the following corollary.

Corollary 2.5.6. *Let $(G_i)_{i\in I}$ be a family of groups. For every $g \in *_{i\in I} G_i$ there exists a unique reduced form over $(G_i)_{i\in I}$.* □

As another corollary of Theorem 2.5.5, we obtain the existence of monomorphisms $\psi_i \colon G_i \to *_{A,i\in I} G_i$.

Corollary 2.5.7. *Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i \colon A \to G_i$ be a monomorphism. Then there exist canonical monomorphisms $\psi_i \colon G_i \to *_{A,i\in I} G_i$.*

Proof. Obviously, there are canonical homomorphisms $\varphi_i \colon G_i \to *_{A,i\in I} G_i$. Let X_i be a transversal of $\iota_i(A)$ in G_i. Since there exists for every $g \in G_i$ exactly one $a \in A$ and $x \in X_i$ with $\iota_i(a)x = g$ and since ax is a non-trivial normal form of $\varphi_i(g)$, we obtain $\varphi_i(g) \neq 1$. Thus, φ_i is injective. □

We obtain additional properties for the free product with amalgamation directly from Theorem 2.3.4, the universal property for group presentations and Corollary 2.3.5.

Theorem 2.5.8 (universal property)**.** *Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism and let $\psi_i\colon G_i \to *_{A,i\in I}G_i$ be the canonical monomorphisms. Let G be a group and let $\varphi_i\colon G_i \to G$ for all $i \in I$ be homomorphisms such that $\varphi_i\iota_i = \varphi_j\iota_j$ for all $i, j \in I$. Then there exists exactly one homomorphism $\varphi\colon *_{A,i\in I}G_i \to G$ such that $\varphi\psi_i = \varphi_i$ for all $i \in I$.* □

Corollary 2.5.9. *Let $(G_i)_{i\in I}$ be a family of groups. Let A be a group and, for every $i \in I$, let $\iota_i\colon A \to G_i$ be a monomorphism. Then $*_{A,i\in I}G_i$ is uniquely determined up to isomorphisms.* □

Definition. Let $(G_i)_{i\in I}$ be a family of groups. A reduced form $g_1 \dots g_n$ is **cyclically reduced** if $n = 1$ or if g_1 and g_n do not lie in the same G_i.

Lemma 2.5.10. *Let $(G_i)_{i\in I}$ be a family of group.*

(1) *Every element of $*_{i\in I}G_i$ is conjugated to a cyclically reduced form.*

(2) *If $g = g_1 \dots g_n$ and $h = h_1 \dots h_m$ are two cyclically reduced forms such that g and h are conjugated in $*_{i\in I}G_i$, then $m = n$ and each reduced form is a cyclic permutation of the other.*

Proof. Statement (1) follows directly by iterated conjugations with g_i^{-1} as long as necessary. This process terminates since the length of the reduced form gets strictly smaller for each conjugation.

Let $f \in *_{i\in I}G_i$ with $g = h^f$ and let $f_1 \dots f_k$ be a normal form of f. If $k = 0$, then (2) is a consequence of Corollary 2.5.6. Since f and h are in reduced form and g is in cyclically reduced form, and thus in normal form, and since

$$g_1 \dots g_n = f_k^{-1} \dots f_1^{-1} h_1 \dots h_m f_1 \dots f_k,$$

Corollary 2.5.6 implies that $f_k^{-1} \dots f_1^{-1} h_1 \dots h_m f_1 \dots f_k$ is not a normal form. So either f_1 and h_1 or f_1 and h_m lie in the same factor G_i, which contains neither f_2 nor h_2 nor h_{m-1}. Then we must have $f_1 = h_1$ or $h_m = f_1^{-1}$: otherwise we obtain a contradiction to the uniqueness of reduced form for the two cases $k = 1$ and $k \neq 1$. Thus, we have

$$g_1 \dots g_n = f_k^{-1} \dots f_2^{-1} h_2 \dots h_m h_1 f_2 \dots f_k$$

or

$$g_1 \dots g_n = f_k^{-1} \dots f_2^{-1} h_m h_1 \dots h_{m-1} f_2 \dots f_k$$

and by induction, we have $m = n$ and the two reduced forms $g_1 \dots g_n$ and $h_1 \dots h_n$ are cyclic permutations of each other. □

Definition. An element of a group is a **torsion element** if it has finite order. A group is **torsion free** if its only torsion element is 1.

We will prove two results on torsion elements or their absence in (sub-)groups of free products.

Theorem 2.5.11. *Let $(G_i)_{i\in I}$ be a family of groups. Every torsion element of $*_{i\in I}G_i$ is conjugated to a torsion element of one of the G_i.*

Proof. Let $g \in *_{i\in I}G_i$ be conjugated to an element h with cyclically reduced form $h = h_1 \dots h_n$. The element h exists by Lemma 2.5.10 (1). It suffices to prove $n = 1$. If $n > 1$, then $h_1 \dots h_n \dots h_1 \dots h_n$ is the normal form of h^k. It is distinct from 1 and thus h and g have infinite order. □

Theorem 2.5.12. *Let G and H be finite groups. Then every torsion free subgroup of $G * H$ is a free group.*

Proof. First, we will construct a tree that admits an action of $G * H$. Let T be the graph with vertex set

$$V(T) = \{gG, gH \mid g \in G * H\},$$

i. e., the vertices are the cosets of G and H. The edge set of T is

$$E(T) = \{\{gG, gH\} \mid g \in G * H\}.$$

To prove that T is connected, it suffices to find a path from G to gG or gH for every $g \in G$. Let $g_1 \dots g_n$ be a normal form of g. We may assume that $g_1 \in H$. Then

$$G, H = g_1H, g_1G = g_1g_2G, g_1g_2H = g_1g_2g_3H, \dots, (g_1 \dots g_n)G$$

is a path that starts at G and ends at $(g_1 \dots g_n)G = gG$ (or at $(g_1 \dots g_n)H = gH$). Thus, T is connected. Every path from G to gG defines a sequence $h_1 \dots h_m$ with $h_m \notin G$ such that two consecutive h_i, h_{i+1} are not both in G or not both in H. Thus, $h_1 \dots h_m$ is a normal form of an element of gG and there exists $h_{m+1} \in G$ with $h_1 \dots h_{m+1} = g$. The uniqueness of the normal form of g (Theorem 2.5.5) implies that the path from G to gG in T is uniquely determined. Thus, T is a tree. By Example 1.1.1 (3′) the group $G * H$ acts on T by multiplication.

Let us show that every vertex and every edge has a finite stabiliser in $G * H$. First, we have a look at the vertices. Since $G * H$ acts transitive on the cosets of G as well as on the cosets of H, it suffices to that the stabilisers of G and of H are finite by Lemma 1.1.10. The stabiliser of G in $G * H$ is G, since $gG = G$ holds if and only if $g \in G$. Thus, it is finite. Analogously, the stabiliser of H is finite. By the definition of the edges, we directly get that $G * H$ acts transitively on the edges. Thus, by Lemma 1.1.10, it suffices to show that the stabiliser of $\{G, H\}$ is finite. Theorem 2.5.5 implies that neither $gG = H$ nor $gH = G$ holds for any $g \in G * H$. Thus, the stabiliser of $\{G, H\}$ is a subgroup of G and of H; in particular, it must be finite. Thus, all stabilisers of vertices and edges are finite.

Let F be a torsion free subgroup of $G * H$. Then F acts on T and this action must be free, since the elements in the stabilisers of vertices or edges have finite order and there are none of such elements in F. Theorem 2.1.9 implies that F is a free group. □

Remark. Theorem 2.5.12 also holds for free products of any finite number of groups, also with amalgamation. But in that proof, the construction of the tree has to be altered a bit. (How?)

2.5.2 HNN extensions

In this section, we will define an extension of groups that is not a product. The idea of this extension is to realise an isomorphism between subgroups as conjugation in the larger group.

Definition. Let $G = \langle S \mid R \rangle$ be a group and let $A, B \leq G$. Let $\varphi\colon A \to B$ be an isomorphism. Then the group $G*_\varphi$ with presentation

$$\langle S \cup \{t\} \mid R \cup \{a^t = \varphi(a) \mid a \in A\}\rangle$$

is the **HNN extension**[3] of G.

Remark 2.5.13. In view of Theorems 2.3.3 and 2.3.4 and Corollary 2.3.5, we obtain the existence of HNN extensions, their universal property and their uniqueness (up to isomorphisms).

Next, we will define a normal form for HNN extension similar to the normal form for free products with amalgamations.

Definition. Let $G*_\varphi$ with $\varphi\colon A \to B$ be an HNN extension of G. Let X be a transversal of A and let Y be a transversal of B in G. A **reduced form** is a finite word $g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n$ with $n \geq 0$ and $\varepsilon_i = \pm 1$ such that no subword $t^{-1} g_i t$ with $g_i \in A$ and no subword $t g_i t^{-1}$ with $g_i \in B$ exists. A **normal form** over G and t is a finite word $g_0 t^{\varepsilon_1} g_1 \ldots t^{\varepsilon_n} g_n$ with $n \geq 0$ and $\varepsilon_i = \pm 1$ such that $g_0 \in G$ and such that the following hold.

(i) If $\varepsilon_i = 1$, then $g_i \in Y$.

(ii) If $\varepsilon_i = -1$, then $g_i \in X$.

(iii) If $g_i = 1$ for some $i > 0$, then $\varepsilon_i \neq -\varepsilon_{i+1}$.

We call n the **length** of the reduced form or normal form. A reduced form or normal form is trivial if $n = 1$ and $g_0 = 1$.

We will show that every element of $G*_\varphi$ has a unique normal form. This will allow us to prove that we can embed G into $G*_\varphi$ canonically.

[3]HNN stands for the authors of the article in which this extension was treated in depth first: Graham Higman, Bernhard H. Neumann and Hanna Neumann.

Theorem 2.5.14. *Let $G*_\varphi = \langle S \cup \{t\} \mid R \cup \{a^t = \varphi(a) \mid a \in A\}\rangle$ be an HNN extension of the group $G = \langle S \mid R\rangle$ with isomorphism $\varphi\colon A \to B$. Then every $g \in G*_\varphi$ has a unique normal form.*

Proof. Let X be a transversal of A and let Y be a transversal of B in G. We divide the proof into two parts: the existence and the uniqueness of the normal form. First, we will show the existence of a normal form for each $g \in G*_\varphi$. We can write g as product of the generators of $G*_\varphi$. So there exists $g_i \in G$ and $\varepsilon_i = \pm 1$ such that $g = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$. We may assume that $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$ is a reduced form, since we can replace every a^t or $b^{t^{-1}}$ for $a \in A$, $b \in B$ by $\varphi(a)$ or $\varphi^{-1}(b)$, respectively. Let us first consider the case $\varepsilon_n = -1$. Let $h_n \in X$ and $a \in A$ with $ah_n = g_n$. Then there exists $b \in B$ with $b = a^t$. Thus, we have $t^{-1}ah_n = t^{-1}att^{-1}h_n = bt^{-1}h_n$ and set $g'_{n-1} := g_{n-1}b$. The case $\varepsilon_n = 1$ is analogous. By induction on n, we know that $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_{n-1}} g'_{n-1}$ has a normal form $h_0 t^{\varepsilon_1} \dots h_m$. Thus, g has the normal form $h_0 t^{\varepsilon_1} \dots h_m t^{\varepsilon_{n-1}} g'_{n-1}$; note that the case $h_m = 1$ and $\varepsilon_m = -\varepsilon_{n-1}$ cannot happen: if $h_m = 1$, then $g'_{n-1} \in B$ (if $\varepsilon_n = -1$) or $g'_{n-1} \in A$ (if $\varepsilon_n = 1$) by induction and hence we have $g_{n-1} \in B$ or $g_{n-1} \in A$, which contradicts that $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$ is a reduced form. We note that we have $m = n-1$ since the number of t or t^{-1} does not change during the induction.

Let us now show the uniqueness of the normal form. For this, we apply the same method as in the proofs of Theorems 2.1.3 and 2.5.5. Let Ω be the set of normal forms over G and t. We define an action of $G*_\varphi$ on Ω. For $g \in G$ we define the map $\varphi_g\colon \Omega \to \Omega$,

$$g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \mapsto (gg_0) t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n,$$

for t we define the map $\varphi_t\colon \Omega \to \Omega$,

$$g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \mapsto \begin{cases} ag_1 t^{\varepsilon_2} g_2 \dots t^{\varepsilon_n} g_n, & \text{if } y = 1 \text{ and } \varepsilon_1 = -1, \\ atyt^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n, & \text{if } y \neq 1 \text{ and } \varepsilon_1 = -1, \\ atyt^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n, & \text{if } \varepsilon_1 = 1, \end{cases}$$

where $g_0 = by$ with $b \in B$, $y \in Y$ and $a^t = b$, and for t^{-1} we define the map $\varphi_{t^{-1}}\colon \Omega \to \Omega$,

$$g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n \mapsto \begin{cases} bg_1 t^{\varepsilon_2} g_2 \dots t^{\varepsilon_n} g_n, & \text{if } x = 1 \text{ and } \varepsilon_1 = 1, \\ bt^{-1}xt^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n, & \text{if } x \neq 1 \text{ and } \varepsilon_1 = 1, \\ bt^{-1}xt^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n, & \text{if } \varepsilon_1 = -1, \end{cases}$$

where $g_0 = ax$ with $a \in A$, $x \in X$ and $b^{t^{-1}} = a$. Obviously, all φ_g are elements of S_Ω, since φ_g and $\varphi_{g^{-1}}$ are maps that are inverse to each other. Also, it is easy to see that φ_t and $\varphi_{t^{-1}}$ are inverse to each other, so they lie in S_Ω, too. We consider the subgroup $H = \langle \varphi_g \mid g \in G \cup \{t\}\rangle$ of S_Ω. We note that the image $\varphi_g \in S_\Omega$ is defined for every $g \in G$ and that $\varphi_b = \varphi_{t^{-1}} \varphi_a \varphi_t$ holds for all $a \in A$ and for $b = \varphi(a)$. As in the proof of Theorem 2.5.5, we can extend the canonical map $S \cup \{t\} \to H, g \mapsto \varphi_g$ via the universal property for presentations of groups (Theorem 2.3.4) to a homomorphism $G*_\varphi \to H$. Thus, for every $g \in G*_\varphi$ its image $\varphi_g \in H$ is uniquely determined. If $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$ is a normal form of g, then

$$\varphi_g(1) = \varphi_{g_0}\varphi_{t^{\varepsilon_1}}\varphi_{g_1}\dots\varphi_{t^{\varepsilon_n}}\varphi_{g_n}(1) = g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n.$$

If $h_0 t^{\delta_1} h_1 \dots t^{\delta_m} h_m$ is another normal form of g, then

$$\begin{aligned} & h_0 t^{\delta_1} h_1 \dots t^{\delta_m} h_m \\ =\; & \varphi_{h_0}\varphi_{t^{\delta_1}}\varphi_{h_1}\dots\varphi_{t^{\delta_m}}\varphi_{h_m}(1) \\ =\; & \varphi_g(1) \\ =\; & \varphi_{g_0}\varphi_{t^{\varepsilon_1}}\varphi_{g_1}\dots\varphi_{t^{\varepsilon_n}}\varphi_{g_n}(1) \\ =\; & g_1 t^{\varepsilon_1} g_2 \dots t^{\varepsilon_n} g_n. \end{aligned}$$

The two elements $h_0 t^{\delta_1} h_1 \dots t^{\delta_m} h_m$ and $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$ must be the same element in Ω and thus the same normal form. So we have $m = n$, $g_i = h_i$ and $\varepsilon_i = \delta_i$ for all $0 \le i \le n$. This shows the uniqueness of the normal form. □

Corollary 2.5.15. *Let $G*_\varphi$ be an HNN extension for a group G and an isomorphism $\varphi\colon A \to B$. The the following statements hold.*

(1) *The canonical map $\psi\colon G \to G*_\varphi$ is a monomorphism and t generates an infinite subgroup.*

(2) (Britton's lemma) *Let $w := g_1 t^{\varepsilon_1} g_2 \dots t^{\varepsilon_{n-1}} g_n$ be a reduced form over G and t. If $n > 1$, then $w \neq 1$.*

Proof. For every $g \in G$, the word g is a normal form. If $g \in \ker(\psi)$, then $g = 1$ in $G*_\varphi$. So 1 would have two distinct normal forms: 1 and g, which contradicts Theorem 2.5.14. For $n \in \mathbb{N}$, every t^n or t^{-n} has the normal form $1t1t\dots1t1$ or $1t^{-1}1t^{-1}\dots1t^{-1}1$, respectively. Thus, t has infinite order. This shows (1).

Let $g_0 t^{\varepsilon_1} g_1 \dots t^{\varepsilon_n} g_n$ be a reduced form over G and t. By our construction of the normal form from the reduced form in the proof of Theorem 2.5.14, we have not changed the number of occurrences of t in that process. Since $1 \in G*_\varphi$ has no non-trivial normal form by Theorem 2.5.14, we obtain (2). □

Corollary 2.5.16. *Let $G*_\varphi$ be an HNN extension of G for the isomorphism $\varphi\colon A \to B$. Then the subgroup of $G*_\varphi$ that is generated by G and G^t is isomorphic to the free product of those two groups with amalgamation over A, where $\iota_1 = \varphi$ and ι_2 is the conjugation with t, i. e. $\langle G, G^t\rangle \cong G *_A G^t$.*

Proof. First, we note that the relators in the definition of the free product with amalgamation of G and G^t are already satisfied in $\langle G, G^t\rangle$. Thus, $K := \langle G \cup G^t\rangle$ is a homomorphic image of $G *_A G^t$. Let $ag_1h_1^t g_2 \dots g_m h_m^t$ be a non-trivial normal form in $G *_A G^t$ with $m \ge 1$, where $g_1 = 1$ or $h_m = 1$ may hold. This is a reduced form in $G*_\varphi$ and by Britton's lemma (Corollary 2.5.15 (2)) it is distinct from 1. Thus, K is already the free product with amalgamation of G and G^t. □

Theorem 2.5.17 (Higman-Neumann-Neumann)**.** *Let G be a countable group. Then there exists a group H that has a generating set consisting of two elements such that $G \leq H$.*

Proof. Let $G = \{g_0, g_1, \ldots\}$ with $g_0 = 1$ and with repetitions if necessary. Let F be a free group with free generating set $\{a, b\}$. The sets $\{b^{-i}ab^i\}$ and $\{a^{-i}ba^i\}$ both freely generate free subgroups A and B of F (cf. Exercise 2 on Sheet 2). We consider the subgroup K of $G * F$ generated by $\{g_i a^{-i} b a^i \mid i \in \mathbb{N}\}$. Then the extension of the maps $\varphi\colon G \to F, g \mapsto 1$ and the identity on F extends to a homomorphism from $G * F \to F$ by Theorem 2.5.8, the projection to F. Thus, any non-trivial reduced word in K that represents 1 is mapped onto a non-trivial reduced word in B. Since B is free, this contradiction shows that K must be free as well and has $\{g_i a^{-i} b a^i \mid i \in \mathbb{N}\}$ as a free generating set.

We consider the map $\psi\colon A \to K$ that is induced uniquely by $\psi(b^{-i}ab^i) = g_i a^{-i} b a^i$. Let H be the HNN extension of $G * F$ with the isomorphism ψ. By Corollaries 2.5.15 and 2.5.7, we find a canonical isomorphic image of G in $G * F$ and in H. Since the image of every g_n is generated by t, a, b, we have $\langle t, a, b\rangle = H$. Since $g_0 = 1$, we have

$$tat^{-1} = tg_0 b^{-0} a b^0 t^{-1} = a^{-0} b a^0 = b.$$

So we have $H = \langle t, a\rangle$ and thus (an isomorphic image of) G lies in a group generated by two elements. □

Chapter 3

Quasi-Isometries

3.1 Word metric and quasi-isometries

Definition. Let G be a group and let S be a generating set of G. The **word metric** d_S of G with respect to S is the metric of the Cayley graph of G and S.

Remark 3.1.1. Let G be a group and let S be a generating set of G. Then $d_S(g,h)$ is the length of a shortest word that represents $g^{-1}h$ for all $g,h \in G$, i. e., we have

$$d_S(g,h) = \min\{n \in \mathbb{N} \mid \exists s_1 \dots s_n \in S \cup S^{-1} \colon g^{-1}h = s_1 \dots s_n\}.$$

Remark 3.1.2. Let G be a group and let S be a generating set of G. Left / right multiplication is an action of G on the metric space (G, d_S). In particular, the multiplication with an element induces an isometry on G.

We directly observe that distinct generating sets can lead to distinct word metrics.

Example 3.1.3. Let $S_1 = \{1\}$ and $S_2 = \mathbb{Z}$ be two generating sets of $\mathbb{Z}$. Then we have $d_{S_1}(g,h) = |g-h|$ and $d_{S_2}(g,h) = 1$ for all $g \neq h \in \mathbb{Z}$.

If we look at distinct locally finite Cayley graphs for the same finitely generated group, then the word metrics are 'essentially' the same. To make this precise, we introduce the notion of quasi-isometries.

Definition. Let (X, d_X) and (Y, d_Y) be two metric spaces.

(1) Let $f \colon X \to Y$ be a map.

- The map f is a **quasi-isometric embedding** if there are constants $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{\geq 0}$ such that

$$\frac{1}{\gamma} d_X(x,x') - c \leq d_Y(f(x), f(x')) \leq \gamma d_X(x,x') + c$$

for all $x, x' \in X$.

- The map f is **quasi-dense** if there is a constant $c \in \mathbb{R}_{\geq 0}$ such that

$$d_Y(y, f(X)) \leq c$$

for all $y \in Y$.

- The map f is a **quasi-isometry** if it is a quasi-dense quasi-isometric embedding.

(2) The metric spaces X and Y are **quasi-isometric** if there exists a quasi-isometry $f\colon X \to Y$. Then we write $X \sim_{QI} Y$.

(3) Let $f\colon X \to Y$ and $g\colon Y \to X$ be quasi-isometries. They are **quasi-inverses** of each other if there exists $c \geq 0$ such that for all $x \in X$ and all $y \in Y$ we have $d(x, g(f(x))) \leq c$ and $d(y, f(g(y))) \leq c$.

Proposition 3.1.4. (i) *The relation $\sim_{QI}$ is an equivalence relation on the class of metric spaces.*[1]

(ii) *For every quasi-isometry there exists a quasi-inverse.*

Proof. Exercise □

Proposition 3.1.5. *Let G be a finitely generated group and let S_1, S_2 be two finite generating sets of G. The identity $id_G\colon (G, d_{S_1}) \to (G, d_{S_2})$ is a quasi-isometry between these two metric spaces.*

Proof. We may assume that G is not trivial. Thus, neither S_1 nor S_2 is empty. Since id_G is surjective, it is obviously quasi-dense and it remains to show that it is a quasi-isometric embedding. Set

$$\gamma_1 := \sup\{d_{S_2}(1, s) \mid s \in S_1 \cup S_1^{-1}\}.$$

Since S_1 is finite but not empty, we have $\gamma_1 \in \mathbb{N} \setminus \{0\}$. Let $g, h \in G$. Let $s_1, \ldots, s_n \in S_1 \cup S_1^{-1}$ for $n = d_{S_1}(g, h)$ such that $gs_1 \ldots s_n = h$. Then we have

$$\begin{aligned} d_{S_2}(g, h) &= d_{S_2}(g, gs_1 \ldots s_n) \\ &\leq d_{S_2}(g, gs_1) + d_{S_2}(gs_1, gs_1s_2) \\ &\quad + \ldots + d_{S_2}(gs_1 \ldots s_{n-1}, gs_1 \ldots s_n) \\ &= d_{S_2}(1, s_1) + d_{S_2}(1, s_2) + \ldots + d_{S_2}(1, s_n) \\ &\leq \gamma_1 n \\ &= \gamma_1 d_{S_1}(g, h). \end{aligned}$$

Analogously, there exists

$$\gamma_2 := \sup\{d_{S_1}(1, s) \mid s \in S_2 \cup S_2^{-1}\}$$

and we have $d_{S_1}(g, h) \leq \gamma_2 d_{S_2}(g, h)$. Set $\gamma := \max\{\gamma_1, \gamma_2\}$. Then id_G is a quasi-isometric embedding for the constants γ and $c = 0$. □

[1] Formally, relations are defined only on sets; here we think of their canonical definition for classes.

Because $\sim_{QI}$ is an equivalence relation and distinct word metrics of finitely generated groups and finite generating sets lead to quasi-isometric metric spaces, the following definition is well-defined.

Definition. Let G and H be finitely generated groups.

(1) We call G **quasi-isometric** to a metric space X if for one[2] finite generating set S of G the metric space (G, d_S) (and thus the Cayley graph of G and S) is quasi-isometric to X.

(2) The groups G and H are **quasi-isometric** if there is a metric space X that is quasi-isometric to both groups.

Example 3.1.6. For every $n \in \mathbb{N}$, the group $\mathbb{Z}^n$ is quasi-isometric to the euclidean space $\mathbb{R}^n$, since the canonical embedding is a quasi-dense map that is a quasi-isometric embedding with respect to the word metric for the standard generating set S of $\mathbb{Z}^n$.

Example 3.1.7. Let G and H be finite groups. Then G and H are quasi-isometric for constants $\gamma = 1$ and $c = \max\{|G|, |H|\}$.

3.2 Švarc-Milnor lemma

Definition. Let X be a metric space.

(1) Let $\ell \in \mathbb{R}_{\geq 0}$. A **geodesic** of **length** ℓ is an isometric embedding $f\colon [0, \ell] \to X$. Its **starting point** is $f(0)$ and its **end point** is $f(\ell)$.

(2) The metric space X is **geodesic** if there exists a geodesic of length $d(x, y)$ with starting point x and end point y for all $x, y \in X$.

(3) A **quasi-geodesic** is a quasi-isometric embedding $f\colon I \to X$ of a closed interval $I = [t_1, t_2] \subseteq \mathbb{R}$. Then $f(t_1)$ is the **starting point** and $f(t_2)$ is the **end point**.

(4) The metric space X is **quasi-geodesic** if there are two constants $c \in \mathbb{R}_{\geq 1}$ and $\gamma \in \mathbb{R}_{\geq 0}$ such that there is a quasi-geodesic with constants γ and c and with starting point x and end point y for all $x, y \in X$.

Remark. (1) Every geodesic metric space ist quasi-geodesic.

(2) Every quasi-geodesic metric space with constants $\gamma = 1$ and $c = 0$ is geodesic.

Example 3.2.1. (1) Let Γ be a graph. Between every two of its vertices x, y, there exists a path of length $d(x, y)$. Thus, graphs are quasi-geodesic metric

[2] and hence (by Proposition 3.1.5) for every

spaces for $\gamma = 1 = c$. In general, Γ is not a geodesic metric space. Nevertheless, we interpret them as geodesic metric space: by interpreting edges as isometric copies of $[0,1]$,[3] the graph becomes a geodesic metric space.[4]

(2) The space $\mathbb{R}^2$ with the euclidean metric is a geodesic metric space.

(3) The space $\mathbb{R}^2 \setminus \{0\}$ with the metric induced by the euclidean metric on $\mathbb{R}^2$ is not geodesic, but it is quasi-geodesic for $\gamma = 1$ and every $c > 0$.

Theorem 3.2.2 (Švarc-Milnor lemma)**.** *Let G be a group acting on a metric space X.*[5] *Let X be quasi-geodesic for $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{>0}$ and assume that there exists a subset $B \subseteq X$ with the following properties.*

(i) *the diameter of B is finite;*

(ii) $\bigcup_{g \in G} gB = X$*;*

(iii) *For $B' := \{x \in X \mid d(x,B) \leq 2c\}$, the set $S := \{g \in G \mid B' \cap gB' \neq \emptyset\}$ is finite.*

The following statements are true.

(1) *The set S generates G; in particular, G is finitely generated.*

(2) *For all $x \in X$, the map $\psi_x \colon G \to X$, $g \mapsto gx$ is a quasi-isometry.*

Proof. We will show (1) in a similar way as used for Theorem 1.3.2. Let $g \in G$. We want to write g as a finite product of elements of S. Let $x \in B$ and let $\varphi \colon [0,\ell] \to X$ be a quasi-geodesic for constants γ and c that starts at x and end at gx. Set $n := \lceil \frac{\gamma \ell}{c} \rceil$. Set $t_j := j \cdot \frac{c}{\gamma}$ for all $j \in \{0,\dots,n-1\}$ and $t_n := \ell$ and set $x_j := \varphi(t_j)$ for all $j \in \{0,\dots,n\}$. By (ii) there exists for every $0 \leq j \leq n$ some $g_j \in G$ with $x_j \in g_jB$. We may assume that $g_0 = id$ and $g_n = g$, because $x_0 = x$ and $x_n = gx$.

We have

$$d(x_j, x_{j+1}) \leq \gamma \cdot |t_j - t_{j+1}| + c \leq \gamma \cdot \frac{c}{\gamma} + c = 2c.$$

Thus, x_{j+1} lies in g_jB'. Since it is also contained in $g_{j+1}B'$, we obtain

$$g_jB' \cap g_{j+1}B' \neq \emptyset$$

and hence $s_j := g_j^{-1}g_{j+1} \in S$. So we have $g = s_0 \cdots s_{n-1} \in \langle S \rangle$, which implies (1). The additional statement is an immediate consequence of (iii).

For the proof of (2), we may assume $x \in B$ by (ii). We immediately obtain from (i) and (ii) that ψ_x is quasi-dense in X for the constant $\operatorname{diam}(B)$: for

[3] e. g. as in the case of planar graphs

[4] This will not play a major role for us. It will not be important whether we consider them as geodesic of just quasi-geodesic metric spaces.

[5] Note that every $g \in G$ induces an isometry on X.

every $y \in X$ there exists by (ii) a $g \in G$ with $y \in gB$ and thus we obtain $d(y, \psi_x(G)) \leq d(y, gx) \leq \mathrm{diam}(B)$ by (i).

Let $\varphi\colon [0,\ell] \to X$ be a quasi-geodesic starting at x and ending at gx for constants γ and c as in the first part of the proof and let $n \in \mathbb{N}$ as we defined it in that part. Then we obtain

$$\begin{aligned} d(\psi_x(1), \psi_x(g)) &= d(x, gx) \\ &= d(\varphi(0), \varphi(\ell)) \\ &\geq \frac{1}{\gamma}\ell - c \\ &\geq \frac{1}{\gamma} \cdot \frac{c(n-1)}{\gamma} - c \\ &= \frac{c}{\gamma^2} n - \left(\frac{c}{\gamma^2} + c\right) \\ &\geq \frac{c}{\gamma^2} d_S(1,g) - \left(\frac{c}{\gamma^2} + c\right). \end{aligned}$$

For the second inequality, let $s_1 \ldots s_n \in S$ with $g = s_1 \ldots s_n$ and $n = d_S(1,g)$. Because of $s_j B' \cap B' \neq \emptyset$ for all $1 \leq j \leq n-1$, we have

$$\begin{aligned} d(\psi_x(1), \psi_x(g)) &= d(x, gx) \\ &\leq d(x, s_1 x) + d(s_1 x, s_1 s_2 x) + \ldots \\ &\quad + d(s_1 \ldots s_{n-1} x, s_1 \ldots s_n x) \\ &= d(x, s_1 x) + d(x, s_2 x) + \ldots + d(x, s_n x) \\ &\leq n \cdot 2\mathrm{diam}(B') \\ &\leq 2\mathrm{diam}(B') \cdot d_S(1,g). \end{aligned}$$

For $\gamma' := \max\{\frac{\gamma^2}{c}, 2\mathrm{diam}(B')\}$ and $c' := \frac{c}{\gamma^2} + c$, the map ψ_x is a quasi-isometric embedding. Thus, we obtain (2). □

Corollary 3.2.3. *Let G be a finitely generated group and let H be a subgroup of G of finite index. Then H is finitely generated and $H \sim_{QI} G$.*

Proof. Let S be a finite generating set of G. The left multiplication of H on G is an action of H on the metric space (G, d_S). By definition of d_S, the space (G, d_S) is a quasi-geodesic metric space for $\gamma = c = 1$ according to Example 3.2.1 (1). Let B be a transversal of the right cosets of H in G. Since $|G : H|$ is finite, B is finite as well. Since B and S are finite, also the set $B' := \{g \in G \mid d_S(g, B) \leq 2\}$ is finite. Hence and since H acts freely on G, the set $\{h \in H \mid B' \cap hB' \neq \emptyset\}$ is finite. Because of $HB = G$, all assumptions of Theorem 3.2.2 are satisfied and obtain that H is finitely generated and that the embedding $id\colon H \to G$ is a quasi-isometry. □

Corollary 3.2.4. *Let G be a group and let H be a subgroup of G of finite index. Then G is finitely generated if and only if H is finitely generated.* □

3.3 Quasi-isometry invariants

In the rest of this chapter, we are interested in properties that are preserved by quasi-isometries. These are algebraic properties as well as geometric ones. For the geometric properties, we will also have a look at what algebraic results for groups they imply.

Definition. A **quasi-isometry invariant** (with values in a set U) is an assignment $\mathcal{P}$ of finitely generated groups in U with $\mathcal{P}(G) = \mathcal{P}(H)$ for all finitely generated groups $G \sim_{QI} H$.

We have seen in Example 3.1.7 that being finite is a quasi-isometry invariant. Now we will show that finite presentability is one, too.

Theorem 3.3.1. *Finite presentability is a quasi-isometry invariant for finitely generated groups.*

Proof. Let G be a finitely presented group and let H be a finitely generated group. Let S_G and S_H be finite generating sets of G and H, respectively, and let R_G be a finite sets of relators of G such that $G = \langle S_G \mid R_G \rangle$. Let $\varphi\colon G \to H$ be a quasi-isometry and let $\psi\colon H \to G$ be a quasi-inverse of φ, where $\gamma \geq 1$ and $c \geq 0$ are the constants for the quasi-isometries and c is the constant for the quasi-inverse. We may assume that $\varphi(1) = 1$ and $\psi(1) = 1$. For all $g, h \in G$, let $w_{g,h}$ be a shortest word over $S_H \cup S_H^{-1}$ such that $\varphi(g)w_{g,h} = \varphi(gh)$. We choose $w_{g,h}$ such that $w_{gh,h^{-1}}$ is the inverse word[6] of $w_{g,h}$. Analogously, we define words $v_{h,s}$ over $S_G \cup S_G^{-1}$ for $g, h \in H$.

Let $w = s_1 \dots s_n$ be a word over $S_H \cup S_H^{-1}$ with $w = 1$. We replace every letter s_i by $w_{s_1\dots s_{i-1},s_i}$. Thereby, we obtain a word $v = v_1 \dots v_k$ over $S_G \cup S_G^{-1}$ with $v = 1$. Note that there are subwords $v_1 \dots v_{i_j}$ for all $j \leq n$ such that $v_1 \dots v_{i_j} = \psi(s_1 \dots s_j)$ and such that $i_j < i_{j'}$ for $j < j'$. We say that v visits all $\psi(\emptyset), \dots, \psi(s_1 \dots s_n)$ in that order.

Since v lies in the normal subgroup generated by R_G in the free group generated by S_G, there are $r_1, \dots r_m \in R_G$ and words $p_1, \dots p_m$ such that $p_1^{-1} r_1 p_1 \dots p_m^{-1} r_m p_m$ has v as a reduction. We apply the same method we used to obtain v from w to all r_i and p_i using the words $v_{h,s}$ in order to get words r_i' and p_i' over $S_H \cup S_H^{-1}$. Then $w' := {p_1'}^{-1} r_1' p_1' \dots {p_m'}^{-1} r_m' p_m'$ is a word over $S_H \cup S_H^{-1}$ that represents 1. Note that w' visits $\varphi(\psi(\emptyset)), \dots, \psi(\varphi(s_1 \dots s_n))$ in that order.

For $1 \leq i \leq n$, let x_i be a shortest word such that $s_1 \dots s_i x_i = \varphi(\psi(s_1 \dots s_i))$. Note that the length of x_i is at most c, since φ and ψ are quasi-inverse. Let y_i be the subword of w' from the word that represents $\varphi(\psi(s_1 \dots s_{i-1}))$ to $\varphi(\psi(s_1 \dots s_i))$. Note that the length of y_i is at most $\gamma + c$. Then $z_i := x_i y_{i+1} x_{i+1}^{-1} s_{i+1}^{-1}$ is a word of length at most $\gamma + 3c + 1$ that represents 1. If we consider the word

$$w'' = z_0(s_1 z_1 s_1^{-1}) \dots (s_1 \dots s_{n-1} z_{n-1} s_{n-1}^{-1} \dots s_1^{-1}) s_1 \dots s_n,$$

[6] If $s_1^{\varepsilon_1} \dots s_n^{\varepsilon_n}$ is a word, then we call the word $s_n^{-\varepsilon_n} \dots s_1^{-\varepsilon_1}$ its **inverse**.

then it can be reduced to w'. Thus, w lies in the normal subgroup of the free group generated freely by S_H that is generated by all z_i and all r_i'. Since each z_i has length at most $\gamma + 3c + 1$ and each r_i' has length at most $\ell(\gamma + c)$, where ℓ denotes the length of the longest relator in R_G. Since there are only finitely many words over $S_H \cup S_H^{-1}$ of length at most $\max\{\ell(\gamma + c), \gamma + 3c + 1\}$, we obtain that H is finitely presented. □

3.4 Ends of groups

In this section, we will look at ends of (finitely generated) groups. For this definition, we rely on the notion of ends of graphs, but will avoid making precise what an end of group is but instead just define the number of ends.

Definition. Let $\Gamma = (V, E)$ be a graph. A **ray** in Γ is a one-way infinite path. For every ray R in Γ and every finite set $U \subseteq V$ of vertices there is a unique component C of $\Gamma - U$ that contain infinitely many vertices of R. Then we say that R lies in C **eventually**. Two rays in Γ are **equivalent** if there is no finite subset $U \subseteq V$ such that the rays lie in distinct components of $\Gamma - U$ eventually. It follows easily that this defines an equivalence relation. Its equivalence classes are the **ends** of Γ.

We shall show first that ends behave well with respect to quasi-isometries.

Lemma 3.4.1. *Let Γ and Δ be two locally finite graphs. If $f\colon \Gamma \to \Delta$ is a quasi-isometry, then f induces a bijection on the ends of the graphs.*

In particular, both graphs have the same number of ends.

Proof. Let $\gamma \geq 1$ and $c \geq 0$ such that f is a (γ, c)-quasi-isometry. Let R be a ray of Γ. By joining every two vertices of $f(R)$ by a path of length at most $\gamma + c$, we obtain a one-way infinite walk W. Note that the distance between occurrences of the same vertex is bounded by some constant κ, since f is a quasi-isometry. Thus, every two rays in W are equivalent.

Let Q be a ray that is equivalent to R. Then for every $r \in \mathbb{N}$ they are connected by a path that lies outside the balls of radius r around the first vertex of R. This implies that we find paths outside of every ball of radius $r/\gamma - c$ around the f-image of the first vertex of R between every two rays R', Q' that are defined by $f(R)$ and $f(Q)$ in Δ. Thus, every end of Γ is mapped to an end of Δ.

Since f has a quasi-inverse g, every two equivalent rays Δ define equivalent rays in Γ, too. Thus, the map induced on the ends is bijective. □

Even though we will not talk about specific ends of groups Lemma 3.4.1 shows that the number of ends for each Cayley graph of a locally finite group and any of its finite generating sets is the same.

Definition. Let G be a finitely generated group. The **number of ends** of G is the number of ends of each of its locally finite Cayley graphs. We denote this number by $e(G)$.

We obtain from Lemma 3.4.1 more than just the basis of our definition of numbers of ends of groups, as we will se in the following corollary.

Corollary 3.4.2. *The number of ends is a quasi-isometry invariant for finitely generated groups.* □

Natural questions that arise now are e. g. which values $e(G)$ can have and whether, for given number of ends, we can characterise the groups that have this number of ends.

Lemma 3.4.3. *Let Γ be a transitive connected locally finite graph.*[7] *If Γ has at least three ends, then it has infinitely many ends.*

Proof. Let us suppose that Γ has finitely many but more than two ends. Then there exists a finite subgraph Δ of Γ such that for every component C of $\Gamma - \Delta$ all rays in C are equivalent. Since Γ is locally finite, there exists in every component C of $\Gamma - \Delta$ a vertex x such that $d(x, \Delta)$ is larger than the diameter of Δ. Mapping $y \in V(\Delta)$ to x by an automorphism φ implies that $\Delta \cap \varphi(\Delta)$ is empty by the choice of x. Now there are at least three infinite components of $\Gamma - \varphi(\Delta)$ that contain ends. Since two of these components must lie in the same component of $\Gamma - \Delta$, this contradicts the choice of Δ that is separates all ends. □

We directly obtain the following theorem from Lemma 3.4.3.

Theorem 3.4.4. *If G is a finitely generated group, then $e(G) \in \{0, 1, 2, \infty\}$.* □

Example 3.4.5. Let G be a finitely generated group.

(1) We have $e(G) = 0$ if and only if G is finite.

(2) If $G = \mathbb{Z}^n$ for some $n \in \mathbb{N}_{\geq 2}$, then $e(G) = 1$.

(3) If $G = \mathbb{Z}$, then $e(G) = 2$.

(4) If G is a free group of rank at least 2, then $e(G) = \infty$.

We will prove in a later chapter that a finitely generated group with more than one end is either a free product with amalgamation or an HNN extension over a finite groups (Stallings' theorem).

In the rest of this section, we will characterise finitely generated groups that have exactly two ends.

Definition. A group is **virtually cyclic** if it has a cyclic subgroup of finite index.

Theorem 3.4.6. *Let G be a finitely generated infinite group. Then the following statements are equivalent.*

(1) *G is virtually cyclic;*

[7] I. e. there is a group acting transitively on Γ.

(2) $G \sim_{QI} \mathbb{Z}$;

(3) $e(G) = 2$.

Proof. The implication (1)⇒(2) is a consequence of Corollary 3.2.3. Corollary 3.4.2 and Example 3.4.5 (3) imply the direction (2)⇒(3).

Let us assume that $e(G) = 2$. Let $\Gamma = (V, E)$ be a Cayley graph of G and some finite generating set S of G. Then there exists a finite connected subgraph $\Delta \subseteq \Gamma$ such that $\Gamma \setminus \Delta$ has exactly two components C_1, C_2 both of which are infinite.

Claim 1. For every $g \in G$ either $C_1 \cap gC_1$ and $C_2 \cap gC_2$ or $C_1 \cap gC_2$ and $C_2 \cap gC_1$ are infinite. The other two intersections are finite.

> *Proof of Claim* 1. Since $\Delta \cup g\Delta$ separates the four involved intersections, but it covers together with them covers the vertex set of Γ and since Δ separates two infinite components, there are precisely two infinite intersections. But not both of them can lie in any of C_1, gC_1, C_2 or gC_2, since its complement is infinite. Thus, we obtain the assertion. □

Set

$$H := \{g \in G \mid C_1 \cap gC_1 \text{ and } C_2 \cap gC_2 \text{ are infinite}\}.$$

Claim 2. The set H is a subgroup of G with $|G : H| \leq 2$.

> *Proof of Claim* 2. Obviously, H contains for every elements also the inverse one. If $g, h \in H$, then $g(C_1 \cap hC_1)$ is infinite. Since $C_2 \cap g(C_1 \cap hC_1)$ is finite by Claim 1, we obtain that $C_1 \cap g(C_1 \cap hC_1)$ and hence $C_1 \cap ghC_1$ is infinite. Thus, H is closed under multiplication. Hence, it is a subgroup.
>
> Let us assume $G \neq H$. We shall show $|G : H| = 2$. Let $g, h \in G \setminus H$. By Claim 1 and the definition of H, the sets $C_1 \cap gC_2$ and $C_2 \cap gC_1$ are infinite and the same holds if we replace g by h. Thus, the set $C_1 \cap g(C_1 \cap hC_2) \subseteq C_1 \cap gC_1$ must be finite. Since $C_1 \cap g(C_2 \cap hC_2)$ is finite, too, $C_1 \cap ghC_2$ must be finite as well. By Claim 1 the element gh lies in H. Thus, g and h are in the same coset of H, which implies $|G : H| = 2$. □

Claim 3. For $h \in H$ with $\Delta \cap h\Delta = \emptyset$ and for $\overline{C}_1 := V \setminus C_1$, we have either

(i) $C_1 \cap h\overline{C}_1 = \emptyset$ and $\overline{C}_1 \cap hC_1 \neq \emptyset$ or

(ii) $C_1 \cap h\overline{C}_1 \neq \emptyset$ and $\overline{C}_1 \cap hC_1 = \emptyset$.

> *Proof of Claim* 3. The claim follows directly from the connectedness of Δ. □

By the choice of H and by Claim 1, the set $|C_1 \cap h\overline{C}_1| - |\overline{C}_1 \cap hC_1|$ is finite for all $h \in H$. We define the function

$$\varphi\colon H \to \mathbb{Z},\ \varphi(h) := |C_1 \cap h\overline{C}_1| - |\overline{C}_1 \cap hC_1|.$$

Claim 4. The function φ is a homomorphism.

Proof of Claim 4. Let $g, h \in H$. We have

$$\begin{aligned}
\varphi(gh) &= |C_1 \cap gh\overline{C}_1| - |\overline{C}_1 \cap ghC_1| \\
&= |C_1 \cap g\overline{C}_1 \cap gh\overline{C}_1| + |C_1 \cap gC_1 \cap gh\overline{C}_1| \\
&\quad - |\overline{C}_1 \cap gC_1 \cap ghC_1| - |\overline{C}_1 \cap g\overline{C}_1 \cap ghC_1| \\
&= |C_1 \cap g\overline{C}_1 \cap gh\overline{C}_1| + |C_1 \cap gC_1 \cap gh\overline{C}_1| \\
&\quad - |\overline{C}_1 \cap gC_1 \cap ghC_1| - |\overline{C}_1 \cap gC_1 \cap gh\overline{C}_1| \\
&\quad + |C_1 \cap g\overline{C}_1 \cap ghC_1| + |\overline{C}_1 \cap gC_1 \cap gh\overline{C}_1| \\
&\quad - |\overline{C}_1 \cap g\overline{C}_1 \cap ghC_1| - |C_1 \cap g\overline{C}_1 \cap ghC_1| \\
&= |C_1 \cap g\overline{C}_1| - |\overline{C}_1 \cap gC_1| \\
&\quad + |gC_1 \cap gh\overline{C}_1| - |g\overline{C}_1 \cap ghC_1| \\
&= \varphi(g) + |C_1 \cap h\overline{C}_1| - |\overline{C}_1 \cap hC_1| \\
&= \varphi(g) + \varphi(h).
\end{aligned}$$

Thus, φ is a homomorphism. □

Claim 5. The kernel of φ is finite.

Proof of Claim 5. Since Δ is a finite subgraph of Γ and since G acts freely on Γ, there are only finitely many $h \in H$ with $\Delta \cap h\Delta \neq \emptyset$. For all other $h \in H$, we obtain by Claim 3 that $\varphi(h)$ and 0 are distinct. □

Let $h \in H \setminus \varphi^{-1}(0)$. Then $\varphi(h)$ and hence h have infinite order. Thus, we have $\langle h \rangle \cong \mathbb{Z}$. Since the index of $\langle \varphi(h) \rangle$ in $\mathbb{Z}$ is finite and $\ker(\varphi)$ is also finite by Claim 5, the subgroup $\langle h \rangle$ has finite index in H and thus in G by Claim 2. □

3.5 Growth of groups

Definition. Let G be a finitely generated group and let S be a finite generating set of G. For $r \in \mathbb{N}$ and $g \in G$ we set

$$B_r^{G,S}(g) := \{h \in G \mid d_S(g,h) \leq r\}.$$

Then

$$\beta_{G,S} \colon \mathbb{N} \to \mathbb{N},\ r \mapsto |B_r^{G,S}(1)|$$

is the **growth function** of G with respect to S.

Note that $|B_r^{G,S}(1)| = |B_r^{G,S}(g)|$ for all $g \in G$.

Example 3.5.1. (1) Let G be a finitely generated group and let S be a finite generating set of G. Then G is finite if and only if $\beta_{G,S}$ becomes stationary.

(2) Let S_1 be the standard generating set of $\mathbb{Z}$. Then $\beta_{\mathbb{Z},S_1}(r) = 2r+1$ for all $r \in \mathbb{N}$.

(3) Let $S_2 := \{2,3\}$ be another generating set of $\mathbb{Z}$. Then

$$\beta_{\mathbb{Z},S_2}(r) = \begin{cases} 1, & \text{if } r = 0, \\ 5, & \text{if } r = 1, \\ 6r+1 & \text{otherwise.} \end{cases}$$

(4) Let S be the standard generating set of $\mathbb{Z}^2$. Then

$$\beta_{\mathbb{Z}^2,S}(r) = 1 + 4 \cdot \sum_{j=1}^{r} j = 2r^2 + 2r + 1.$$

(5) Let F be a finitely generated free group of rank at least 2 and let S be a free generating set of F. Then $\beta_{F,S}$ is an exponential function.[8]

Example 3.5.1 shows that distinct generating sets of the same group lead to distinct growth functions. We will see later that all these functions are similar for each groups.

Proposition 3.5.2. *Let G be a finitely generated group and let S be a finite generating set of G.*

(1) (Sub-multiplicativity) *For all $r, r' \in \mathbb{N}$ we have*

$$\beta_{G,S}(r + r') \leq \beta_{G,S}(r) \cdot \beta_{G,S}(r').$$

(2) *Let F be a free group with free generating set S. For all $r \in \mathbb{N}$ we have*

$$\beta_{G,S} \leq \beta_{F,S}.$$

Proof. Exercise ☐

Definition. Let $f, g \colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be maps.

(i) If f is increasing then it is a **generalised growth function**.

(ii) Let f and g be generalised growth functions. The map g **dominates** f if there are $c \in \mathbb{R}_{\geq 0}$ and $\gamma \in \mathbb{R}_{>0}$ such that

$$f(r) \leq \gamma g(\gamma r + c) + c$$

for all $r \in \mathbb{R}_{\geq 0}$. Then we write $f \preccurlyeq g$.

(iii) Let f and g be generalised growth functions. They are **equivalent** if $f \preccurlyeq g$ and $g \preccurlyeq f$. Then we write $f \sim g$.

[8] Proof: Exercise

Example and Definition 3.5.3. Let G be a finitely generated group and let S be a finite generating set of G. Then the map

$$f\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0},\ r \mapsto \beta_{G,S}\left(\lceil r \rceil\right)$$

is a generalised growth function. If H is another finitely generated group and T a finite generating set of H, then $\beta_{G,S}$ **dominates/is equivalent to** $\beta_{H,T}$ if the same holds for their generalised growth functions.

Lemma 3.5.4. (1) *Domination of (generalised) growth functions is a quasi-order.*[9]

(2) *Equivalence of (generalised) growth functions is an equivalence relation.*

Proof. Exercise □

Now we are going to prove that distinct finite generating sets of the same group essentially lead to the same growth functions: they are equivalent.

Proposition 3.5.5. *Let G and H be finitely generated groups with finite generating sets S of G and T of H. If there is a quasi-isometric embedding $\varphi\colon (G, d_S) \to (H, d_T)$, then*

$$\beta_{G,S} \prec \beta_{H,T}.$$

Proof. Let $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{\geq 0}$ be the constants for the quasi-isometric embedding φ. Let $e := \varphi(1_G)$. Then we have

$$d_T(e, \varphi(g)) \leq \gamma d_S(1_G, g) + c \leq \gamma r + c$$

for all $g \in B_r^{G,S}(1_G)$ and hence

$$\varphi(B_r^{G,S}(1_G)) \subseteq B_{\gamma r+c}^{H,T}(e).$$

Let $g, g' \in G$ with $\varphi(g) = \varphi(g')$. Then we have

$$\frac{1}{\gamma} d_S(g, g') - c \leq d_T(\varphi(g), \varphi(g'))$$

and hence

$$d_S(g, g') \leq \gamma(d_T(\varphi(g), \varphi(g')) + c) = \gamma c.$$

Thus, we have

$$\begin{aligned}
\beta_{G,S}(r) &= |B_r^{G,S}(1_G)| \\
&\leq |B_{\gamma c}^{G,S}(1_G)| \cdot |B_{\gamma r+c}^{H,T}(e)| \\
&\leq |B_{\gamma c}^{G,S}(1_G)| \cdot |B_{\gamma r+c}^{H,T}(1_H)| \\
&= \beta_{G,S}(\gamma c) \cdot \beta_{H,T}(\gamma r + c).
\end{aligned}$$

Since the first factor does not depend on r, we obtain $\beta_{G,S} \prec \beta_{H,T}$. □

[9]A quasiorder is a reflexive and transitive relation.

Proposition 3.1.5 implies the following two corollaries.

Corollary 3.5.6. *Distinct growth functions of the same finitely generated group are equivalent.* □

Corollary 3.5.7. *Quasi-isometric groups have equivalent growth functions.* □

Definition. Let G be a finitely generated group. The **growth type** of G is the equivalence class of the generalised growth functions that contains all growth functions of G (with respect to finite generating sets. The groups G has ...

(i) ... **exponential growth** if the growth type contains the map $x \mapsto e^x$;

(ii) ... **polynomial growth** if, for every finite generating set S of G, there exists an $a \in \mathbb{R}_{\geq 0}$ such that

$$\beta_{G,S} \prec (x \mapsto x^a);$$

(iii) ... **intermediate growth** if it has neither exponential nor polynomial growth.

Example 3.5.8. (1) Let $n \in \mathbb{N}$. The growth type of the group $\mathbb{Z}^n$ is polynomial.[10]

(2) Let F be a finitely generated free group of rank $n \geq 2$. Then the growth type of F is exponential.

Remark 3.5.9. (1) By Corollary 3.5.6, the growth type of a finitely generated group is a quasi-isometry invariant.

(2) Since free groups of rank at least 2 have exponential growth, we obtain by Proposition 3.5.2 that every group has at most exponential growth. A theorem of van den Dries and Wilkies implies that every polynomial function is dominated by the growth functions of finitely generated groups of intermediate growth.

Remark 3.5.10. We already know groups with polynomial and with exponential growths. There are examples of groups of intermediate growth, e. g. the so-called Grigorchuk group.

Theorem 3.5.11. *Let G be a finitely generated group with finite generating set S and let H be a finitely generated subgroups of G with finite generating set T. Then*

$$\beta_{H,T} \prec \beta_{G,S}.$$

Proof. The set $S' := S \cup T$ is a finite generating set of G. Let $r \in \mathbb{N}$. Then we have

$$d_{S'}(1,h) \leq d_T(1,h) \leq r$$

for all $h \in B_r^{H,T}(1)$. Thus, we have $B_r^{H,T}(1) \subseteq B_r^{G,S'}(1)$. This implies together with Corollary 3.5.6

$$\beta_{H,T} \prec \beta_{G,S'} \prec \beta_{G,S}.$$ □

[10] Proof: exercise

Corollary 3.5.12. *If a finitely generated group G has a free subgroup of rank 2, then G has exponential growth.* □

Definition. Let G be a group.

(i) Let $G_1 := G$. For $n \in \mathbb{N}$, we define G_n recursively as **commutator**

$$[G_{n-1}, G] := \{h^{-1}g^{-1}hg \mid h \in G_{n-1}, g \in G\}$$

of G_{n-1} and G. We call G **nilpotent** if there exists $n \in \mathbb{N}$ such that $G_n = 1$. (The sequence $(G_n)_{n\in\mathbb{N}}$ is called a **central series**.)

(ii) The group G is **virtually nilpotent** if it has a nilpotent subgroup of finite index.

We cite the main theorem in the area of growth of groups without proof.

Theorem 3.5.13 (Gromov)**.** *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Corollary 3.5.14. *Being virtually nilpotent is a quasi-isometry invariant for finitely generated groups.* □

Chapter 4

Bass-Serre theory

Definition. A group G acts **without (edge-)inversion** on a graph if every element of G that fixes an edge xy fixes already the two incident vertices x and y.

4.1 Group actions on trees

We have already seen that every finite Group that acts on a tree fixes either a vertex or an edge but that we cannot expect the same if we drop the assumption of finiteness. In this section, we will prove an analogue for infinite groups.

First, we need a notion from infinite graph theory.

Definition. Let $\Gamma = (V, E)$ be a graph. A two-way infinite sequence $\ldots x_{-1}x_0x_1 \ldots$ of pairwise distinct vertices and such that $x_ix_{i+1} \in E$ for all $i \in \mathbb{Z}$ is a **double ray**.

Definition. Let the group G act on the tree T without inversion. For $g \in G$, let $R = \ldots x_{-1}x_0x_1 \ldots$ be a g-invariant double ray, i. e. $gR = R$. Then g acts by **translation** on R if there exists $z \in \mathbb{Z}$ with $gx_i = x_{i+z}$ for all $i \in \mathbb{Z}$ and g acts by **reflection** on R if there exists $z \in \mathbb{Z}$ with $gx_{z-i} = x_{z+i}$ for all $i \in \mathbb{Z}$.

For every $g \in G$ we set $|g| := \min\{d(v, gv) \mid v \in V(T)\}$ and call it the **translation length** of g. We call g **elliptic** if $|g| = 0$ and **hyperbolic** otherwise.

Remark 4.1.1. Every elliptic element has a fixed vertex.

Notation. For two vertices x, y in a tree, we denote by $[x, y]$ the unique path between them.

Let us obtain some easy properties of hyperbolic group elements.

Lemma 4.1.2. *Let the group G act on the tree T without inversion. Then the following hold for all hyperbolic $g \in G$.*

(i) *There exists a unique g-invariant double ray R in T. Furthermore, g acts on R by translation.*

(ii) *The order of g is infinite.*

(iii) *We have $d(v, g^z v) = |z| \cdot |g| + 2d(v, R)$ for all $v \in V(T)$ and $z \in \mathbb{Z} \setminus \{0\}$.*

(iv) *We have $|g^z| = |z| \cdot |g|$ for all $z \in \mathbb{Z}$.*

Proof. Let $v \in V(T)$ with $d(v, gv) = |g|$ and let $R := \bigcup_{z\in\mathbb{Z}} [g^z v, g^{z+1} v]$. First, we will show that R is a double ray. It suffices to prove that $[g^{z-1}v, g^z v]$ and $[g^z v, g^{z+1} v]$ meet only in $g^z v$ and thus it suffices to show that $[g^{-1}v, v]$ and $[v, gv]$ meet only in v. Let us suppose that there exists a vertex in the intersection of these two paths that is not v. Then the neighbour w of v on $[v, gv]$ lies in the intersection $[v, g^{-1}v] \cap [v, gv]$. Then $g^{-1}w$ lies in $[g^{-1}v, gv]$ and because of $d(w, gw) = d(w, g^{-1}w) \leq d(v, gv)$ the choice of v implies $w = g^{-1}v$ and $g^{-1}w = v$, which is a contradiction to the action without inversion of G on T, since g fixes the edge vw but neither of the two incident vertices. This, R is a double ray.

Obviously, R is g-invariant. Thus and since g acts on R by translation, it remains to show the uniqueness of R in order to show (i). So let R' be a double ray that is distinct from R. Then there is a vertex u on R that has minimum distance to R' and, since R and R' are distinct, there is a vertex on R of arbitrarily large distance to R' that lies in the same g-orbit as u. But then R' cannot be g-invariant. This shows (i).

Since R is a double ray, infinitely many $g^z v$ must be distinct. Thus, g cannot have finite order, which shows (ii).

We note that the definition of R implies that (iii) holds for all vertices on R. Let $x \in V(T)$ and $z \in \mathbb{Z} \setminus \{0\}$. There exists a unique vertex $y \in R$ with $d(x, y) = d(x, R)$. So we have $[x, g^z x] = [x, y] \cup [y, g^z y] \cup [g^z y, g^z x]$ and thus (iii).

It remains to show (iv), which follows immediately from (iii). □

In the proof of Lemma 4.1.2 we used a property of hyperbolic group elements which is even sufficient for a characterisation of these elements.

Lemma 4.1.3. *Let the group G act without inversion on the tree T. Let $g \in G$. Then g is hyperbolic, if and only if there exists a vertex $v \in V(T)$ with $v \neq gv$ and such that $[v, gv] \cap [gv, g^2 v]$ contains only gv.*

Proof. If g is hyperbolic, then we have already seen in the proof of Lemma 4.1.2 (i) that for every $v \in V(T)$ with $d(v, gv) = |g|$ the intersection $[v, gv] \cap [gv, g^2 v]$ contains only gv. For the other direction, we obtain directly from the assumption that $\bigcup_{z\in\mathbb{Z}} [g^z v, g^{z+1} v]$ is a double ray and g acts on it as translation. In particular, g cannot fix any vertex and we obtain $|g| > 0$. Thus, g is hyperbolic. □

If two elliptic elements have a common fixed vertex, then their product must fix that vertex, too. This obvious obstacle for two elliptic elements to have a hyperbolic product is the only one, as we will see now.

Lemma 4.1.4. *Let the group G acts on the tree T without inversion. Let g, h be elliptic elements of G. Then gh is hyperbolic, if and only if g and h have no common fixed vertex.*

Proof. It suffices to prove that gh is hyperbolic if g and h have no common fixed vertex. Let x be a fixed vertex of g and let y be a fixed vertex of h such that $d(x, y)$ is minimal. By assumption, we have $d(x, y) > 0$. Then $[x, y] \cap [gx, gy] = [x, y] \cap [x, gy]$ only contains the vertex x by minimality of $d(x, y)$. Analogously, $[x, y] \cap [hx, hy] = [x, y] \cap [hx, y]$ only contains the vertex y and both statements also hold for g^{-1} instead of g and h^{-1} instead of h. Since x separates y from $g^{-1}y$, we obtain that $h^{-1}x$ separates $h^{-1}y = y$ from $h^{-1}g^{-1}y$. Thus, and since y separates $h^{-1}x$ from x and x separates y from $gy = ghy$, we obtain that y separates $h^{-1}g^{-1}y$ from $ghy = gy$. Together with Lemma 4.1.3 this implies the assertion. □

Definition. Let the group G act on X. We denote by $\mathrm{Fix}(g)$ the set of **fixed points** of $g \in G$, i. e. $\mathrm{Fix}(g) = \{x \in X \mid gx = x\}$.

Lemma 4.1.5. *Let G be a finitely generated group that act on the tree T without inversion. If each element of G is elliptic, then there exists $x \in V(T)$ with $Gx = \{x\}$.*

Comment. We will see later (Lemma 4.1.9) that Lemma 4.1.5 is wrong if we drop the assumption on G being finitely generated.

Beweis von Lemma 4.1.5. Let S be a finite generating set of G. For every $g \in G$, the set $\mathrm{Fix}(g)$ forms a non-empty subtree of T. Lemma 4.1.4 implies $\mathrm{Fix}(g) \cap \mathrm{Fix}(h) \neq \emptyset$ for all $g, h \in G$. Thus, the finite intersection $\bigcap_{s \in S} \mathrm{Fix}(s)$ is non-empty as well and every element of this intersection is fixed by each element of G. □

Lemma 4.1.6. *Let the group G act on the tree T without inversion. Let $g, h \in G$ be hyperbolic. Let R_g be the unique g-invariant double ray and let R_h be the unique h-invariant double ray. If $R_g \cap R_h$ is finite, then there are $m, n \in \mathbb{N}$ such that g^m and h^n freely generate a free group of rank 2.*

Proof. We set $P := R_g \cap R_h$. Let $m, n \in \mathbb{N}$ with $|P| + 2 \leq \min\{|g^m|, |h^n|\}$. If $P \neq \emptyset$, then let x_g, y_g be the two neighbours of the end vertices of P on $R_g \setminus P$ such that $g^m x_g$ lies in the component of $R_g \setminus P$ that contains y_g. Otherwise, let a be on R_g and b on R_h such that $d(a, b)$ is smallest possible and let x_g and y_g be neighbour of a with the corresponding property. If $P \neq \emptyset$, then let A_g be the component of $T \setminus P$ that contains y_g and let B_g be the component of $T \setminus P$ that contains x_g. Otherwise, let A_g and B_g be the corresponding components of $T - x_g y_g$. Analogously, we choose vertices x_h, y_h on R_h and components A_h, B_h of $T \setminus P$. Since g^m acts on R_g as translation with translation length $|g^m| \geq d(x_g, y_g)$, we obtain $(T \setminus B_g)g^m \subseteq A_g$. Similarly, we obtain $(T \setminus B_h)h^n \subseteq A_h$. Lemma 2.1.12 implies that g^m and h^n freely generate a free subgroup of G. □

Theorem and Definition 4.1.7. *Let the group G act on the tree T without inversion. Then exactly one of the following hold.*

(1) *G acts* **trivially** *on T, i. e., there exists $v \in V(T)$ with $Gv = \{v\}$. We call this action* **elliptic**.

(2) *There are two hyperbolic elements in G that freely generate a free subgroup of G of rank 2 and such that the g-invariant and the h-invariant double rays meet only in a finite subpath. We call this action* **hyperbolic**.

(3) *The action of G on T is not elliptic and there exists a G-invariant double ray in T such that all elements of G act on it as translations. We call this action* **cyclic**.

(4) *The action of G on T is neither elliptic nor cyclic and there exists a G-invariant double ray in T such that all elements of G act on it as translations and reflections. We call this action* **dihedral**.

(5) *The action of G is neither elliptic nor cyclic and there exists a ray R such that, for every $g \in G$, the intersection $R \cap Rg$ is a subray of R. We call this action* **parabolic**.

Proof. Obviously, no two of these five statements can hold simultaneously. Thus, we just have to show that one of the statements holds for the action of G on T.

First, we consider the case that all elements of G are elliptic. We assume that (1) does not hold and show (5). For this, we construct to sequences: one sequence $(x_i)_{i\in\mathbb{N}}$ of vertices and one sequence $(g_i)_{i\in\mathbb{N}}$ of group elements. Let $g_0 \in G$ and $x_0 \in \mathrm{Fix}(g_0)$ be arbitrary. For $i \in \mathbb{N}$, let $g_i \in G$ such that $g_i x_{i-1} \neq x_{i-1}$ and let $x_i \in V(T)$ such that $g_j x_i = x_i$ for all $j \leq i$ and such that $d(x_i, x_{i-1})$ is minimal with this property. Since G is not finitely generated by Lemma 4.1.5 but since the finite intersection $\bigcap_{j\leq i} \mathrm{Fix}(g_j)$ is not empty by the same lemma, we find these two sequences. We set

$$R = [x_0, x_1] \cup [x_1, x_2] \cup \ldots$$

Then R is a ray by minimality of $d(x_i, x_{i-1})$ and by Lemma 4.1.4. Let us now show that for every $g \in G$ the intersection $R \cap gR$ is a ray again. If this does not hold for some $g \in G$, then $R \cap gR$ must be finite. Since g is elliptic, there exists a fixed vertex of g. Let x be such a vertex that has minimal distance to R and let y be the vertex on R that realises this distance. Let $i \in \mathbb{N}$ with $gx_i \neq x_i$ and $d(x_0, x_i) > d(x_0, y)$. We have already mentioned in the proof of Lemma 4.1.5 that $\mathrm{Fix}(g_{i+1})$ and $\mathrm{Fix}(g)$ each span subtrees of T. Since $g_{i+1}x_i \neq x_i \neq gx_i$ and since x_{i+1} and x lie in distinct components of $T - x_i$, the elements g and g_{i+1} have no common fixed vertex. Thus, Lemma 4.1.4 implies that gg_{i+1} is hyperbolic, a contradiction to our assumption. Thus, the action of G on T is parabolic.

Let us now consider the case that G contains hyperbolic elements. Thus, the action of G on T cannot be elliptic. If there are two hyperbolic elements g, h such that the intersection of their two invariant double rays is finite, then Lemma 4.1.6 implies (2). Thus, we may assume that for every two hyperbolic elements g, h the intersection of their invariant double rays R_g and R_h is infinite.

Since $R_g \cap R_h$ must be connected, it is either a ray or a double ray. If it is a double ray for any two hyperbolic elements, then all these double rays must be the same. Since g^f is hyperbolic for all elements $f \in G$ and $R_{g^f} = f^{-1}R_g$ holds because of $g^f(f^{-1}R_g) = f^{-1}gR_g = f^{-1}R_g$ for the unique g^f-invariant double ray R_{g^f}, also the elliptic elements leave R_g invariant. Thus, we have either (3) or (4). So let us assume that there are g and h such that $R' := R_g \cap R_h$ is a ray. We set $f := g^h$. Then f is hyperbolic and we have as before $R_f = h^{-1}R_g$. Since $h^{-1}R' \cap R'$ is infinite and lies in $h^{-1}R_g$, the double ray R_f contains a subray of R'. Let $R := R' \cap R_f$. Let us suppose that (5) does not hold, i. e., there exists $e \in G$ such that $eR \cap R$ is finite. We have $eR = R_{f^{e^{-1}}} \cap R_{g^{e^{-1}}} \cap R_{h^{e^{-1}}}$ and thus one of the three double rays $R_{f^{e^{-1}}}$, $R_{g^{e^{-1}}}$ or $R_{h^{e^{-1}}}$ has only finite intersection with R and thus also finite intersection with one of the three (distinct!) double rays R_f, R_g or R_h. This contradiction shows (5). □

We have already seen the following.

Lemma 4.1.8. *Actions without inversions of finite groups on trees are elliptic.*

Proof. Let G be a finite groups that acts on a tree T without inversion. Let $t \in V(T)$. Then the orbit of t is finite. Thus, the minimal subtree T' of T that contains this orbit is finite, too, and G acts on T'. The middle vertex or edge (depending on the parity of the diameter of T') of a longest path in T' must be fixed by G: otherwise we would obtain a contradiction to the maximal length of that path. Since the action of G on T and thus on T' is without inversion, this fixed vertex or edge must be a vertex and hence the action of G on T' and hence on T is elliptic. □

Lemma 4.1.9. *For every countable group G that is not finitely generated, there is a tree T such that G acts on T without inversion parabolically and every element of G is elliptic.*

Proof. There exists countably many subgroups $U_0 < U_1 < \ldots$ with $\bigcup_{i\in\mathbb{N}} U_i = G$: since G is countable, there exists a countable generating set $S = \{s_i \mid i \in \mathbb{N}\}$ of G; we set $V_i := \langle s_j \mid j \leq i\rangle$ and choose a stricly ascending infinite subsequence. If this sequence would not exists, there would exist some $n \in \mathbb{N}$ such that $\langle s_1, \ldots, s_n\rangle = G$, which is impossible by our assumption.

We consider the graph T whose vertex set is the set of cosets of the subgroups U_i, i. e. $V(T) = \{gU_i \mid g \in G, i \in \mathbb{N}\}$. Two vertices gU_m and hU_n are adjacent if and only if $|m - n| = 1$ and either $gU_m \subseteq hU_n$ or $hU_n \subseteq gU_m$. Let us show that T is connected. Because of $\bigcup_{i\in\mathbb{N}} U_i = G$, there exists $i \in \mathbb{N}$ with $g, h \in U_i$. Then

$$gU_m, gU_{m+1}, \ldots, gU_i = U_i = hU_i, hU_{i-1}, \ldots, hU_n$$

contains a path from gU_m to hU_n.

Every gU_0 has exactly one neighbour, since the cosets of U_1 form a partition of G and since gU_0 and gU_1 are adjacent. Additionally, every gU_i has a unique neighbour in the cosets of U_{i+1}. Thus, T contains no cycle.

Obviously, G acts by left multiplication on T. Since there exists for every $g \in G$ an $i \in \mathbb{N}$ with $g \in U_i$, we have $gU_i = U_i$ and hence g is elliptic. Also, for every U_i there exists g in $U_{i+1} \setminus U_i$. We have $gU_i \neq U_i$ for g. Furthermore, $h^{-1}(hU_i) \neq hU_i$ for every coset hU_i that is distinct from U_i. Thus, there is no vertex fixed by all of G. By the proof of Theorem 4.1.7, the action of G on T must be parabolic. □

In the rest of this section. we will use knowledge of all action of a group on all trees to gain informations about the group.

Definition. A group is **noetherian** if it contains no infinite strictly ascending sequence of subgroups.

A group has property (**AR**) if each of its actions without inversion on trees is either elliptic, cyclic or dihedral.

Theorem 4.1.10. *Let G be a group. The following are equivalent.*

(a) *G is noetherian.*

(b) *Every subgroup of G is finitely generated.*

(c) *Every subgroup of G has property* (AR).

Proof. The implication (a)⇒(b) follows immediately, since every group that is not finitely generated has an infinite strictly ascending sequence of subgroups.

To prove (b)⇒(c), we suppose that some subgroup H of G does not have property (AR). So there exists a tree T such that H acts on T without inversion either hyperbolically or parabolically. First, we consider the case that the action of H on T hyperbolic. Then U contains a free subgroup F of rank 2 and thus a subgroup of F that is not finitely generated in contradiction to (b). So let us assume that the action of H on T is parabolic. Let $R = x_0x_1 \ldots$ be a ray such that for all $h \in H$ the intersection $hR \cap R$ is a ray again. The subgroup $U := \bigcup_{i\in\mathbb{N}} H_{x_i}$ of H is finitely generated by assumption. Thus, there exists $n \in \mathbb{N}$ with $H_{x_i} = H_{x_n}$ for all $i \geq n$. We may assume $n = 0$. If $g \in H$ is elliptic, then it fixes a vertex $v \in V(T)$. Since $R \cap gR$ is a ray and $d(gx_i, v) = d(x_i, v)$ holds for that $i \in \mathbb{N}$ with minimum distance to v, we have $gx_j = x_j$ for all $j \geq i$. Thus, we have $g \in U$ and U is the group of all elliptic elements of H. Since T has no vertex that is fixed by all of G but $Ux_i = \{x_i\}$, there exists a hyperbolic element h in H. Let R_h be the h-invariant double ray in T. For every vertex x on R_h there exists $z \in \mathbb{Z} \setminus \{0\}$ such that $h^z x$ and h^{2zx} lie on R. We have $H_x = U$ because of $H_{h^z x} = H_{h^{2z} x} = U$. Thus, R_h is invariant under U. Let g be another hyperbolic element and let R_g be the unique g-invariant double ray in T. By replacing g by g^{-1} or h by h^{-1}, if necessary, we may assume $gx_j = x_{j+|g|}$ for all x_j on $R \cap R_g$ and $hx_j = x_{j+|h|}$ for all x_j on $R \cap R_h$. We set $f := h^{-|g|}g^{|h|}$. Then f is elliptic, so it lies in U. In particular, we have $R_h = h^{|g|} f R_h = g^{|h|} R_h$. Hence, R_h is $g^{|h|}$-invariant and thus g-invariant. Thus, R_h is invariant under H and the action of H on T is not parabolic. This shows the implication (b)⇒(c).

It remains to show the implication (c)$\Rightarrow$(a). Let us assume that G is not noetherian. So we find an infinite strictly ascending sequence $(H_i)_{i\in\mathbb{N}}$ of subgroups of G. Let $s_0 \in H_0$ and, for $i \geq 1$, let $s_i \in H_i \setminus H_{i-1}$. Then $(U_i)_{i\in\mathbb{N}}$ with $U_i := \langle s_j \mid j \leq i\rangle$ is an infinite strictly ascending sequence of countable subgroups of G and $U := \bigcup_{i\in\mathbb{N}} U_i$ is a countable subgroup of G. Since every finite subset of U lies in some U_i, the group U is not finitely generated. By Lemma 4.1.9 there exists a tree T such that U acts without inversion on T and this action is parabolic. We obtain that U does not have property (AR). $\square$

Now we will look at connections between free products and actions without inversion on trees. For this, we first consider the case that our group is a free product (Proposition 4.1.11) and afterwards we look at the situation of a group action with certain properties in which we will show that the group that we are considering is a free product (Propositions 4.1.13 and 4.1.14).

Proposition 4.1.11. *Let A and B be groups. Then there exists a tree T such that $G := A * B$ acts on T and such that this action has the following properties.*

(1) *The action induced on the edges is free and transitive.*

(2) *The action is without inversion.*

(3) *There are exactly two orbits on the vertex set.*

(4) *There is an edge $uv \in E(T)$ with $A = G_u$ and $B = G_v$.*

Proof. Let T be the graph whose vertex set consists of the left cosets of A and of B and such that two vertices gA and hB are adjacent if $g = h$. The proof that T is a tree is analogous to the corresponding part of the proof of Theorem 2.5.12: we just have to note that we did not use finiteness of the involved groups in that part of the proof.

We also verified $G_A = A$ and $G_B = B$ in that proof. (Again, we did not use finiteness of the involved groups.) Since A and B are adjacent, we obtain (4). The action of G on T has exactly two orbits on the vertices: the left cosets of A form one orbits and the left cosets of B form the other. This implies (2) and (3). It remains to show (1). Note that transitivity directly follows from the definition of the edges. The stabiliser of an edge uv must lies in the intersection of G_u and G_v because of (3). Since this intersection is trivial, G acts freely on the edges of T. $\square$

We need the following version of the ping-pong lemma.

Lemma 4.1.12. *Let the group G act on X. Let $H_1, H_2 \leq G$ with $|H_1| \geq 3$. Let A, B be two non-empty disjoint subsets of X. We assume $gB \subseteq A$ for all $g \in H_1$ with $g \neq 1$ and $gA \subseteq B$ for all $g \in H_2$ with $g \neq 1$. Then the subgroup of G generated by H_1 and H_2 is isomorphic to $H_1 * H_2$.*

Proof. Exercise $\square$

Proposition 4.1.13. *Let T be an infinite tree. Let the group G act on T without inversion with the following properties.*

(1) *G acts transitively and free on the edges of T.*

(2) *G acts transitively on the vertices of T.*

Let $vw \in E(T)$ and $g \in G$ with $gv = w$. Then $G \cong G_v * \langle g \rangle$.

Proof. Since the action of G on T is without inversion and because of $ge \neq e$ for all $e \in E(T)$, Lemma 4.1.3 implies that g is hyperbolic and thus has infinite order by Lemma 4.1.2 (ii).

Let $e = vw$. Obviously, $(\{v\}, \emptyset)$ is a fundamental domain of the action of G on T. Thus, Theorem 1.3.2 implies

$$G = \langle G_v \cup \{h \in G \mid vhv \in E(T)\}\rangle.$$

Note that there are at most two orbits of G_v on the edges incident with v and thus, there are at most two G_v-orbits on the neighbours of v. If $g^{-1}v$ and gv lie in the same G_v-orbit, then we obtain a contradiction since the existence of $h \in G_v$ with $hgv = g^{-1}v$ implies that hg fixes e, which is impossible since $hg \neq 1$. Thus, there are exactly two G_v-orbits on the neighbours of v. Those in the same orbit as w are obtained as the image of v under hg for some suitable $h \in G_v$ and those in the same orbit as $g^{-1}v$ are obtained as the image of v under hg^{-1} for some suitable $h \in G_v$. Thus, we have shown

$$G = \langle G_v \cup \{g\}\rangle.$$

Let A be the set of all those vertices that can be reached by a path from v that contains either gv or $g^{-1}v$ and let B be the set of all those vertices that can be reached by a path from v that contains neither gv nor $g^{-1}v$. Obviously, we have $g^z B \subseteq A$ for all $z \in \mathbb{Z} \setminus \{0\}$. Since G acts freely on the edges of T and because of $g \notin G_v$, we have $hgv \notin \{gv, g^{-1}v\}$ and $hg^{-1}v \notin \{gv, g^{-1}v\}$ for all $h \in G_v \setminus \{1\}$. Thus, we obtain $hA \subseteq B$. Lemma 4.1.12 implies $G \cong G_v * \langle g \rangle$. □

A modification of the proof of Proposition 4.1.13 leads to the following proposition.

Proposition 4.1.14. *Let T be an infinite tree that is not a double ray. Let the group G act on T without inversion with the following properties.*

(1) *G acts transitively and free on the edges of T.*

(2) *There are exactly two g-orbit on the vertex set of T.*

Then we have $G \cong G_v * G_w$ for adjacent vertices $v, w \in V(T)$.

Proof. Exercise □

4.2 Fundamental groups of graphs

Definition. A **graph with involution** is an oriented multigraph Γ together with a map $\bar{\cdot}\colon E(\Gamma) \to E(\Gamma)$ with $\bar{e} \neq e$ and $\bar{\bar{e}} = e$ and such that $\bar{e}$ is an edge from v to u if e is an edge from u to v. We denote by $i(e)$ the initial vertex of e and by $t(e)$ its terminal vertex. (So we have $t(e) = i(\bar{e})$ and $t(\bar{e}) = i(e)$.)

Example 4.2.1. In a (multi-)graph, we can replace every edge by two inversely directed edges between the same vertices. That way, we obtain a graph with involution.

Definition. Let Γ be a graph with involution. Let $K = v_0e_0v_1 \dots e_{k-1}v_k$ be a directed walk in Γ, i. e., the edge e_i satisfies $i(e_i) = v_i$ and $t(e_i) = v_{i+1}$. If $\bar{e}_i = e_{i+1}$, then we call $v_iv_{i+1}v_{i+2}$ a **spike**. If K has no spike, then it is **spikeless**. If $v_iv_{i+1}v_{i+2}$ is a spike, then

$$K' = v_0e_0 \dots e_{i-1}v_ie_{i+2} \dots v_k$$

is obtained from K by **removing** a spike. Let $K_1 = v_0e_0v_1 \dots e_{k-1}v_k$ and $K_2 = w_0f_0w_1 \dots f_{\ell-1}w_\ell$ be two directed walks with $v_k = w_0$. Then

$$K_1K_2 := v_0e_0v_1 \dots e_{k-1}v_kf_0w_1 \dots f_{\ell-1}w_\ell$$

is a directed walk as well, the **composition** of K_1 and K_2.

Remark 4.2.2. In a graph with involution, every directed walk can be transferred by removing spikes to a spikeless directed walk.

Definition. Let K, K' be two directed walks of a graph Γ with involution. We write $K \sim K'$ if there exists a sequence $K = K_0, \dots, K_n = K'$ of directed walks such that with K_i is obtained from K_{i-1} or K_{i-1} is obtained from K_i by removing a spike.

Remark 4.2.3. The relation $\sim$ is an equivalence relations on the directed walks of a graph with involutions.

Lemma 4.2.4. *Let $\Gamma = (V, E)$ be a connected graph with involution. Then every equivalence class of $\sim$ contains exactly one spikeless walk.*

Proof. Exercise □

Lemma 4.2.5. *Let K_1, K_2, L_1, L_2 be four directed walks of a graph with involution such that the compositions K_1K_2 and L_1L_2 exist. If $K_1 \sim L_1$ and $K_2 \sim L_2$, then we have $K_1K_2 \sim L_1L_2$.*

Proof. Let $K_1 = M_1, \dots, M_m = L_1$ be a sequence of directed walks that verify the equivalence $K_1 \sim L_1$ and let $K_2 = N_1, \dots, N_n = L_2$ be an analogous sequence for $K_2 \sim L_2$. Then $M_1N_1, \dots, M_mN_1, \dots, M_mN_n$ is a sequence that shows $K_1K_2 \sim L_1L_2$. □

Definition. Let Γ be a graph with involution. Let $\pi_1(\Gamma, v)$ with $v \in V(\Gamma)$ be the set of equivalence classes of $\sim$ on the directed walks in Γ that start and end at v. We define a multiplication on $\pi_1(\Gamma, v)$ by $[K][L] := [KL]$. Lemma 4.2.5 implies that this multiplication is well-defined.

Lemma 4.2.6. *Let Γ be a connected graph with involution and let $v \in V(\Gamma)$.*

(1) $\pi_1(\Gamma, v)$ *is a group.*

(2) *If $u \in V(\Gamma)$, then $\pi_1(\Gamma, v)$ and $\pi_1(\Gamma, u)$ are isomorphic.*

Proof. Since the multiplication is well-defined, we obtain (1).

To prove (2), we choose a directed v–u walk K and note that every element $[L]$ of $\pi_1(\Gamma, v)$ can be transferred into an element $[K^{-1}LK]$ of $\pi_1(\Gamma, u)$ by 'conjugation with K', where K^{-1} is the reverse of the walk K obtained by replacing each edge e on K by $\bar{e}$. Conversely, every element $[M]$ of $\pi_1(\Gamma, u)$ can be transferred into an element $[KMK^{-1}]$ of $\pi_1(\Gamma, v)$ by 'conjugation with K^{-1}'. The corresponding maps are inverse to each other. Furthermore, these maps are homomorphisms, so we obtain (2). □

Definition. The **fundamental group** $\pi_1(\Gamma)$ of a connected graph Γ with involution is an element of the isomorphism class of the groups $\pi_1(\Gamma, v)$ for $v \in V(\Gamma)$.

Comment. Eventhough the elements of the fundamental group are equivalence classes of directed walks, we usually only look at representatives of such equivalence classes.

Remark 4.2.7. Accordingly to Example 4.2.1, our definitions can be transferred directly to (multi-)graphs.

Example 4.2.8. If T is a tree, then $\pi_1(T) = 1$.

Definition. For a (directed) multigraph Γ and a subset $F \subseteq E(\Gamma)$ of the edge set, let Γ/F be the (directed) multigraph whose vertex set is the set of components of $(V(\Gamma), F)$. For every edge e in $E(\Gamma) \setminus F$, we add an edge between the components that contain the incident vertices of e. We note that loops and multi-edges may be created that way.

Lemma 4.2.9. *Let Γ be a connected multigraph and let T be a subtree of Γ. Then we have $\pi_1(\Gamma) \cong \pi_1(\Gamma/E(T))$.*

Proof. It suffices to prove the assertion for graphs with involutions, in which case the 'tree' then contains for every edge e also the edge $\bar{e}$. We consider the fundamental group $\pi_1(\Gamma, x)$ with respect to a vertex $x \in V(T)$ and the fundamental group $\pi_1(\Gamma/E(T))$ with respect to the vertex v_T of $\Gamma/E(T)$ that contains all vertices of T. We define a map $\varphi_T \colon \pi_1(\Gamma) \to \pi_1(\Gamma/E(T))$. For a closed directed walk $K = v_0e_0v_1 \ldots v_k$ with $v_0 = x = v_k$ in Γ, let $\varphi_T(K)$ be the canonical image of K in $\Gamma/E(T)$: we replace every maximal subwalk in T by v_T and replace every edge incident with exactly one vertex of T by its canonical image in $\Gamma/E(T)$. Obviously, φ_T is a well-defined group homomorphism. It remains to show that φ_T is bijective.

Let $K = v_0e_0v_1 \dots e_{k-1}v_k$ be a closed directed walk in $\Gamma/E(T)$. By replacing the vertex $v_i = v_T$ for $i \neq 0$ and $i \neq k$ by a directed walk hat connects the end vertices of the edges e_i and e_{i+1} in T and adding a directed walk from x to the initial vertex of e_0 in $V(\Gamma)$ and one from the terminal vertex of the edge e_{k-1} in $V(\Gamma)$ to x, we obtain a closed directed walk in Γ that starts and ends at x. Obviously, this will be mapped by φ_T to K. Thus, φ_T is surjective.

Let $K = v_0e_0v_1 \dots e_{k-1}v_k$ with $v_0 = x = v_k$ be a spikeless directed walk in the kernel of φ_T. Then we can view K as composition $K_1L_1K_2 \dots L_{m-1}K_m$ of directed walks, where the walks K_i lies in T and the walks L_i lie in $\Gamma \setminus T$. Let us suppose that K is non-trivial. Since $L_1L_2 \dots L_m$ in $\Gamma/E(T)$ is equivalent to the trivial walk and, by Lemma 4.2.4, the trivial walk is the unique spikeless directed walk in its equivalence class, the walk $L_1L_2 \dots L_m$ must contain a spike. This spike cannot lie in any of the L_i, so it is created by the composition of L_i an L_{i+1} for some $1 \leq i \leq m-1$. This spike corresponds to a directed walk $vew\bar{e}v$ in Γ. Thus, K_{i+1} is a closed directed walk in T with starting and end vertex w. Since $\pi_1(T)$ is trivial by Example 4.2.8 and since K_{i+1} is spikeless, K_{i+1} is the trivial walk. This contradicts the choice of K having no spike and thus φ_T is injective. □

Lemma 4.2.10. *For every connected multigraph $\Gamma = (V, E)$, the fundamental group $\pi_1(\Gamma)$ is a free group.*

If Γ is finite, then $|E| - |V| + 1$ is the rank of $\pi_1(\Gamma)$.

Proof. As before, it suffices for the first part to show the assertion for graphs with involution. So let us assume that Γ is a graph with involution. Let T be a spanning tree of Γ. (Again, T contains for every edge e also the edge $\bar{e}$.) By Lemma 4.2.9, we have $\pi_1(\Gamma) \cong \pi_1(\Gamma/T)$ and we may assume that Γ has exactly one vertex.

Let S be a minimal subset of $E(\Gamma)$ such that

$$E(\Gamma) = S \cup \{\bar{s} \mid s \in S\}.$$

We will show that $\pi_1(\Gamma)$ and the free group freely generated by S are isomorphic. Every element of $\pi_1(\Gamma)$ contains a unique directed walk without spikes as representative by Lemma 4.2.4. By replacing each edge $\bar{s}$ by s^{-1} and by dropping the vertices, this walk corresponds to a reduced word over $S \cup S^{-1}$, which is trivial if and only if the walk is trivial. Conversely, every reduced word over $S \cup S^{-1}$ corresponds to a directed walk in Γ by replacing s^{-1} by $\bar{s}$ and inserting the correct vertices. Since these correlations respect compositions of walks and concatenations of words, $\pi_1(\Gamma)$ and the free group freely generated by S must be isomorphic.

Now let Γ be a connected multigraph. Then we have

$$|E| - |V| + 1 = |E(\Gamma/E(T))| - |V(\Gamma/E(T))| + 1$$

by the considerations for the first part and hence, we obtain the second part. □

Comment. The existence of a spanning tree can be shown using Zorn's lemma. Furthermore, the existence of spanning trees for all (multi-)graphs is equivalent to the axiom of choice over the system ZF.

Definition. Let a group G act on a graph Γ. Then the **quotient graph** Γ/G is defined as multigraph whose vertex set consists of the orbits of G in $V(\Gamma)$ and whose edge set is induced by the orbits of G in $E(\Gamma)$.[1]

Example 4.2.11. Let G be a group with generating set S and let Γ be the Cayley graph of Γ and S. Then Γ/G is a graph with exactly one vertex and at most $|S|$ loops. (Note that there may be fewer loops if $S \cap S^{-1}$ is not empty.)

Remark 4.2.12. Let G be a group acting on the graph Γ. Then the canonical projection $\varrho\colon \Gamma \to \Gamma/G$ is a surjective graph homomorphism, i. e., adjacent vertices are mapped onto adjacent vertices by ϱ.

Proposition and Definition 4.2.13. *For every connected graph Γ there exists a tree T_Γ and a free action of $\pi_1(\Gamma)$ on T_Γ such that $T_\Gamma/\pi_1(\Gamma) \cong \Gamma$. The tree T_Γ is the* **universal cover** *of Γ.*

Proof. Let T be a spanning tree of Γ. If S is an orientation of the edges of $\Gamma - T$, then we have seen in the proof of Lemma 4.2.10 that $\pi_1(\Gamma)$ is isomorphic to the free group that is freely generated by S. In particular, there exists a canonical map $\varphi\colon S \to \pi_1(\Gamma)$ that maps each $s \in S$ to the uniquely determined cycle in $T + s$. (Note that, formally, we have to replace each edge of T by two conversely oriented edges to obtain a directed cycle.) We define a graph T_Γ as follows: let

$$\bigcup_{g \in \pi_1(\Gamma)} \{(g, v) \mid v \in V(T)\}$$

be its vertex set and let the union of the two sets

$$E_1 := \bigcup_{g \in \pi_1(\Gamma)} \{\{(g, u), (g, v)\} \mid \{u, v\} \in E(T)\}$$

and

$$E_2 := \bigcup_{s=(u,v) \in S} \bigcup_{g \in \pi_1(\Gamma)} \{\{(g, u), (gs, v)\}\}$$

be its edge set. Note that E_1 implies that T_Γ consists of copies of T and E_2 describes the edges between these copies of T.

To show that T_Γ satisfies the assertion will be left as exercise. □

We have already verified the following corollary in the proof of Proposition 4.2.13.

Corollary 4.2.14. *Let Γ be a graph, T a spanning tree of Γ and T_Γ the universal covering of Γ (constructed with respect to the spanning tree T). Then T_Γ contains an isomorphic copy of T that is mapped onto T by the canonical projection $\varrho\colon T_\Gamma \to \Gamma$.* □

[1] I. e., there exists a bijection between the edges of Γ/G and the orbits of edges in Γ such that each edge e in Γ/G is incident with those vertices that are the orbits of the incident vertices of any edge f in the image of e. Note that this is independent of the choice of f.

4.3 Graphs of groups

Remark 4.3.1. Let us consider an action without inversion of a group G on a graph Γ. Then the following holds.

(1) Γ/G is a multigraph $\widehat{\Gamma}$.

(2) For every vertex $v \in V(\widehat{\Gamma})$ there exists a group G^v such that $G^v \cong G_x$ for all $x \in V(\Gamma)$ with $Gx = v$.

(3) For every edge $e \in E(\widehat{\Gamma})$ there exists a group G^e such that $G^e \cong G_f$ for all $f \in E(\Gamma)$ that are mapped onto e.

(4) For every edge $e \in E(\widehat{\Gamma})$ there are two injective group homomorphisms $\iota_{e,i(e)} \colon G^e \to G^{i(e)}$ and $\iota_{e,t(e)} \colon G^e \to G^{t(e)}$.

In view of this remark, let us make the following definition.

Definition. A **graph of groups** is a triple $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$, where Γ is a connected graph with involution and $\mathcal{G}$ is a map that that assigns to each vertex v a group G_v and to each edge e a group G_e such that $G_e = G_{\bar{e}}$ and Λ is a family of monomorphisms $\alpha_e \colon G_e \to G_{i(e)}$, one for every edge e. We call the groups G_v the **vertex groups** and the groups G_e the **edge groups**.

Example 4.3.2. Let G be a group that acts without inversion on a graph Γ.

(1) The quotient graph Γ/G defines a graph of groups according to Remark

(2) If $\mathcal{G}$ maps each vertex v and each edge e to their stabiliser G_v or G_e and Λ is the family of the canonical embeddings of the edge stabilisers into the stabilisers of the vertices incident with that edge, then $(\mathcal{G}, \Gamma, \Lambda)$ is a graph of groups. 4.3.1.

Remark 4.3.3. Generally, we will denote by G_v and G_e the vertex and edge groups. Even though this collides with the notion for the stabilisers of vertices and edges , we stick to it, in particular, since the stabilisers will be the important examples and thus play a major role for us. In case it is not obvious which notion we mean, we will explicitly name it.

Next, we will define the fundamental group of graphs of groups in two different ways such that the groups obtained by each definition are isomorphic in a canonical way. Compared to Section 4.2, we need a new definition of the fundamental group to take the function $\mathcal{G}$ and the family Λ into account.

Definition. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of group and let T be a spanning tree of Γ. This time, T contains at most one element of $\{e, \bar{e}\}$ for every $e \in E(\Gamma)$. For every vertex group G_v, let $\langle S_v \mid R_v \rangle$ be a presentation of G_v and, for every edge group G_e, let S_e be a generating set of G_e. Then the **fundamental group** of $\mathbb{G}$ (with respect to T) is defined by the presentation

$$\pi_1(\mathbb{G}, T) := \left\langle \dot{\bigcup}_{v \in V(\Gamma)} S_v \cup \{g_e \mid e \in E(\Gamma)\} \mid \bigcup_{v \in V(\Gamma)} R_v \cup N \right\rangle,$$

where the g_e are new generators and the set N of relators is defined as follows:

$$N := \{g_e \mid e \in E(T)\} \cup \{g_e g_{\bar{e}} \mid e \in E(\Gamma)\} \\ \cup \{g_e \alpha_{\bar{e}}(s) g_e^{-1} (\alpha_e(s))^{-1} \mid e \in E(\Gamma), s \in S_e\}.$$

Let us look at some examples of these fundamental groups, examples where the graph essentially only has one edge. That mean, that its edge set is $\{e, \bar{e}\}$ for some e.

Example 4.3.4. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups with $E(\Gamma) = \{e, \bar{e}\}$. Let T be a spanning tree of Γ and let $\langle S_v \mid R_v \rangle$ be a presentation of G_v for every vertex v and let S_e be a generating set of G_e for every edge e

(1) If Γ has exactly two vertices u, v, then T contains an edge of Γ and in the presentation of $\pi_1(\mathbb{G}, T)$ all g_e with $e \in E(\Gamma)$ are trivial. Using Tietze transformations (removing the generators g_e) we obtain the presentation

$$\pi_1(\mathbb{G}, T) = \left\langle S_u \cup S_v \mid R_u \cup R_v \cup \{\alpha_{\bar{e}}(s)(\alpha_e(s))^{-1} \mid e \in E(\Gamma), s \in S_e\} \right\rangle.$$

We directly obtain

$$\pi_1(\mathbb{G}, T) \cong G_u *_{G_e} G_v,$$

where the monomorphisms for the free product with amalgamations are α_e and $\alpha_{\bar{e}}$.

(2) If Γ has exactly one vertex v, then the edges of Γ are loops. Using Tietze transformations, we can remove the generator $g_{\bar{e}}$ (but not at the same time g_e) from the presentation of the fundamental group and we obtain

$$\pi_1(\mathbb{G}, T) = \left\langle S_v \cup \{g_e\} \mid R_v \cup \{g_e \alpha_{\bar{e}}(s) g_e^{-1} (\alpha_e(s))^{-1} \mid s \in S_e\} \right\rangle.$$

Thus, we have

$$\pi_1(\mathbb{G}, T) \cong G_v *_{\alpha_{\bar{e}}^{-1} \alpha_e},$$

where the isomorphism for the HNN extensions is $\alpha_{\bar{e}}^{-1} \alpha_e$ that maps the images of G_e under α_e onto those of $\alpha_{\bar{e}}$.

Let us now move to the second definition of the fundamental group of a graph of groups. While the first definition depends on the choice of a spanning tree, the second one will depend on the choice of a vertex. Later, we will show the equivalence of these two definitions and thereby show that the groups are isomorphic for any choice of spanning trees or vertices. Our second definition of the fundamental group follows the strategy of Section 4.2 that we still have to adapt to to our new situation.

Definition. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups. A **$\mathbb{G}$-walk** (of **length** $|P| = k$) from $u \in V(\Gamma)$ to $v \in V(\Gamma)$ is a sequence $P = g_0 e_1 \dots e_k g_k$, where $e_1 \dots e_k$ induces a directed walk in Γ and such that $g_0 \in G_u$, $g_k \in G_v$ and $g_i \in G_{t(e_i)} = G_{i(e_{i+1})}$ for all $0 < i < k$. For $0 \leq i \leq j \leq k$, the sequence

$g_i e_{i+1} \dots e_j g_j$ is a $\mathbb{G}$-**subwalk** of P. If $P = g_0 e_1 \dots e_k g_k$ is a $\mathbb{G}$-walk from u to v and $Q = h_0 f_1 \dots f_\ell h_\ell$ is a $\mathbb{G}$-walk from v to w, then their **concatenation** is the $\mathbb{G}$-walk

$$PQ = g_0 e_1 \dots e_k (g_k h_0) f_1 \dots f_\ell h_\ell$$

from u to w. Two $\mathbb{G}$-walks P and Q are **elementarily equivalent** if Q can be obtained from P by one of the following operations or their reverses:

(i) Replace a $\mathbb{G}$-walk $ge(\alpha_{\bar{e}}(c))\bar{e}g'$ with $e \in E(\Gamma)$, $c \in G_e$ and $g, g' \in G_{i(e)}$ by $g(\alpha_e(c))g'$.

(ii) Replace a $\mathbb{G}$-walk geg' with $e \in E(\Gamma)$, $g \in G_{i(e)}$ and $g' \in G_{t(e)}$ by

$$(g(\alpha_e(c)))e((\alpha_{\bar{e}}(c))^{-1}g'),$$

where c is an element of G_e.

Let $\sim$ be a relation on the $\mathbb{G}$-walks such that for two $\mathbb{G}$-walks P, Q we have $P \sim Q$ if there exists a sequence $P = P_1 \dots P_k = Q$ of $\mathbb{G}$-walks such that P_i and P_{i+1} are elementarily equivalent for all $1 \leq i < k$. Obviously, $\sim$ is an equivalence relation.

Later, we will take a closer look at the elements of the equivalence classes and show that the minimal elements with respect to the first operation, which lie in the same equivalence class, are equivalent by using only the second operation. But first, we are interested in the second definition of the fundamental group and the equivalence of both definitions.

Definition. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. Then the equivalence classes of $\mathbb{G}$-walks from v to v form the **fundamental group** $\pi_1(\mathbb{G}, v)$ of $\mathbb{G}$ (with respect to v), where the multiplication is defined by concatenation: $[P][Q] := [PQ]$.

Remark 4.3.5. That the multiplication on $\pi_1(\mathbb{G}, v)$ is well-defined follows by an argumentation similar to the one we used in the proof of Lemma 4.2.5. This then directly implies that the fundamental group with respect to a vertex $v \in V(\Gamma)$ is a group.

Proposition 4.3.6. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, let $v \in V(\Gamma)$ and let T be a spanning tree of Γ. Then the map*

$$\varphi \colon \pi_1(\mathbb{G}, v) \to \pi_1(\mathbb{G}, T), [g_0 e_1 \dots e_k g_k] \mapsto g_0 g_{e_1} \dots g_{e_k} g_k$$

is a group isomorphism.

Proof. From the definition of the relation $\sim$ and the third set in the definition of N in the definition of $\pi_1(\mathbb{G}, T)$, we obtain that φ is well-defined. It is a homomorphism by the definition of the concatenation of $\mathbb{G}$-walks. Thus, it remains to show that φ is bijective. For this, we construct the inverse map of φ.

Let us construct a map from the generating set of $\pi_1(\mathbb{G}, T)$ to $\pi_1(\mathbb{G}, v)$. For $u \in V(\Gamma)$, let $P_u = x_0 e_1 x_1 \dots e_k x_k$ be the walk corresponding to the unique path in T from v to u and let $P_{u,\mathbb{G}} = 1e_1 1 \dots e_k 1$ be the corresponding $\mathbb{G}$-walk. For $g \in G_u$ with $u \in V(\Gamma)$ we define the image of g as the equivalence class of $P_{u,\mathbb{G}} g P_{u,\mathbb{G}}^{-1}$. For $e \in E(\Gamma)$ we define the image of g_e as the equivalence class of $P_{i(e),\mathbb{G}} e P_{t(e),\mathbb{G}}^{-1}$.

It is easy to verify that the relators in the definition of $\pi_1(\mathbb{G}, T)$ are all mapped onto the equivalence class of the trivial $\mathbb{G}$-walk. Using the universal property of group presentations (Theorem 2.3.4), we obtain that the map we just defined can be extended to a homomorphism

$$\psi\colon \pi_1(\mathbb{G}, T) \to \pi_1(\mathbb{G}, v).$$

Obviously, this is the reverse map of φ. □

Corollary 4.3.7. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, let $v \in V(\Gamma)$ and let T be a spanning tree of Γ. Using the notations from the proof of Proposition 4.3.6, the set*

$$\{[P_{u,\mathbb{G}} g P_{u,\mathbb{G}}^{-1}] \mid u \in V(\Gamma), g \in G_u\} \cup \{[P_{i(e),\mathbb{G}} e P_{t(e),\mathbb{G}}^{-1}] \mid e \in E(\Gamma \smallsetminus T)\}$$

is a generating set of $\pi_1(\mathbb{G}, v)$.

Proof. The assertion follows directly from the proof of Proposition 4.3.6, since the given set is the image of the generating set of $\pi_1(\mathbb{G}, T)$. □

By multiple applications of Proposition 4.3.6, we obtain the following corollary.

Corollary 4.3.8. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, let $v, w \in V(\Gamma)$ and let T, T' be spanning tree of Γ. Then we have*

$$\pi_1(\mathbb{G}, v) \cong \pi_1(\mathbb{G}, w) \cong \pi_1(\mathbb{G}, T) \cong \pi_1(\mathbb{G}, T').$$ □

Definition. The **fundamental group** $\pi_1(\mathbb{G})$ of a graph of groups $\mathbb{G}$ is a group of the isomorphism class of the fundamental groups with respect to an arbitrary spanning tree or an arbitrary vertex.

Definition. We call a $\mathbb{G}$-walk **$\mathbb{G}$-reduced** if we cannot apply operations of type (i) from the definition of elementary equivalence. For every edge group G_e, let X_e be a transversal (of the right cosets) of $\alpha_e(G_e)$ in $G_{i(e)}$. A $\mathbb{G}$-reduced $\mathbb{G}$-walk $g_0 e_1 g_1 \dots e_k g_k$ is a **normal form** if $g_i \in X_{\bar{e}_i}$ for all $0 < i \leq k$.

We will show (similar to previous situations) that every equivalence class contains a unique normal form.

Theorem 4.3.9. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $P = g_0 e_1 \dots e_k g_k$ and $Q = h_0 f_1 \dots f_\ell h_\ell$ be two $\sim$-equivalent $\mathbb{G}$-reduced $\mathbb{G}$-walks. Then we have $k = \ell$ and $e_i = f_i$ for all $1 \leq i \leq k$ and there are $a_i \in G_{e_i}$ such that*

1. $g_0 = h_0(\alpha_{e_1}(a_0))$ *and*

2. $g_i = (\alpha_{\bar{e}_i}(a_i^{-1}))h_i(\alpha_{e_{i+1}}(a_{i+1}))$ *for all* $1 \leq i < k$ *and*

3. $g_k = (\alpha_{\bar{e}_k}(a_k^{-1}))h_k$.

In particular, every equivalence class contains a unique normal form.

Proof. It suffices to show the additional statement since then every two equivalent $\mathbb{G}$-reduced $\mathbb{G}$-walks are equivalent (by operations of type (ii)) to the same normal form. The according operations put after another (with inverting the second one) show that both $\mathbb{G}$-reduced $\mathbb{G}$-walks are equivalent using only the operation (ii) – which is exactly what we claim in the assertion of this theorem.

Obviously, every $\mathbb{G}$-walk is equivalent to a $\mathbb{G}$-reduced $\mathbb{G}$-walk and also to a normal form, where the second claim follows by replacing g_i by $\alpha_{\bar{e}_i}(c_i)x_i$ for $x_i \in X_{\bar{e}_i}$ and $c_i \in G_{\bar{e}_i} = G_{e_i}$ and then pushing $\alpha_{\bar{e}_i}(c_i)$ backwards across the edge e_i using the second operation. Inductively, we obtain the second claim.

For vertices $u, v \in V(\Gamma)$, let $P_{\mathbb{G}}(u,v)$ be the set of $\mathbb{G}$-walks from u to v and let $N_{\mathbb{G}}(u,v)$ be the set of normal forms from u to v. For a normal form $P = g_0e_1 \dots e_kg_k$ from u to v and for $g \in G_u$, we set

$$\varphi(g, g_0e_1 \dots e_kg_k) := (gg_0)e_1 \dots e_kg_k$$

and, if e is an edge with $t(e) = u$, we set

$$\varphi(1e1, g_0e_1 \dots e_kg_k) := \begin{cases} \alpha_e(g_e)g_2e_2 \dots e_kg_k, & \text{if } e = \bar{e}_1 \text{ and } x_e = 1, \\ \alpha_e(g_e)ex_ee_1 \dots e_kg_k, & \text{otherwise,} \end{cases}$$

where $g_e \in G_e$ and $x_e \in X_e$ with $\alpha_{\bar{e}}(g_e)x_e = g_0$. Every $\mathbb{G}$-walk can be written as concatenation of $\mathbb{G}$-walks $g \in G_w$ or $1e1$. For a normal form N from u to v and a $\mathbb{G}$-walk P with $P = P_1 \dots P_n$, where the P_i are of the just described form, we define recursively

$$\varphi(P, N) := \varphi(P_1, \varphi(P_2 \dots P_n, N)).$$

Using this definition, we obviously obtain

$$\varphi(N, 1) = N$$

for every normal form N from u to v and for the trivial $\mathbb{G}$-walk 1 from v to v. We want to verify the following for every two equivalent $\mathbb{G}$-walks P_1, P_2 from u to v and every normal form N from v to w:

$$\varphi(P_1, N) = \varphi(P_2, N).$$

If we have this, then we obtain for every two equivalent normal forms N_1, N_2 from u to v:

$$N_1 = \varphi(N_1, 1) = \varphi(N_2, 1) = N_2$$

And hence every equivalence class of $\sim$ contains exactly one normal form. It suffices to verify

$$\varphi(P_1, N) = \varphi(P_2, N)$$

for all $\mathbb{G}$-walks P_1, P_2 that can be transferred to each other by a single operation of the elementary equivalence. First, let $P = P_1 ge(\alpha_{\bar{e}}(c))\bar{e}g'P_2$ be a $\mathbb{G}$-walk. If we prove

$$\varphi(ge\alpha_{\bar{e}}(c)\bar{e}g', N) = \varphi(g\alpha_e(c)g', N)$$

for normal forms N from v to $t(e)$, then we obtain the following for normal forms N from v to the end vertex of P:

$$\begin{aligned}
&\varphi(P_1 ge\alpha_{\bar{e}}(c)\bar{e}g'P_2, N)\\
&= \varphi(P_1, \varphi(ge\alpha_{\bar{e}}(c)\bar{e}g', \varphi(P_2, N)))\\
&= \varphi(P_1, \varphi(g\alpha_e(c)g', \varphi(P_2, N)))\\
&= \varphi(P_1 g\alpha_e(c)g'P_2, N).
\end{aligned}$$

So let us prove in this situation the remaining equation. For this, let $N = g_N e_1 g_1 \dots e_k g_k$ be a normal form and set $N^- := 1e_1 g_1 \dots e_k g_k$. Let $ge\alpha_{\bar{e}}(c)\bar{e}g'$ be as just described and let $x_e \in X_{\bar{e}}$ be an element of the transversal of $\alpha_{\bar{e}}(G_e)$ in $G_{t(e)}$ and let $b \in G_{e'}$ such that $g'g_N = \alpha_e(b)x_e$. If $e \neq e_1$, then we have:

$$\begin{aligned}
&\varphi(ge\alpha_{\bar{e}}(c)\bar{e}g', N)\\
&= \varphi(ge\alpha_{\bar{e}}(c), \varphi(1\bar{e}1, \alpha_e(b)x_e N^-))\\
&= \varphi(ge1, \varphi(\alpha_{\bar{e}}(c)\alpha_{\bar{e}}(b)\bar{e}x_e N^-))\\
&= \varphi(g\alpha_e(c)\alpha_e(b)1\bar{e}1e1, x_e N^-)\\
&= g\alpha_e(c)g'N\\
&= \varphi(g\alpha_e(c)g', N).
\end{aligned}$$

The case $e = e_1$ follows analogously. Also, in case of the second operation we obtain the claim by a similar argumentation. □

Let us get two corollaries from Theorem 4.3.9.

Corollary 4.3.10. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, let $v \in V(\Gamma)$ and let $P = g_0 e_1 \dots e_k g_k$ be a $\mathbb{G}$-reduced $\mathbb{G}$-walk from v to v. Then we have $[P] = 1 \in \pi_1(\Gamma, v)$ if and only if $k = 0$ and $g_0 = 1 \in G_v$.* □

Corollary 4.3.11. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, let $u, v \in V(\Gamma)$ and let $P = g_0 e_1 \dots e_k g_k$ be a $\mathbb{G}$-walk from v to u. Then the map*

$$G_u \to \pi_1(\mathbb{G}, v), \; g \mapsto [PgP^{-1}]$$

is a group monomorphism. □

4.4 Structure theorem of the Bass-Serre theory

Now we are aiming at obtaining an analogue for the universal covering of graphs via trees (using the fundamental group of graphs, see Proposition 4.2.13) in the situation of graph of groups and their fundamental group.

Definition. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. On the set of $\mathbb{G}$-walks that start at v we will define a relation $\approx$ via $P_1 \approx P_2$ if and only if

- P_1 and P_2 end at the same vertex w and
- there exists $g \in G_w$ with $P_1 \sim P_2 g$.

Obviously, $\approx$ is an equivalence relation on the $\mathbb{G}$-walks that start at v. We denote by $P_w^{\mathbb{G}}$ the equivalence class of a $\mathbb{G}$-walk P from v to w. Similar to the proof of Theorem 4.3.9, every equivalence class $P_w^{\mathbb{G}}$ of $\approx$ contains exactly one representative of the form $x_0 e_1 x_1 \dots x_{k-1} e_k 1$ with $x_i \in X_i$, where X_i is a transversal of $\alpha_{\bar{e}_i}(G_{e_i})$ in $G_{i(\bar{e}_i)}$ for $i > 0$.

Let us define a graph $\widetilde{\mathbb{G}}_v$: its vertex set is the set of equivalence classes of $\approx$. Two vertices[2] $P_u^{\mathbb{G}}, Q_w^{\mathbb{G}}$ are adjacent by the edge $f := (P_u^{\mathbb{G}}, e, Q_w^{\mathbb{G}})$ with $e \in E(\Gamma)$ if $i(e) = u$ and $t(e) = w$ and if there exists $g \in G_u$ with $Pge1 \in Q_w^{\mathbb{G}}$. (Note that this definition does not depend on the choice of P.) Let $i(f) = P_u^{\mathbb{G}}$ and $t(f) = Q_w^{\mathbb{G}}$. The involution is defined by $\overline{(P_u^{\mathbb{G}}, e, Q_w^{\mathbb{G}})} := (Q_w^{\mathbb{G}}, \bar{e}, P_u^{\mathbb{G}})$.

Remark 4.4.1. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. Let $P = g_0 e_1 \dots e_k g_k$ and $Q = h_0 f_1 \dots f_\ell h_\ell$ be two $\mathbb{G}$-reduced $\mathbb{G}$-walks from v to u and w such that $P_u^{\mathbb{G}}$ and $Q_w^{\mathbb{G}}$ are adjacent in $\widetilde{\mathbb{G}}_v$. We may assume that $k \leq \ell$. Let $(P_u^{\mathbb{G}}, e, Q_w^{\mathbb{G}})$ be the corresponding edge. Then there exists $g \in G_u$ such that $Pge1 \approx Q$. Since P and Q are $\mathbb{G}$-reduced, Theorem 4.3.9 implies $k + 1 = \ell$ and $f_\ell = e$.

Theorem and Definition 4.4.2. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. Then $\widetilde{\mathbb{G}}_v$ is a tree. It is called the* **universal covering tree** *or* **Bass-Serre tree**.

Proof. Obviously, every vertex lies in the same component of $\widetilde{\mathbb{G}}_v$ as $1_v^{\mathbb{G}}$. Thus, the graph is connected and it remains to show that it contains no cycle.

Let
$$P = (P_0)_{v_0}^{\mathbb{G}} e_1 \dots e_k (P_k)_{v_k}^{\mathbb{G}}$$
be a closed non-trivial walk in $\widetilde{\mathbb{G}}_v$, where every P_i is a $\mathbb{G}$-reduced $\mathbb{G}$-walk. (Note that this is not a restriction to the walk itself.) If we show that P contains a spike
$$(P_{i-1})_{v_{i-1}}^{\mathbb{G}} e_i (P_i)_{v_i}^{\mathbb{G}} e_{i+1} (P_{i+1})_{v_{i+1}}^{\mathbb{G}},$$
then we directly obtain that $\widetilde{\mathbb{G}}_v$ contains no cycle. Let $0 \leq i \leq k$ such that the length of P_i is maximum. By cyclic permutations of the walk P we may assume that $0 < i < k$. We consider $(P_{i-1})_{v_{i-1}}^{\mathbb{G}}$ and $(P_{i+1})_{v_{i+1}}^{\mathbb{G}}$. Remark 4.4.1 implies that both vertices contain a common $\mathbb{G}$-walk. Thus, they lie in the same vertex in $\widetilde{\mathbb{G}}_v$ and remark 4.4.1 implies
$$e_i = ((P_{i-1})_{v_{i-1}}^{\mathbb{G}}, f_i, (P_i)_{v_i}^{\mathbb{G}})$$

[2] Using the notation $P_u^{\mathbb{G}}$ for vertices implies that we directly choose a representative P of the equivalence class and choose u as last vertex of the $\mathbb{G}$-walk P.

and

$$e_{i+1} = ((P_i)^{\mathbb{G}}_{v_i}, \bar{f}_i, (P_{i+1})^{\mathbb{G}}_{v_{i+1}}).$$

Thus, we have $e_i = \bar{e}_{i+1}$ and so P contains a spike, which implies that $\widetilde{\mathbb{G}}_v$ contains no cycle. □

Remark. It is easily verifiable that Bass-Serre trees for distinct vertices v, w from Γ are isomorphic: Choose a $\mathbb{G}$-walk P from v to w. Then the map $Q \mapsto PQ$ from the set of $\mathbb{G}$-walks starting at w to the set of $\mathbb{G}$-walks starting at v defines an isomorphism of the corresponding Bass-Serre trees $\widetilde{\mathbb{G}}_w$ and $\widetilde{\mathbb{G}}_v$.

Lemma 4.4.3. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. Then by setting*

$$[P]Q^{\mathbb{G}}_w := (PQ)^{\mathbb{G}}_w$$

and

$$[Q]((P_1)^{\mathbb{G}}_u, e, (P_2)^{\mathbb{G}}_w) := ([Q](P_1)^{\mathbb{G}}_u, e, [Q](P_2)^{\mathbb{G}}_w).$$

we obtain an action without inversion from $\pi_1(\mathbb{G}, v)$ on $\widetilde{\mathbb{G}}_v$.

Proof. We obtain from the definition of the vertices of $\widetilde{\mathbb{G}}_v$ that the assignment is a well-defined action on the vertex set and on the edge set. Since we directly obtain from the definition that edges and non-edges are preserved by this map, we obtain the assertion. □

We will state two small facts on the stabilisers of the vertices and edges of the universal covering tree in the fundamental group of the graph of groups.

Lemma 4.4.4. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. The action of $\pi_1(\mathbb{G}, v)$ on $\widetilde{\mathbb{G}}_v$ as defined in Lemma 4.4.3 satisfies the following properties.*

(1) *The stabiliser of a vertex $P^{\mathbb{G}}_w$ is $\{[PgP^{-1}] \mid g \in G_w\}$.*

(2) *The stabiliser of an edge $(P^{\mathbb{G}}_w, e, (Pgeg')^{\mathbb{G}}_u)$ with $g \in G_w$ is*

$$\{[Pg\alpha_e(c)g^{-1}P^{-1}] = [Pge\alpha_{\bar{e}}(c)\bar{e}g^{-1}P^{-1}] \mid c \in G_e\}.$$

Proof. Simple calculation. □

Analogously to the universal covering of graphs, we obtain also for graphs of groups the following statement whose proof is similar to the case of graphs and remains as exercise.

Lemma 4.4.5. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups and let $v \in V(\Gamma)$. Let $G := \pi_1(\mathbb{G}, v)$ and $T := \widetilde{\mathbb{G}}_v$. Then $T/G \cong \Gamma$.* □

Definition. Let G be a group acting on a tree T and let H be a group acting on a tree T'. Then T und T' are **isomorphic with respect to the actions of *G* and *H*** if there is a group isomorphism $\varphi \colon G \to H$ and a graph isomorphism $f \colon T \to T'$ with $f(gv) = \varphi(g)f(v)$ for all $g \in G$ and $v \in V(T)$. We call (φ, f) an **isomorphism** from (G, T) to (H, T').

Let G be a group that acts without inversion on a tree T. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be the graph of groups from Example 4.3.2 (1) with $\Gamma = T/G$ and let $v \in V(\Gamma)$. Set $H := \pi_1(\Gamma, v)$. Let $\mathcal{T}$ be a spanning tree of Γ. By an exercise, there exists a monomorphism $\iota\colon \mathcal{T} \to T$. Thus, ι is defined on all vertices of Γ but only on the edge of $\mathcal{T}$. We want to extend ι to all of Γ. We have to emphasise that we are just defining *images* of vertices and edges but in general this will not lead to a homomorphism. For every $e \in E(\Gamma) \smallsetminus E(\mathcal{T})$ we set $\iota(e)$ so that $\iota(\bar{e}) = \overline{\iota(e)}$ is satisfied and also either[3] $i(\iota(e)) = \iota(i(e))$ or $t(\iota(e)) = \iota(t(e))$ and additionally that e is the G-orbit of $\iota(e)$. If $i(\iota(e)) \neq \iota(i(e))$, then let $g_e \in G$ with $i(\iota(e)) = g_e\iota(i(e))$ and, if $i(\iota(e)) = \iota(i(e))$, then let $g_e := g_{\bar{e}}^{-1}$. For every spikeless walk $P = v_0e_1 \dots e_kv_k$ in $\mathcal{T}$ we set $P_{\mathbb{G}} := 1e_11 \dots e_k1$.

We will define a map from H to G and a map from $\widetilde{\mathbb{G}}_v$ to T and then prove that these maps are isomorphisms. We define $\varphi\colon H \to G$. First, we set

$$\varphi([P_{\mathbb{G}}gP_{\mathbb{G}}^{-1}]) := \psi_u(g)$$

for all spikeless walks P in $\mathcal{T}$ from v to u, for all $g \in G_u$ and for the canonical isomorphism $\psi_u\colon G_u \to G_{\iota(u)}$; additionally, we set

$$\varphi([P_{i(e),\mathbb{G}}eP_{t(e),\mathbb{G}}^{-1}]) := g_e$$

for all $e \in E(\Gamma \smallsetminus \mathcal{T})$, where $g_e \in G$ is chosen as above (i. e. with $i(\iota(e)) = g_e\iota(i(e))$). By Corollary 4.3.7 we have defined φ at a generating set of H. Let ϕ be the isomorphism from $\pi_1(\mathbb{G}, \mathcal{T})$ to $\pi_1(\mathbb{G}, v)$ as constructed in Proposition 4.3.6. Obviously, the images under ϕ of the relators in the definition of $\pi_1(\mathbb{G}, \mathcal{T})$ are mapped to 1 by φ.[4] The universal property for group presentations, Theorem 2.3.4, implies that we can extend φ to a homomorphism $H \to G$.

Let us define a map $f\colon \widetilde{\mathbb{G}}_v \to T$ by $f(P_u^{\mathbb{G}}) := \iota(u)$ for all $u \in V(\Gamma)$ and all $\mathbb{G}$-reduced $\mathbb{G}$-walks P in $\mathcal{T}$ from v to u and set $f(hP_u^{\mathbb{G}}) := \varphi(h)f(P_u^{\mathbb{G}})$ for all $u \in V(\Gamma)$, all $\mathbb{G}$-reduced $\mathbb{G}$-walks P in $\mathcal{T}$ from v to u and all $h \in H$. Obviously, f is well-defined and preserves edges and non-edges; thus, f is a graph homomorphism.

Note that φ induces canonically isomorphisms of vertex and edge stabilisers.

Proposition 4.4.6. *Let the group G act without inversion on the tree T. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be the graph of groups from Example* 4.3.2 (1) *with $\Gamma = T/G$ and let $v \in V(\Gamma)$. Then there exists an isomorphism from $(\pi_1(\mathbb{G}, v), \widetilde{\mathbb{G}}_v)$ to (G, T).*

Proof. We choose $\mathcal{T}$, φ and f as in the discussion before the proposition and want to show that (φ, f) is an isomorphism from $(\pi_1(\mathbb{G}, v), \widetilde{\mathbb{G}}_v)$ to (G, T). For this, it only remains to show that φ and f are bijective maps, since by definition of f we have $f(hP_u^{\mathbb{G}}) = \varphi(h)f(P_u^{\mathbb{G}})$ for all $h \in H$ and all $P_u^{\mathbb{G}} \in V(\widetilde{\mathbb{G}}_v)$.

[3] Note that we do not ask for both equalities here.

[4] Here, we mean the following: if $s_1 \dots s_n$ is a relator, then $\varphi(\phi(s_1)) \dots \varphi(\phi(s_n)) = 1$ lies in G.

Let us show that f is surjective. By definition we have $\varphi(H)\iota(V(\mathcal{T})) = f(\widetilde{\mathbb{G}}_v)$. Thus, it suffices to prove $\varphi(H)\iota(V(\mathcal{T})) = V(T)$. Let us suppose $\varphi(H)\iota(V(\mathcal{T})) \neq V(T)$. Then there exists an edge $e_T \in E(T)$ with $i(e_T) \in \varphi(H)\iota(V(\mathcal{T}))$ and $t(e_T) \notin \varphi(H)\iota(V(\mathcal{T}))$. We may replace e_T by $\varphi(h)e_T$ in order to assume $i(e_T) \in \iota(V(\mathcal{T}))$. Let $e \in E(\Gamma)$ with $G(\iota(e)) = Ge_T$. Then we have either $i(e_T) = i(\iota(e))$ or $g_{e_T} i(e_T) = i(\iota(e))$. This implies either $ge_T = \iota(e)$ or $gg_{e_T}e_T = \iota(e)$ for some $g \in G_{i(\iota(e))} \leq \varphi(H)$ and hence we obtain $e_T \in \varphi(H)\iota(E(\Gamma))$ and in particular $t(e_T) \in \varphi(H)\iota(V(\Gamma))$. This contradiction shows that f is surjective.

Let us show that f is injective. First, we show that no two edges $e_1, e_2 \in E(\widetilde{\mathbb{G}}_v)$ with $i(e_1) = i(e_2)$ exist such that $f(e_1) = f(e_2)$. Suppose such edges e_1, e_2 exist. Then there exists $h \in H$ with $he_1 = e_2$. We obtain $\varphi(h) \in G_{f(e_1)} = G_{f(e_2)}$. This contradicts our observation that φ induces isomorphisms between edge stabilisers. Thus, we have $f(e_1) \neq f(e_2)$. Since $\widetilde{\mathbb{G}}_v$ and T are trees (by Theorem 4.4.2 and assumption), we directly obtain that f is injective.

Let us show that φ is surjective. Since φ induces isomorphisms on the vertex stabilisers, we have $G_v \leq \varphi(H)$ for all $v \in V(T)$. Let $g \in G$ and $w \in \iota(V(\mathcal{T}))$. Since $\varphi(H)\iota(V(\Gamma)) = V(T)$, there exists $h \in \varphi(H)$ with $hgw \in V(\Gamma)$. Every two distinct vertices of $\iota(V(\Gamma))$ lie in distinct G-orbits. Thus, we have $hgw = w$ and $hg \in G_w \leq \varphi(H)$. We obtain $g = h^{-1}(hg) \in \varphi(H) \cdot G_w = \varphi(H)$. Thus, φ is surjective.

Let us show that φ is injective. For this, we show that the kernel of φ is trivial. Let $h \in H \setminus \{1\}$. If h has a fixed point, then the observation that φ induces isomorphisms on the stabilisers implies $\varphi(h) \neq 1$. So let us assume that h has no fixed point. Then we have $hv \neq v$ for all $v \in V(\Gamma)$ and thus $\varphi(h)f(v) = f(hv) \neq f(v)$, since f is injective. Thus, $\varphi(h) \neq 1$ and φ is injective. □

Theorem 4.4.7 (structure theorem)**.** *Let the group G act on a graph X without inversion. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be the graph of groups from Example* 4.3.2 (1) *with $\Gamma = X/G$, let $v \in V(\Gamma)$ and let φ and f as in the discussion before Proposition* 4.4.6*. Then the following statements are equivalent.*

(1) *X is a tree;*

(2) *$f\colon \widetilde{\mathbb{G}}_v \to X$ is an isomorphism;*

(3) *$\varphi\colon \pi_1(\mathbb{G}, v) \to G$ is an isomorphism.*

Proof. If X is a tree, then (2) and (3) holds by Proposition 4.4.6. Since $\widetilde{\mathbb{G}}_v$ is a tree by Theorem 4.4.2, the implication from (2) to (1) follows. The remaining implication (3)⇒(2) follows directly from the definition of f. □

Remark 4.4.8. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups with $G_v = 1$ for all $v \in V(\Gamma)$. Then $\pi_1(\mathbb{G}) \cong \pi_1(\Gamma)$.

As corollary of Theorem 4.4.7 using Remark 4.4.8, we obtain the following.

Corollary 4.4.9. *A group that acts freely on a tree is free.* □

4.5 Minimal actions

Definition. A group G acts on a tree T **minimally** if there is no non-empty G-invariant proper subtree of T. A graph of group is **minimal** if its fundamental groups acts minimally on its Bass-Serre tree.

Remark 4.5.1. Let G be a group acting on a tree T. If there exists a hyperbolic element $g \in G$, then there exists a g-invariant double ray R. Thus, the intersection of all G-invariant subtrees of T that contain R is not empty and according to Lemma 4.1.2 (iii) the ray R must lie in the intersection of all non-empty G-invariant subtrees. Then G acts minimally on this intersection. If G contains only elliptic elements, then Theorem 4.1.7 implies that the action of G on T is either elliptic, where we may choose for our G-invariant subtree one that has only one vertex, or it is parabolic. In the second case, T does not contain any minimal G-invariant subtree, since the intersection of all G-invariant subtrees is empty (exercise).

This discussion implies that for a minimal parabolic action, the group must contain hyperbolic elements.

Proposition 4.5.2. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a minimal graph of groups and let $v \in V(\Gamma)$. Then the action of $\pi_1(\mathbb{G}, v)$ on $\widetilde{\mathbb{G}}_v$ is ...*

(i) *elliptic if and only if Γ consists of a unique vertex and no edge;*

(ii) *cyclic if and only if Γ is a cycle and all monomorphisms from the edge groups into the vertex groups are isomorphisms;*

(iii) *dihedral if and only if Γ is a non-trivial path and the monomorphisms from the edge groups into the vertex groups of the inner vertices are surjective and the image of the monomorphisms in the vertex groups for the end vertices of the path are subgroups of index 2 each;*

(iv) *parabolic if and only if Γ is a cycle and for some closed spikeless walk $v_0 e_0 \dots e_{k-1} v_k$ the maps α_{e_i} are surjective for all $1 \leq i \leq n$ but at least on $\alpha_{\bar{e}_i}$ is not surjective;*

(v) *hyperbolic otherwise.*

Proof. In all cases, the backward implication is easy. That is, why we restict ourselves to the forward direction. Let $G := \pi_1(\mathbb{G}, v)$ and $T := \widetilde{\mathbb{G}}_v$. If the action of G on T is elliptic, then T has a unique vertex by the minimal action. Thus, Γ also has a unique vertex and no edge.

Let the action of G on T be cyclic. By the minimality of the action, T is a double ray and G acts as translations on T. Thus, T/G is a cycle. Since every group element that fixes a vertex of T must fix the two incident edge (since it acts as a translation on T), the monomorphisms from the edge groups into the vertex groups must be surjective.

Let us assume that the action of G on T is dihedral. As in the previous case, T is a double ray. Since there exists an element that acts as a reflection on that double ray, we obtain that T/G is a path that is non-trivial (so it contains at least two vertices) as the action is without inversion. Every group element that stabilises an edge must already fix all of T. Since the stabilisers of the end vertices of T/G have index 2 in the fix group of T (an element from such a stabiliser must either fix all of T or act on T as a reflection), we obtain the assertion. This proves (iii).

Now let the action of G on T be parabolic. In the case that every element of G is elliptic on T, Remark 4.5.1 implies that the action cannot be minimal. Thus, there exists a hyperbolic element. Let $g \in G$ be such an element of minimal translation length. Then, T contains a G-invariant double ray R_g by Lemma 4.1.2 (i). This double ray contains a subray R such that $R \cap hR$ is a ray again for all $h \in G$. Thus, the subgraph $\bigcup_{h \in G} hR_g$ is connected, so it is a subtree. Since it is G-invariant, the minimality of the action implies that it is already T. Thus, every edge of T lies in a common orbit with an edge from R_g. The minimality of $|g|$ implies that T/G is a cycle with $|g|$ edges. Obviously, the stabiliser of a vertex u on R_g must fix the incident edge that separates u from infinitely many vertices of R. Thus, there exists an orientation of the cycle Γ such that the monomorphisms for those edge groups are surjective into the vertex groups. Furthermore, for one edge e of those, the map $\alpha_{\bar{e}}$ is not surjective since otherwise (ii) implies that the action would be cyclic.

The final case follows from Theorem 4.1.7. □

Definition. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups. A pair (v, e) with $v \in V(\Gamma)$ and $e \in E(\Gamma)$ such that v has degree 2[5] and $i(e) = v \neq t(e)$ is **inessential** if α_e is surjective. A graph of groups is **reduced** if it contains no inessential pair.

Remark 4.5.3. If $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ is a graph of groups and (v, e) an inessential pair, then we can suppress v and e, i. e., we delete v as well as e and $\bar{e}$ from the vertex or and edge set and we set $t(f) = t(e)$ for all edges f in Γ with $t(f) = v$ and $i(f) = i(\bar{e})$ for all edges f in Γ with $i(f) = v$.[6] For the monomorphism $\alpha_f \colon G_f \to G_v$, we set the new monomorphism as $\alpha'_f := \alpha_{\bar{e}} \alpha_e^{-1} \alpha_f$. It is easy to verify that this operation preserves the fundamental group – or more precise: that the fundamental groups before and after the operation are isomorphic. One way to see that is to consider a spanning tree of Γ that contains e. Then the relators $g_e \alpha_{\bar{e}}(s) g_e^{-1} (\alpha_e(s))^{-1}$ reduce to $\alpha_{\bar{e}}(s)(\alpha_e(s))^{-1}$ and we can put this information into the relators of the other edge with initial vertex $i(e)$. Applying Tietze transformations lead to the fundamental group obtained after the operation.

In particular, we can transfer a graph of groups $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ with finite graph Γ to a reduced graph of groups by multiple such operations while keeping an isomorphic fundamental group.

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[5]For us, this means that there are precisely two edges ending in v and two edges starting at v.

[6]This corresponds in the case of graph exactly the situation G/e. But here, we also have to take case of the involution $\bar{\ }$, the directions of the edges and their edge groups.

Using Remark 4.5.3, we can sharpen the formulation of the possibilities in Proposition 4.5.2 in the cases (ii)–(iv). We will note these new version, where we also follow the statements on the group structure from Example 4.3.4.

Proposition 4.5.4. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a minimal reduced graph of groups and let $v \in V(\Gamma)$. Let $G := \pi_1(\mathbb{G}, v)$ and $T := \widetilde{\mathbb{G}}_v$. Then the action of G on T is ...*

(i) *elliptic if and only if Γ consists of a unique vertex and no edge. Then G is exactly the kernel of the action.*[7]

(ii) *cyclic if and only if Γ consists of a unique vertex and a unique edge*[8] *and the monomorphisms of both edge groups in the vertex groups are isomorphisms. Then we have $G \cong G_v *_\varphi$, where v is the vertex of Γ, e is an edge of Γ and $\varphi = \alpha_{\bar{e}}\alpha_e^{-1}$.*

(iii) *dihedral if and only if Γ consists of exactly two vertices and a unique edge and the image of the monomorphisms in the vertex group are subgroups of index 2 each. Then we have $G \cong G_u *_{G_e} G_v$ with $|G_u : G_e| = 2 = |G_v : G_e|$, where u and v are the two vertices of Γ and $e \in E(\Gamma)$ an edge.*

(iv) *parabolic if and only if Γ consists of a unique vertex and a unique edge such that exactly one of the monomorphisms into the vertex groups is surjective. Then we have $G \cong G_v *_\varphi$, where v is the vertex of Γ, e its edge such that $\alpha_{\bar{e}} \colon G_{\bar{e}} \to G_v$ is not surjective and $\varphi = \alpha_{\bar{e}}\alpha_e^{-1}$.*

(v) *hyperbolic otherwise.* □

Definition. A free product with amalgamation $A *_C B$ is **proper** if none of the monomorphisms $\iota_A \colon C \to A$ and $\iota_B \colon C \to B$ is surjective.

Proposition 4.5.5. *Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a minimal graph of groups such that Γ has at least one edge. Then $\pi_1(\mathbb{G})$ is either a proper free product with amalgamation or an HNN extension.*

Proof. Exercise □

Definition. A group has property (**FA**) if every action without inversion of it on every tree is elliptic.

We have already seen that finite groups always have the property (FA), see Lemma 4.1.8. Now, we want to give a group theoretic characterisation of the groups with property (FA).

Theorem 4.5.6. *A countable group G has property* (FA) *if and only if the following statements hold.*

[7] The **kernel** of the action consists of those elements that fix every vertex and every edge of T.

[8] up to its image under the involution

(1) *G is finitely generated;*

(2) *G is not a proper free product with amalgamation;*

(3) *G is not an HNN extension.*

Proof. By Lemma 4.1.9 we obtain that G is finitely generated since it has property (FA). Let us suppose that that G is a proper free product with amalgamation $G \cong A *_C B$ for groups A, B, C. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be the graph of groups with exactly two vertices and one edge, where the edge groups are C and the vertex groups are A and B. By Example 4.3.4 (1) we have $G \cong \pi_1(\mathbb{G})$. The action of $\pi_1(\mathbb{G})$ on $\widetilde{\mathbb{G}}_v$ with $v \in V(\Gamma)$ is without inversion but not elliptic. Thus G is no proper free product with amalgamation. Analogously, we obtain that G is not an HNN extension.

For the reverse direction, we assume that (1)–(3) hold. Let T be a tree such that G acts on T without inversion. By Lemma 4.1.5, it suffices to prove that every element of G is elliptic. Let us suppose that G contains a hyperbolic element. According to Remark 4.5.1 there exists a minimal G-invariant subtree of T and we may assume that the action of G on T is minimal. By Theorem 4.4.7, we may assume that $G = \pi_1(\mathbb{G})$ and $T = \widetilde{\mathbb{G}}_v$ with $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ and $v \in V(\Gamma)$, where $\mathbb{G}$ is the graph of groups with $\Gamma = T/G$ as defined in Example 4.3.2 (1). Since G contains a hyperbolic element, the action of G on T is not elliptic and there is an edge in Γ by Proposition 4.5.4 (i). Thus, Proposition 4.5.5 leads to a contradiction to (2) or (3). So the action of G on T is elliptic. □

4.6 Kurosh's theorem

Example 4.6.1. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups with $G_e = 1$ for all $e \in E(\Gamma)$ and let T be a spanning tree of Γ. Let $G_v = \langle S_v \mid R_v \rangle$ be presentation of the vertex groups. Then we have

$$\pi_1(\mathbb{G}, T) = \Bigg\langle \bigcup_{v \in V(\Gamma)} S_v \cup \{g_e \mid e \in E(\Gamma) \setminus E(T)\} \mid \bigcup_{v \in V(\Gamma)} R_v \cup \{g_e g_{\bar{e}} \mid e \in E(\Gamma) \setminus E(T)\} \Bigg\rangle .$$

Thus, there exists a free group F with $\pi_1(\mathbb{G}, T) \cong (*_{v \in V(\Gamma)} G_v) * F$.

Example 4.6.2. Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups, where Γ is a star, and let A be a group. Let $G_e = A$ for all $e \in E(\Gamma)$ and $G_z = A$ for the central vertex z of Γ. Let $G_v = \langle S_v \mid R_v \rangle$ be the presentation of the vertex groups. Then $\pi_1(\mathbb{G}, \Gamma)$ has the presentation

$$\Bigg\langle \bigcup_{v \in V(\Gamma), v \neq z} S_v \mid \bigcup_{v \in V(\Gamma), v \neq z} R_v \cup \left\{ \alpha_e(a)(\alpha_{\bar{e}}(a))^{-1} \mid e \in E(\Gamma) \text{ und } i(e) = z; a \in A \right\} \Bigg\rangle .$$

So we have $\pi_1(\mathbb{G}, \Gamma) \cong *_A G_v$.

Theorem 4.6.3. *Let $G = *_{A,i\in I} G_i$ be a free product with amalgamation over A of a family $(G_i)_{i\in I}$ of groups. Let $H \le G$ be a subgroup whose intersection with each A^g with $g \in G$ is trivial. For $x \in G$ and $i \in I$, set $H_{i,x} := H \cap xG_ix^{-1}$. Let X_i be a set of representatives of the double cosets HgG_i. Then there exists a free group F such that*

$$H \cong (*_{i\in I, x\in X_i} H_{i,x}) * F.$$

Proof. Let $\mathbb{G} = (\mathcal{G}, X, \Lambda)$ be a graph of groups, where X is a star with centre v_A and there exists a leaf v_i for each $i \in I$. The edge groups are all A and the vertex groups of the centre is A, too. Let G_i be the vertex group for the leaf v_i. The maps α_e are the identity if $i(e) = v_A$ and the monomorphism from A to G_i given by the free product with amalgamation otherwise. Set $T := \widetilde{\mathbb{G}}_{v_A}$. By Example 4.6.2, we have $\pi_1(\mathbb{G}) \cong G$. Thus, G and hence H act without inversion on T and we may think of the vertex and edge stabilisers of vertices or edges of T in $\pi_1(\mathbb{G})$ as stabilisers in G (cf. Lemmas 4.4.3 and 4.4.4). In particular, the edge stabilisers are subgroups of G that are conjugated to A.

Let $\Gamma := T/H$ and let $\mathcal{T}$ be a spanning tree of Γ. Let $\mathbb{G}_H = (\mathcal{G}_H, \Gamma, \Lambda_H)$ be the graph of groups that is induced by the action of H on T: we have $G_v^H = G_v \cap H$ and $G_e^H = G_e \cap H$ for all vertices v and edges e of Γ. Then Theorem 4.4.7 implies $H \cong \pi_1(\mathbb{G}_H, \mathcal{T})$. Since H has trivial intersection with each subgroup of G that is conjugated to A, we obtain $G_e^H = 1$ for all edge groups of Γ. By Example 4.6.1 we obtain

$$H \cong \pi_1(\mathbb{G}_H, \mathcal{T}) \cong (*_{v\in V(\Gamma)} G_v^H) * F,$$

where F is a free group.

The vertices of T correspond to the set[9]

$$G/A \cup \bigcup_{i\in I} G/G_i,$$

and thus the vertices of Γ and thus $\mathcal{T}$ correspond to the set[10]

$$H\backslash G/A \cup \bigcup_{i\in I} H\backslash G/G_i,$$

since we just combine the H-orbits. The embedding of $\mathcal{T}$ in T defines a set of representatives $X_A \subseteq G/A$ of $H\backslash G/A$ and a set of representatives $X_i \subseteq G/G_i$ of $H\backslash G/G_i$. If $x \in X_A$, then the corresponding group G_{v_A} is exactly $H \cap xAx^{-1}$ and, if $x \in X_i$, then the corresponding group G_{v_i} is exactly $H \cap xG_ix^{-1}$. Since $H \cap xAx^{-1} = 1$, we obtain the assertion. □

[9] Note that the normal form of $\mathbb{G}$-walks correspond (by removing the edges in the sequence) reduced forms in the free product G, canonically.

[10] For subgroups H, U of a group G, the set $H\backslash G/U$ is the set of double cosets HgU with $g \in G$.

Theorem 4.6.3 for $A = 1$ implies the subgroup theorem of Kurosh.

Corollary 4.6.4 (Kurosh's subgroup theorem)**.** *Let $G = *_{i\in I} G_i$ be a free product of a family $(G_i)_{i\in I}$ of groups. Let $H \leq G$ be a subgroup. For $x \in G$ and $i \in I$ let $H_{i,x} := H \cap xG_ix^{-1}$. Let X_i be a set of representatives of the double cosets HgG_i. Then there exists a free group F such that*

$$H \cong (*_{i\in I, x\in X_i} H_{i,x}) * F.$$

□

Note that Corollary 4.6.4 implies in particular that the order of any element of H must be the order of some element of one of the G_i, if it is finite.

4.7 Stallings' theorem

In this section, we will show that finitely generated groups with more than one end always split over a finite subgroup either as free product with amalgamation or as HNN extension.

Theorem 4.7.1 (Stallings' theorem)**.** *Let G be a finitely generated group with more than one end. Then one of the following holds.*

(1) *There exists three subgroups A, B, H of G such that H is finite and $A *_H B \cong G$ is a proper free product with amalgamation.*

(2) *There exists a subgroup H of G and an isomorphism φ between two finite subgroups of G with $G \cong H*_\varphi$.*

We will obtain Theorem 4.7.1 as corollary of Proposition 4.7.2.

Proposition 4.7.2. *Let G be a finitely generated group with more than one end. Then there exists a tree T such that G acts on T edge-transitively and without inversion such that all edge stabilisers are finite and no vertex stabiliser is G.*

Proof of Theorem 4.7.1. Let T be the tree from Proposition 4.7.2. The graph of groups for the action of G on T consists of a single edge and at most two vertices because of the edge transitivity. Then Example 4.3.4 directly implies Theorem 4.7.1, since the monomorphisms of the edge groups into the vertex groups in Example 4.3.4 cannot be surjective. □

Thus, we want to construct a tree such that G acts on that tree in a suitable way.

Proof of Proposition 4.7.2. Let $\Gamma = (V, E)$ be a locally finite Cayley graph of G. Then Γ has more than one end, since G has more than one end. First, we consider the case that Γ has at least three ends. Then Lemma 3.4.3 implies that Γ has infinitely many ends. We set

$$\mathcal{B}_i := \{U \subseteq V \mid |U| = \infty = |V \smallsetminus U| \text{ and } |E(U, V \smallsetminus U)| \leq i\}.$$

Since Γ has more than one end, there exists a finite set $S \subseteq V$ such that two rays lie in distinct components of $G - S$ eventually. If i is the number of edge from S into one of these components, then we obtain $\mathcal{B}_i \neq \emptyset$. Let $m \in \mathbb{N}$ be minimal with $\mathcal{B}_m \neq \emptyset$. The minimality of m implies that all $U \in \mathcal{B}_m$ are connected. If $U \subseteq V$, then we set $\overline{U} := V \setminus U$.

Claim 1. If $U_1 \supsetneq U_2 \supsetneq \dots$ is a chain in $\mathcal{B}_m$, then we have $\bigcap_{i \in \mathbb{N}} U_i = \emptyset$.

Proof of Claim 1. Let us suppose that there exists a chain $U_1 \supsetneq U_2 \supsetneq \dots$ in $\mathcal{B}_m$ such that $U := \bigcap_{i \in \mathbb{N}} U_i$ is not empty. Then there exists an edge e_1 one of whose incident vertices lies in U and the other lies in U_1 but outside of U and there is an index $i_1 \in \mathbb{N}$ such that

$$e_1 \in E(\overline{U_{i_1}}, U) \setminus E(\overline{U_{i_1 - 1}}, U).$$

Analogously, there exists an edge $e_2 \in E(U_{i_1})$ with exactly one of its incident vertices in U. Let $i_2 \in \mathbb{N}$ such that

$$e_2 \in E(\overline{U_{i_2}}, U) \setminus E(\overline{U_{i_2 - 1}}, U).$$

Then we have $i_2 > i_1$ and e_1 and e_2 lie in $E(\overline{U_{i_2}}, U)$. This way, we can find infinitely many edges and for every $n \in \mathbb{N}$ there exists i_n such that the first n of these edges lie in $E(\overline{U_{i_n}}, U)$. For $n > m$ this contradicts our choice of the sets $U_i \in \mathcal{B}_m$. □

Because of Claim 1, there exists a minimal $W \in \mathcal{B}_m$ with $1 \in W$.

Claim 2. For every $U \in \mathcal{B}_m$ one of the following four sets is finite:

$$U \cap W,\ \overline{U} \cap W,\ U \cap \overline{W},\ \overline{U} \cap \overline{W}.$$

Proof of Claim 2. Let us suppose that all four intersections are infinite. Then each of these sets contains a ray, since Γ is locally finite. For all $A \in \{U, \overline{U}\}$ and $B \in \{W, \overline{W}\}$ we obtain

$$E(A \cap B, \overline{A \cap B}) \geq m$$

by the minimality of m. Every edge of $E(U, \overline{U}) \cup E(W, \overline{W})$ lies in exactly two of the sets $E(A \cap B, \overline{A \cap B})$. We obtain

$$\begin{aligned} & 4m \\ \leq & \sum_{\substack{A \in \{U, \overline{U}\}, \\ B \in \{W, \overline{W}\}}} \left| E\left(A \cap B, \overline{A \cap B}\right) \right| \\ \leq & \ 2|E(U, \overline{U})| + 2|E(W, \overline{W})| \\ \leq & \ 4m, \end{aligned}$$

and thus, all inequalities must be equalities and all infinite components of these intersections must lie in $\mathcal{B}_m$. We may assume that 1 lies in $U \cap W$. Because of $U \cap W \in \mathcal{B}_m$ and the minimality of W with respect to containing 1, we have $U \cap W = W$. This implies $\overline{U} \cap W = \emptyset$. This contradiction proves the claim. □

We define an equivalence relation $\cong$ on $\mathcal{B}_m$ via

$$U_1 \cong U_2 :\Longleftrightarrow U_1 \cap \overline{U_2} \text{ and } \overline{U_1} \cap U_2 \text{ are finite}$$

and a strict order[11] via

$$U_1 \prec U_2 :\Longleftrightarrow \overline{U_1} \cap U_2 \text{ finite, but } U_1 \ncong U_2.$$

It is easy to verify that these two relations have the properties as claimed; we will skip it at this point.

Claim 3. Let $A, U \in \mathcal{B}_m$ with $W \prec A \prec U$. Then there are only finitely many $g \in G$ with $W \prec gA \prec U$.

> *Proof of Claim* 3. Let $\Delta \subseteq \Gamma$ be a finite connected subgraph that contains $E(X, \overline{X})$ for all $X \in \{A, U, W\}$. For all $g \in G$ except for finitely many, this implies $\Delta \cap g\Delta = \emptyset$. Let $g \in G$ with $\Delta \cap g\Delta = \emptyset$ and suppose $W \prec gA \prec U$. Then $U \cap g\overline{A}$ and $gA \cap \overline{W}$ are finite. Since Δ is connected but avoids $E(gA, g\overline{A})$, it lies in either gA or in $g\overline{A}$. First, we assume that Δ lies in gA. Then we have either $W \subseteq gA$ or $\overline{W} \subseteq gA$, since W and $\overline{W}$ are connected. Since $gA \cap \overline{W}$ is finite, we must have $W \subseteq gA$, which implies $g\overline{A} \cap W = \emptyset$ and hence $gA \cong W$, a contradiction to $W \prec gA$. With a similar argument, the case that Δ lies in $g\overline{A}$ leads to a contradiction. □

We set

$$T := \{gW, g\overline{W} \mid g \in G\}/\cong.$$

We extend the definition of the complement and of $\prec$ to T: for $\mathcal{U}_1, \mathcal{U}_2 \in T$ set

$$\overline{\mathcal{U}_1} := \{\overline{U_1} \mid U_1 \in \mathcal{U}_1\}$$

and

$$\mathcal{U}_1 \prec \mathcal{U}_2 :\Longleftrightarrow \exists U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2 \colon U_1 \prec U_2.$$

Claim 4. The triple $(T, \bar{\cdot}, \prec)$ is a **tree set**, i. e., it has the following properties:

(1) $\bar{\bar{t}} = t$ and $t \neq \bar{t}$ for all $t \in T$;

(2) $\prec$ is a strict order on T;

(3) $t_1 \prec t_2 \Longleftrightarrow \overline{t_2} \prec \overline{t_1}$ for all $t_1, t_2 \in T$;

(4) for $t_1, t_2 \in T$ exactly one of the following cases is true:

$$t_1 = t_2,\ \overline{t_1} = t_2,\ t_1 \prec t_2,\ t_1 \prec \overline{t_2},\ \overline{t_1} \prec t_2,\ \overline{t_1} \prec \overline{t_2};$$

(5) for no $t \in T$, the set $T_t^{\prec} := \{t' \in T \mid t' \prec t\}$ contains an infinite chain $t_1 \prec t_2 \prec \cdots$.

[11] that is an asymmetric, transitive relation

Additionally, the tree set has the following property.

(6) there exist no maximal and minimal elements with respect to $\prec$.

Proof of Claim 4. Statement (1) follows directly from the definition of the complement and (3) follows directly from the definition of $\prec$. Since $\prec$ is a strict order on $\mathcal{B}_m$, the same holds for $\prec$ on T. Because of Claim 2, we obtain (4) and Claim 3 implies (5).

Let us suppose that $U \in \mathcal{B}_m$ is maximal with respect to $\prec$. In particular, U is infinite. Let Δ be a finite subgraph of Γ such that $\Gamma - \Delta$ has three infinite components. Let $g \in G$ such that $g\Delta$ lies in U. This element g exists, since Γ is locally finite. Let $h \in G$ such that $hE(W, \overline{W})$ lies in an infinite component of $\Gamma - g\Delta$ that intersects $\overline{U}$ trivially. Then we have either $U \prec hW$ or $U \prec h\overline{W}$, which contradicts the maximality of U. Analogously, we obtains a contradiction, if U is minimal with respect to $\prec$. Thus, T has no maximal and no minimal elements with respect to $\prec$. □

By an exercise, we obtain the existence of a tree $\mathcal{T}$ with edges T and vertices $T/{\sim}$, where

$$t_1 \sim t_2 :\Longleftrightarrow t_1 = t_2 \vee (t_1 \prec \bar{t}_2 \wedge \neg \exists t \in T \colon t_1 \prec t \prec \bar{t}_2).$$

Since T is G-invariant[12], G acts on $\mathcal{T}$. The action of G has at most two orbits on the tree set since it consists of the equivalence classes of elements gW or $g\overline{W}$. In the exercise, we also saw that gW and $g\overline{W}$ form the same edge (up to its direction). Thus, G acts on $\mathcal{T}$ and it acts transitively on the edges of $\mathcal{T}$. If this action is not without inversion, we subdivide each edge once and obtain an action without inversion. The stabiliser of W is finite, since it is the stabiliser of the finite edge set $E(W, \overline{W})$ is and since G acts freely on Γ. In order to show that the stabilisers of the edge of $\mathcal{T}$, that are the elements of T, are finite, too, it suffices to show that only finitely many elements of $\{gW, g\overline{W} \mid g \in G\}$ are equivalent with respect to $\cong$. This is a direct consequence of Claim 3 and Claim 4 (6). Thus, the statement on the edge stabilisers follows from Lemma 1.1.10, since every element of T lies in the orbit of the equivalence class of W or of $\overline{W}$.

Since there are no maximal or minimal elements with respect to $\prec$ by Claim 4 (6), there exists a path of length at least 3 in $\mathcal{T}$. This and the transitivity of the action on the edges implies that there is no vertex that is fixed by all of G.

Let us now briefly discuss the case that Γ has exactly two ends. The reason, why the above proof fails is that the proof of Claim 4 (6) fails, the set T consists of at most two elements and thus we cannot conclude that the splitting is proper. In this situation, we have to work more to find some $U \in \mathcal{B}_m$ such that for U and gU we have one of the cases of Statement (4) in the definition of a tree set, where $\prec$ is now replaced by $\subseteq$. Then we also obtain a tree that contains a path of length at least 3, which implies that the splitting is proper.[13] □

[12] Note that the equivalence relation $\cong$ is invariant under G.

[13] This sharpening of the requirements for an element of $\mathcal{B}_m$ will not be covered in this lecture.

Corollary 4.7.3. *The property of being a proper free product with amalgamation or an HNN extension over a finite group is a quasi-isometry invariant.* □

We have shown that we can split groups with more than one end, e. g. as free product with amalgamation $A *_H B$. But now it can happen that one or two of these groups A and B have more than one end, too. Can we continue this indefinitely? Can it happen that again and again one of the groups involed in the product has more than one end?

Definition. A finitely generated group G is **0-accessible** if it has at most one end. For $n \in \mathbb{N} \setminus \{0\}$, the group is ***n*-accessible** if it is isomorphic either to $A *_H B$ for subgroups $A \neq H \neq B$, where H is finite and, for some $i_A, i_B < n$, the groups A and B are i_A- and i_B-accessible, respectively, or to $A*_\varphi$ for some i-accessible group A with $i < n$ and an isomorphism φ between finite subgroups. We call G **accessible** if it is n-accessible for some $n \in \mathbb{N}$.

Accessibility of groups can be characterised using the Bass-Serre theory as follows.

Proposition 4.7.4. *A finitely generated group is accessible if and only if it is the fundamental group of a finite graph of groups whose edge groups are finite and whose vertex groups are finitely generated have at most one end.*

Proof. Exercise □

Wall conjectured the following.

Conjecture 4.7.5 (Wall 1971)**.** *Every finitely generated group is accessible.*

A first positive result is due to Dunwoody.

Theorem 4.7.6 (Dunwoody 1985)**.** *Every finitely presented group is accessible.*

The general conjecture, however, was disproved a bit later.

Theorem 4.7.7 (Dunwoody 1991)**.** *There are finitely generated groups that are not accessible.*

Another important result for accessible groups follows from a theorem of Thomassen and Woess.

Theorem 4.7.8 (Thomassen and Woess 1993)**.** *A finitely generated group is accessible if and only if one (and hence every) of its locally finite Cayley graphs has the following property: there exists $n \in \mathbb{N}$ such that every two ends can be separated by at most n edges.*

Corollary 4.7.9. *Accessibility is a quasi-isometry invariant.*

Chapter 5

Hyperbolic groups

5.1 Hyperbolic graphs and groups

Definition. Let $\Gamma = (V, E)$ be a graph and let $\delta \in \mathbb{R}_{\geq 0}$. Let $x_1, x_2, x_3 \in V$ and let P_i be a shortest path between x_i and x_{i+1} (mod 3). We call the tupel

$$(x_1, x_2, x_3; P_1, P_2, P_3)$$

a **geodesic triangle**. It is **δ-thin** if for every $x \in V(P_i)$ there exists a $y \in \bigcup_{i \neq j} V(P_j)$ with $d(x, y) \leq \delta$. The graph Γ is **δ-hyperbolic** if every geodesic triangle is δ-thin and **hyperbolic** if it is δ'-hyperbolic for some $\delta' \in \mathbb{R}_{\geq 0}$.

Example 5.1.1. (1) Trees are 0-hyperbolic.

(2) The grid $\mathbb{Z}^2$ is not hyperbolic.

Remark. In the literature, hyperbolicity is usually defined for metric spaces. In this context, edges of graphs will be considered as continuous images of $[0, 1]$, similar to planar graphs. Thus, there are small differences for the involved constants δ. Often, the result *trees are the* 0*-hyperbolic graphs* is mentioned, which is wrong for our definition.

We obtain directly from the definition of δ-thin geodesic triangles that in hyperbolic graphs geodesic paths between the same vertices lie close to each other.

Lemma 5.1.2. *Let Γ be a δ-hyperbolic graph and let $x, y \in V(G)$. Let P_1, P_2 be two geodesic x–y paths. Then we have $d(v, P_i) \leq \delta$ for all $v \in V(P_j)$ with $i, j \in \{1, 2\}$.* □

The previous lemma even holds for quasi-geodesic paths. But before we can show that, we first have to prove a result on the divergence of geodesic paths.

Definition. Let Γ be a graph. A function $f \colon \mathbb{N} \to \mathbb{R}$ is a **divergence function** for Γ if for all geodesic paths $P_1 = x_0 \dots x_n$ and $P_2 = y_0 \dots y_m$ with $x_0 = y_0$

and for all $r, R \in \mathbb{N}$ with $r + R \leq \min\{m, n\}$ we have the following as soon as $d(x_R, y_R) > f(0)$: for all paths Q outside of $B_{R+r-1}(x_0)$ from x_{R+r} to y_{R+r} we have $\ell(Q) > f(r)$.

We say that **geodesic paths diverge exponentially** if there is an exponential divergence function.[1]

Proposition 5.1.3. *A graph is hyperbolic if and only if geodesic paths diverge exponentially in it.*

Proof. First, let Γ be a hyperbolic graphs. Let $P_1 = x_0 \dots x_{R+r}$ and $P_2 = y_0 \dots y_{R+r}$ be two geodesic paths with common first vertex $x_0 = y_0$ and length $R+r$ with $r, R \in \mathbb{N}$ such that $d(x_R, y_R) > 2\delta$. Let Q be a x_{R+r}–y_{R+r} path that lies outside of $B_{R+r-1}(x_0)$ and let Q_g be a geodesic x_{R+r}–y_{R+r} path.

Let $v \in V(Q_g)$ and let $w \in V(Q)$ with

$$d_Q(x_{R+r}, w) = \left\lceil \frac{d_Q(x_{R+r}, y_{R+r})}{2} \right\rceil.$$

Let Q_1 be a geodesic x_{R+r}–w path and Q_2 a geodesic w–y_{R+r} path. Then there exists $u \in V(Q_1) \cup V(Q_2)$ with $d(v, u) \leq \delta$. We may assume that u lies on Q_1. Inductively, we obtain

$$d(v, Q) \leq \delta \log_2(d_Q(x_{R+r}, y_{R+r})).$$

Because of $d(x_R, y_R) > 2\delta$ there is no vertex a on P_2 with $d(x_R, a) \leq \delta$: such a vertex had distance at least $R - \delta$ and at most $R + \delta$ to x_0 and thus distance at most δ to y_R. Thus, there exists a vertex on Q_g with distance at most δ to x_R. We may assume that this vertex is v. We obtain

$$r + R = d(x, Q) \leq d(x, v) + d(v, Q) \leq R - \delta + \delta \log_2(\ell(Q)).$$

Thus, we have

$$\ell(Q) \geq 2^{\frac{r+\delta}{\delta}}$$

and hence geodesic paths diverge exponentially in Γ.

Let us now assume that geodesic paths diverge exponentially in Γ. For this, let f be an exponential divergence function for Γ. Let $(x, y, z; P_1, P_2, P_3)$ be a geodesic triangle in Γ. Let $x_1 \in V(P_1)$ and $x_2 \in V(P_3)$ with $d(x, x_1) = d(x, x_2)$ maximal such that $d(u, v) \leq f(0)$ for all $u \in V(P_1)$ and all $v \in V(P_3)$ with $d(x, x_1) \geq d(x, u) = d(x, v)$. Analogously, we define y_1 and y_2 on P_2 and P_1, respectively, and z_1 and z_2 on P_3 and P_2, respectively.

First, we consider the case that xP_1x_1 and y_2P_1y cover P_1. Then there exists $x_3 \in V(xP_3x_2)$ and $y_3 \in V(yP_2y_1)$ with

$$d(x_3, y_3) \leq 2f(0) + 1.$$

[1]Strictly speaking, we ask for a divergence function that is equivalent to an exponential function in the sense that (generalised) growth function are equivalent to each other.

The paths P_2 and P_3 are geodesic and thus we obtain

$$d(z, x_3) + 2f(0) + 1 \geq d(z, y_3).$$

Since f is an exponential divergence function, the distances $d(z_1, x_3)$ and $d(z_2, y_3)$ are bounded by $f^{-1}(4f(0) + 2)$. Thus, the distances $d(z_1, x_2)$ and $d(z_2, y_1)$ are bounded by the same value. Thus, our geodesic triangle is λ-thin for

$$\lambda = f^{-1}(4f(0) + 2) + 2f(0) + 1.$$

Let us now consider the case that there exists a vertex on P_1 outside of xP_1x_1 and y_2P_1y and that the analogous statement hold for the other two sides. We set

$$\begin{aligned} K_1 &:= d(x_1, y_2), \\ K_2 &:= d(y_1, z_2) \text{ and} \\ K_3 &:= d(z_1, x_2). \end{aligned}$$

We may assume $K_1 \geq K_2 \geq K_3$. Let $v \in V(P_1)$ with $d(x_1, v) = \lceil d(x_1, y_2)/2 \rceil$.

Claim 1. The path x_2P_3z lies outside of $B_{d(y,v)-1}(y)$.

Proof of Claim 1. Let us suppose that there exists a vertex $u \in V(x_2P_3z)$ with $d(u, y) \leq d(y, v) - 1$. Because of

$$d(y, v) + d(x, v) - 1 < d(x, y)$$

we have $u \notin B_{d(x,v)}(x)$ and thus $d(u, x_2) \geq K_1/2$. Because of $K_2 \geq K_3$ there exists $w \in V(P_2)$ with $d(u, z) = d(w, z)$. We obtain

$$\begin{aligned} K_1/2 &\leq d(u, x_2) \\ &= d(x_2, z) - d(u, z) \\ &= d(x_2, z_1) + d(z_1, z) - d(u, z) \\ &\leq d(z_2, y_1) + d(z_2, z) - d(z, w) \\ &= d(z, y_1) - d(z, w) \\ &= d(w, y_1). \end{aligned}$$

Thus, w does not lie in $B_{d(y,v)-1}(y)$. We obtain

$$\begin{aligned} d(y, z) &= d(z, w) + d(w, y) \\ &\leq d(z, u) + d(u, y) \\ &< d(z, w) + d(w, y) \\ &= d(y, z). \end{aligned}$$

This contradiction shows the assertion. □

Let $s \in V(P_2)$ with $d(y,s) = d(y,v)$. Note that Claim 1 implies the existence of s. There exists a v–s path in the complement of $B_{d(y,v)-1}(y)$ of length at most

$$\begin{aligned} & d(v,x_1) + f(0) + K_3 + 3f(0) + d(z_2,s) \\ \leq\ & \lceil K_1/2 \rceil + 4f(0) + K_1 + \lceil K_1/2 \rceil \\ \leq\ & 2K_1 + 4f(0) + 1. \end{aligned}$$

Thus, we have

$$f\left(\left\lfloor \frac{K_1}{2} \right\rfloor\right) \leq 2K_1 + 4f(0) + 1.$$

Since f is an exponential function, there exists $K \in \mathbb{N}$ (independent of K_1) with $K_1/2 \leq K$. Thus, our geodesic triangle in λ'-thin for $\lambda' := 4K + 4f(0) + 1$. For $\delta = \max\{\lambda, \lambda'\}$ we obtain that Γ is δ-hyperbolic. □

Proposition 5.1.4. *Let Γ be a hyperbolic graph and let $x, y \in V(G)$. Let P_1 be a geodesic path and let P_2 be a (γ, c)-quasi-geodesic x–y path with $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{\geq 0}$.*

(1) *There exists $\lambda \in \mathbb{N}$, depending only on (δ, γ, c), such that $d(v, P_2) \leq \lambda$ for all $v \in V(P_1)$.*

(2) *There exists $\kappa \in \mathbb{N}$, depending only on (δ, γ, c), such that $d(v, P_i) \leq \kappa$ for all $v \in V(P_j)$ with $i, j \in \{1, 2\}$.*

Proof. By Proposition 5.1.3 there exists an exponential divergence function $f\colon \mathbb{N} \to \mathbb{R}$. Let $D := \max\{d(v, P_2) \mid v \in V(P_1)\}$ and let $v \in V(P_1)$ with $d(v, P_2) = D$. Let u' be a vertex on xP_1v with $d(u', v) = 2D$, if possible, and $u' = x$ otherwise. Analogously, we choose w' on vP_1y. Note that we have

$$d(v,x) \geq D \leq d(v,y)$$

by the choice of v. We choose $u \in V(u'P_1v)$ with $d(u,v) = D$ and $w \in V(vP_1w')$ with $d(v,w) = D$. By the choice of D, there exists $a \in V(P_2) \cap B_D(u')$ and $b \in V(P_2) \cap B_D(w')$. Thus, we have $d(a,b) \leq 6D$ and $d_{P_2}(a,b) \leq 6\gamma D + c$, since P_2 is a (γ, c)-quasi-geodesic path. Hence, we find a path of length at most $4D + 6\gamma D + c$ from u to w that lie outside of $B_{D-1}(v)$. Since f is an exponential divergence function, but the length of the path is only linear in D, we obtain the existence of an upper bound λ, depending only on (δ, γ, c) with $D \leq \lambda$. This implies (1).

For the proof of (2), let us suppose that P_2 contains vertices that lie outside of $B_\lambda(P_1)$. Let P' be a maximal subpath of P_2 such that its inner vertices lie outside of $B_\lambda(P_1)$. Let u and v be the end vertices of P'. We may assume that u lies on xP_2v. By the choice of P' there exists $a, b \in V(P_1)$ with $a \in B_\lambda(u)$ and $b \in B_\lambda(v)$. By (1) every vertex on aP_1b has distance at most λ to some vertex of P_2, that lies in $xP_2u \cup vP_2y$ by the choice of P'. In particular, there exists a vertex z on aP_1b that has distance at most λ to some vertex z_1 on xP_2u and distance at most $\lambda+1$ to some vertex z_2 on vP_2y. We obtain $d(z_1,z_2)\leq 2\lambda+1$ and hence $d_{P_2}(z_1,z_2)\leq\gamma(2\lambda+1)+c$, since P_2 is a (γ,c)-quasi-geodesic path. Thus, the length of P' is bounded by $\gamma(2\lambda+1)+c$ and we obtain (2) for $\kappa=\lambda+\gamma\lambda+\lceil(\gamma+c)/2\rceil$. □

Lemma 5.1.5. *Let Γ and Δ be two graphs, let $\varphi\colon\Gamma\to\Delta$ be a (γ,c)-quasi-isometric embedding with $\gamma\geq 1$ and $c\geq 0$ and let $P=x_0\ldots x_n$ be a geodesic path in Γ. Then $\varphi(P)$ induces a (γ',c')-quasi-geodesic $x_0\varphi$–$x_n\varphi$ path Q in Δ such that every vertex of Q has distance at most a to some vertex of $\varphi(P)$, where γ', c' and a only depend on γ and c.*

Proof. For every $0\leq i<n$, let Q_i be a geodesic path between $\varphi(x_i)$ and $\varphi(x_{i+1})$. The union of these paths is a $x_0\varphi$–$x_n\varphi$ walk that contains a $x_0\varphi$–$x_n\varphi$ path. That this path satisfies our claim is shown in an exercise. □

Proposition 5.1.6. *Let Γ and Δ be two graphs. If there exists a (γ,c)-quasi-isometric embedding $\varphi\colon\Gamma\to\Delta$ for some $\gamma\geq 1$ and $c\geq 1$ and if Δ is hyperbolic, then Γ is hyperbolic.*

Proof. Let $(x_1,x_2,x_3;P_1,P_2,P_3)$ be a geodesic triangle in Γ. Let $y_i:=\varphi(x_i)$ for all $i\in\{1,2,3\}$. Then $\varphi(P_i)$ induces a (γ',c')-quasi-geodesic y_i–y_{i+1} path P_i' by Lemma 5.1.5, where γ' and c' only depend on γ and c. Let Q_i be geodesic y_i–y_{i+1} paths for all $i\in\{1,2,3\}$. Let $x\in V(P_i)$. By Proposition 5.1.4 (2) there exists κ, depending only on (δ,γ,c), such that there exists $x'\in V(Q_i)$ with $d(\varphi(x),x')\leq\kappa$. Since Δ is δ-hyperbolic, we find $y'\in V(Q_j)$ for some $j\neq i$ with $d(x',y')\leq\delta$ and $y''\in P_j'$ with $d(y',y'')\leq\kappa$. By Lemma 5.1.5 there exists $y\in V(P_j)$ with $d(y'',\varphi(y))\leq\gamma+c$. Thus, we have

$$\frac{1}{\gamma}d(x,y)-\gamma\leq d(\varphi(x),\varphi(y))\leq 2\kappa+\delta+\gamma+c$$

and hence

$$d(x,y)\leq\gamma(2\kappa+\delta+\gamma+2c).$$

Thus, Γ is δ'-hyperbolic for $\delta':=\gamma(2\kappa+\delta+\gamma+2c)$. □

Corollary 5.1.7. *Let Γ and Δ be two quasi-isometric graphs. Then Γ is hyperbolic, if and only if Δ is hyperbolic.* □

Definition. A finitely generated groups is **hyperbolic** if one (and hence by Proposition 3.1.5 and Corollary 5.1.7 every) of its locally finite Cayley graph is hyperbolic.

Example 5.1.8. (1) Finite groups are hyperbolic.

(2) Free groups are hyperbolic.

(3) The group $\mathbb{Z}^2$ is not hyperbolic.

Lemma 5.1.9. *Let* Γ *be a* δ*-hyperbolic graph and* $K = x_0e_0 \dots x_n$ *be a closed walk in* Γ *with* $n > 4\delta+4$. *Then there exist two vertices* x_i, x_j *such that* $d(x_i, x_j)$ *is smaller than the length of each of the two walks* $x_ie_i \dots x_j$ *and* $x_je_j \dots x_i$.

Proof. Let us suppose that the claim does not hold. Then, for all x_i, x_j we have that either $x_ie_i \dots x_j$ or $x_je_j \dots x_i$ is a walk that belongs to a geodesic path. In particular, K corresponds to a cycle C.

Let $y_1, y_2, y_3 \in V(C)$ with

$$\begin{aligned} d(y_1, y_2) &= \lfloor \ell(C)/2 \rfloor, \\ d(y_2, y_3) &= \lceil \ell(C)/4 \rceil \text{ and} \\ d(y_3, y_1) &= \ell(C) - d(y_1, y_2) - d(y_2, y_3). \end{aligned}$$

Let P_i be the subpath of C from y_i to y_{i+1}[2] that realises this distance. Thus, the paths P_i are geodesic paths and $(y_1, y_2, y_3; P_1, P_2, P_3)$ is a geodesic triangle. By the choice of y_1 and y_2 and because of $\ell(C) \geq 4\delta + 4$ there exists a vertex $v \in V(P_1)$ with

$$d(v, y_1) > \delta < d(v, y_2).$$

Since Γ is δ-hyperbolic, there exists $w \in V(P_2) \cup V(P_3)$ with $d(v, w) \leq \delta$. This contradicts our assumption $d(v, w) = d_C(v, w)$. □

Theorem 5.1.10. *Hyperbolic groups are finitely presented.*

Proof. Let $G = \langle S \mid R \rangle$ be a δ-hyperbolic group, where S is a finite generating set. Let Γ be the Cayley graph of G and S. Every relator corresponds to a closed walk in Γ. If R contains a relator w of length more than $4\delta + 4$, then this corresponds to a closed walk $K = x_0e_0 \dots x_n$ of length more than $4\delta + 4$. By Lemma 5.1.9 there exist vertices x_i, x_j on K such that $d(x, y)$ is smaller than the lengths of $x_ie_i \dots x_j$ and $x_je_j \dots x_i$. Let $y_0f_0 \dots y_m$ with $y_0 = x_i$ and $y_m = x_j$ be a shortest walk between x_i and x_j. Then $x_ie_i \dots x_jf_{m-1} \dots f_0y_0$ and $y_0f_0 \dots f_{m-1}y_me_j \dots x_i$ are closed walks that correspond to words whose concatenation allows elementary reductions such that the resulting word is w. Thus, w lies in the normal subgroup generated by these two words of smaller length. Inductively, we obtain that R is generated as normal subgroup by words of length at most $4\delta + 4$. Since there are only finitely many such words over $S \cup S^{-1}$, we found a finite presentation of G. □

5.2 Subgroups of hyperbolic groups

We want to show that infinite hyperbolic groups always contains elements of infinite order.

Definition. Let G be a finitely generated group, S a finite generating set of G and $g \in G$. Then **cone** of g with respect to S is the set

$$Cone_S(g) := \{h \in G \mid d_S(1, gh) \geq d_S(1, g) + d_S(1, h)\}.$$

[2] or to y_1, if $i = 3$

Example 5.2.1. Let F be a free group of rank $n \in \mathbb{N}$ with free generating set S. Then F has exactly $2 \cdot |S| + 1$ cones: besides $Cone_S(1) = F$ there are the cones $Cone_S(s) = \{s_1 \dots s_n \mid s_i \in S \cup S^{-1}, s_1 \neq s^{-1}\}$ for each $s \in S \cup S^{-1}$.

Obviously, these cones are distinct (s^{-1} is the unique element of $S \cup S^{-1}$ that does not lie in $Cone_S(s)$) and for every word $s_1 \dots s_n$ over $S \cup S^{-1}$ with $n \geq 2$ we have $Cone_S(s_1 \dots s_n) = Cone_S(s_n)$.

Definition. A group is a **torsion group** if each of its elements is a torsion element.

Proposition 5.2.2. *Let G be an finitely generated infinite group that has only finitely many cones with respect to a finite generating set S. Then G is not a torsion group.*

Proof. We set

$$k := |\{Cone_S(g) \mid g \in G\}| .$$

Since S is finite, the Cayley graph Γ of G and S is locally finite. Thus and since G is infinite, there exists $g \in G$ with $d(1, g) > k$. Let $1 = g_0, g_1, \dots, g_m = g$ be a shortest path in Γ from 1 to g. Because of $m > k$ there exists two vertices $g_i \neq g_j$ with $i < j$ on this path that have the same cone. We claim that $h := g_i^{-1} g_j$ has infinite order. For this, we will show by Induktion that we have

$$d_S(1, g_i h^n) \geq d_S(1, g_i) + n \cdot d_S(1, h)$$

for all $n \in \mathbb{N}$. The proposition immediately follows, since the previous statement implies that the elements $g_i h^n$ must be distinct for all $n \in \mathbb{N}$.

For $n = 1$, the claim follows directly from the choice of h. So let $n \in \mathbb{N}$ such that the claim holds for n. Then we have $d_S(1, h^n) = n \cdot d_S(1, h)$ because of

$$\begin{aligned} d_S(1, g_i) + d_S(1, h^n) &\geq d_S(1, g_i h^n) \\ &\geq d_S(1, g_i) + n \cdot d_S(1, h) \\ &\geq d_S(1, g_i) + d_S(1, h^n). \end{aligned}$$

Furthermore, we have $h^n \in Cone_S(g_i) = Cone_S(g_i h)$. Thus, we obtain

$$\begin{aligned} d_S(1, g_i h^{n+1}) &= d_S(1, g_i h h^n) \\ &\geq d_S(1, g_i h) + d_S(1, h^n) \\ &= d_S(1, g_i) + d_S(1, h) + n \cdot d_S(1, h) \\ &= d_S(1, g_i) + (n+1) \cdot d_S(1, h). \end{aligned}$$

This finishes the induction. ☐

Proposition 5.2.3. *Let G be a hyperbolic group with finite generating set S. Then G has only finitely many cones with respect to S.*

Proof. For $g \in G$ and $r \in \mathbb{N}$ we define the set

$$P_r^S(g) := \{h \in B_r^{G,S}(1) \mid d_S(1, gh) \leq d_S(1, g)\}.$$

Let Γ be the Cayley-Graph of G and S. By assumption, there exists $\delta \in \mathbb{R}_{\geq 0}$ such that Γ is δ-hyperbolic. Set $r := 2\delta + 1$.

If we can show that the set $P_r^S(g)$ of each group element $g \in G$ already determines its cone, then the fact that each $P_r^S(g)$ is a subset of the finite set $B_r^{G,S}(1)$ implies that there are only finitely many distinct cones. Thus, we want to show that for all $g, g' \in G$ with $P_r^S(g) = P_r^S(g')$ we have $Cone_S(g) = Cone_S(g')$.

Let $g, g' \in G$ with $P_r^S(g) = P_r^S(g')$ and $h \in Cone_S(g)$. By induction on $d_S(1,h)$, we show $h \in Cone_S(g')$.

If $d_S(1,h) = 0$, then we have $h = 1$ and, obviously, we have $h \in Cone_S(g')$. If $d_S(1,h) = 1$, then $h \in Cone_S(g)$ implies together with the definitions of cones and of $P_r^S(g)$ that h does not lie in $P_r^S(g) = P_r^S(g')$. Thus, it must lie in $Cone_S(g')$.

So let $d_S(1,h) > 1$. Then there exists $s \in S \cup S^{-1}$ and $h' \in G$ with $h = h's$ and $d_S(1,h') = d_S(1,h) - 1$. Since $h \in Cone_S(g)$, we have $h' \in Cone_S(g)$ and $h' \in Cone_S(g')$ by induction.

Let us suppose that $h \notin Cone_S(g')$ holds. Then we have

$$d_S(1, g'h) < d_S(1, g') + d_S(1, h).$$

Let $s_1, \ldots, s_n \in S \cup S^{-1}$ with $s_1 \ldots s_n = g'h$ and $n = d_S(1, g'h)$. Set $k_1 := s_1 \ldots s_{d_S(1,g')}$ and $k_2 := k_1^{-1} g'h$. Then we have

$$\begin{aligned} d_S(1, g'h) &= d_S(1, k_1) + d_S(1, k_2) \text{ and} \\ d_S(1, k_1) &= d_S(1, g') \end{aligned}$$

and, since $h \notin Cone_S(g')$, we obtain

$$d_S(1, k_2) \leq d_S(1, h) - 1.$$

We consider the element $h'' := {g'}^{-1} k_1$. We have

$$\begin{aligned} d_S(1, h'') &= d_S(1, {g'}^{-1} k_1) \\ &= d_S(g', k_1) \\ &\leq 2\delta + 1 \\ &\leq r, \end{aligned}$$

where the second last inequality follows similarly as one of the exercises: the $s_1 \ldots s_n$ define a geodesic path and a different one is defined by g' and h'. Its end vertices have distance 1 and we can use an argument as in one of the exercises to obtain $d_S(g', k_1) \leq 2\delta + 1$. Thus, h'' lies in $B_r^{G,S}(1)$.

Furthermore, we have

$$d_S(1, g'h'') = d_S(1, k_1) = d_S(1, g')$$

and hence we obtain $h'' \in P_r^S(g') = P_r^S(g)$. Because of $h \in Cone_S(g)$ we get:

$$\begin{aligned} d_S(1,g) + d_S(1,h) &\le d_S(1,gh) \\ &= d_S(1, gg'^{-1}g'h) \\ &= d_S(1, gg'^{-1}k_1k_2) \\ &\le d_S(1, gh'') + d_S(1,k_2) \\ &\le d_S(1,g) + d_S(1,h) - 1. \end{aligned}$$

This contradiction finishes the induction and thus the proposition. □

As an immediate corollary of Propositions 5.2.2 and 5.2.3 we obtain the following.

Theorem 5.2.4. *Infinite hyperbolic groups are no torsion groups.* □

Next, we want to show that no hyperbolic groups has a subgroup isomorphic to $\mathbb{Z}^2$. Therefore, we first show that every infinite cyclic subgroup of a hyperbolic group is a quasi-isometric embedding of $\mathbb{Z}$.

Proposition 5.2.5. *Let g be an element of infinite order in a hyperbolic group G. Then the function*

$$\psi\colon \mathbb{Z} \to G,\ z \mapsto g^z$$

is a quasi-isometric embedding.

Proof. Let S be a finite generating set of the hyperbolic group G and let Γ be the Cayley graph of G and S. Let $\delta \ge 0$ such that Γ is δ-hyperbolic and set $n := |\{g \in G \mid d_S(1,g) \le 4\delta + 1\}|$. Here, a **midpoint** of a path is one of its (at most two) central vertices. First, we will show $d_S(1, g^{nr}) \ge r$ for all $r \in \mathbb{N}$. For this, let $r \in \mathbb{N}$ with $r > 0$ and $k \in \mathbb{N}$ with

$$d_S(1, g^k) > 8r + 4\delta + 1.$$

Let P be a geodesic 1–g^k path, x a midpoint of P and P_x a subpath of P of length $2r$ such that x is a midpoint of P_x as well. Let us show the following.

Claim 1. If $u \in B_r(1)$ and $v \in B_r(g^k)$, then we have $d_S(y, P_x) \le 4\delta + 1$ for every midpoint y of each geodesic u–v paths P'.

Proof of Claim 1. Let P'' be a geodesic 1–v path. We have

$$|d_S(1, g^k) - d_S(1, v)| \le r$$

and

$$|d_S(1, g^k) - d_S(u, v)| \le 2r.$$

Thus, the midpoint y of P' must have distance more than δ from $B_r(1)$ and $B_r(g^k)$ and, since Γ is δ-hyperbolic, there exists a vertex z on P'' with $d_S(y, z) \le \delta$. Since the lengths of P' and P'' differ by at most r, we have

$d_S(y, z') \leq \lceil r/2 \rceil + \delta$, where z' is a midpoint of P''. Analogously, we find a vertex x' on P of distance at most δ to z and such that

$$d(x, x') \leq 2(\lceil r/2 \rceil + \delta) \leq r + 2\delta + 1.$$

The claim follows. □

Since $n = |B_{4\delta+1}(1)|$, there are most $2nr$ distinct vertices of distance at most $4\delta + 1$ to P_x. Since all g^i are distinct and G acts freely on Γ, the image of x under all g^i are distinct. At most $2nr$ of these images have distance at most $4\delta + 1$ to P_x. Thus and since $d_S(1, g^i) = d_S(1, g^{-i})$, there exists $f(r) \leq nr$ with $0 < f(r)$ such that $g^{f(r)} \notin B_r(1)$ and $g^{k+f(r)} \notin B_r(g^k)$.

Claim 2. We have $d_S(1, g^{nR}) \geq R$ for all $R \in \mathbb{N}$ with $R > 0$.

Proof of Claim 2. Let us suppose that there exists $R \in \mathbb{N}$ with $R > 0$ and $d_S(1, g^{nR}) < R$. For every $m \in \mathbb{N}$ with $m > nR$, let $n_m, r_m \in \mathbb{N}$ such that $m = n_m nR + r_m$ and $0 \leq r_m < nR$. Since n_m can be arbitrarily large but there are only finitely many values for r_m, there exists for every $\varepsilon > 0$ a q_ε with $n_m \varepsilon > d_S(1, g^{r_m})$ for all m with $n_m \geq q_\varepsilon$. Let $m \in \mathbb{N}$ such that $n_m \geq q_\varepsilon$ for $\varepsilon := R - d_S(1, g^{nR})$. We obtain

$$\begin{aligned} d_S(1, g^m) &\leq d_S(1, g^{n_m nR}) + d_S(1, g^{r_m}) \\ &\leq n_m d_S(1, g^{nR}) + d_S(1, g^{r_m}) \\ &\leq n_m(R - \varepsilon) + d_S(1, g^{r_m}) \\ &< n_m R. \end{aligned}$$

Let $M \in \mathbb{N}$ such that $f(M) > nR$ and $n_{f(M)} \geq q_\varepsilon$. Then we have $f(M) \leq nM$ and $d_S(1, g^{f(M)}) > M$ by the choice of $f(M)$. So we obtain

$$d_S(1, g^{f(M)}) < n_{f(M)} R \leq f(M)/n \leq M.$$

This contradicts the choice of $f(M)$ and proves our claim. □

Now we are ready to prove that ψ is a quasi-isometric embedding. For this, let $i, j, m, r_{ij} \in \mathbb{Z}$ with $0 \leq m < n$ and $|i - j| = nr_{ij} + m$ and let $K \in \mathbb{R}_{\geq 0}$ with $d(1, g^{m'}) \leq K$ for all $0 \leq m' < n$. Then we have

$$\begin{aligned} \frac{1}{n}|i - j| - (n + K) &\leq r_{ij} + m - (m + K) \\ &= r_{ij} - K \\ &\leq d_S(1, g^{nr_{ij}}) - K \\ &\leq d_S(1, g^{nr_{ij}+m}) \\ &= d_S(1, g^{|i-j|}) \end{aligned}$$

and

$$\begin{aligned} d_S(1, g^{|i-j|}) &= d_S(1, g^{nr_{ij}+m}) \\ &\leq d_S(1, g^{nr_{ij}}) + d_S(1, g^m) \\ &\leq nr_{ij} d_S(1, g) + m d_S(1, g) \\ &\leq d_S(1, g)(nr_{ij} + m) \\ &= d_S(1, g)|i - j|. \end{aligned}$$

Choosing $\gamma := \max\{n, d_S(1, g)\}$ and $c := n + K$ as constants for the quasi-isometric embedding proves the assertion. □

Now we will take a closer look at the centralisers of elements of infinite order in hyperbolic groups.

Definition. Let G be a group and let $g \in G$. The **centraliser** of g is the subgroup

$$C_G(g) := \{h \in G \mid hg = gh\}.^3$$

Theorem 5.2.6. *Let G be an infinite hyperbolic group and let $g \in G$ be an element of infinite order. Then we have*

$$|C_G(g) : \langle g \rangle| \in \mathbb{N}.$$

Proof. Let S be a finite generating set of G and let $\delta \in \mathbb{R}_{\geq 0}$ such that the Cayley graph Γ of G and S is δ-hyperbolic. By Proposition 5.2.5 there exists $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{\geq 0}$ such that

$$\mathbb{Z} \to G, \ z \mapsto g^z$$

is a (γ, c)-quasi-isometric embedding. Let $h \in C_G(g)$. Since the order of g is infinite, there exists $m \in \mathbb{N}$ such that

$$d_S(1, g^m) > 2d_S(1, h) + 4\delta + 2.$$

We choose geodesic paths

- P_1 between 1 and g^m,
- P_2 between g^m and hg^m,
- P_3 between hg^m and h,
- P_4 between h and 1 and
- P_5 between 1 and hg^m.

[3] It is easy to verify that the centraliser is indeed a subgroup.

Let x be a midpoint of P_1. Then there exists a vertex y on P_5 of distance at most δ to x, since by the choice of the length of P_1 every vertex on P_2 has distance more than δ from x. Analogously, we find a vertex z on P_3 such that $d_S(y, z) \leq \delta$. Thus, we have $d_S(x, z) \leq 2\delta$.

Let κ be the constant of Proposition 5.1.4 (2). Then there exists $i, j \in \{0, \ldots, m\}$ such that $d_S(x, g^i) \leq \kappa$ and $d_S(z, hg^j) \leq \kappa$. We obtain

$$d_S(1, hg^{j-i}) = d_S(g^i, hg^j) \leq 2\kappa + 2\delta$$

and hence the coset $h\langle g\rangle$ contains a vertex of the ball $B_{2\kappa+2\delta}(1)$. Since this holds for all cosets of $\langle g\rangle$ and since this ball is finite, this finishes the proof. □

Corollary 5.2.7. *No hyperbolic group has a subgroup isomorphic to* $\mathbb{Z}^2$. □

We note that, in general, it is false that a group cannot be a subgroup of a hyperbolic group just because it is not hyperbolic itself, as Rips has shown the following result.

Theorem 5.2.8 (Rips)**.** *There exists a hyperbolic group that has a finitely generated subgroup that is not hyperbolic.*

Even stronger, the following was shown.

Theorem 5.2.9 (Brady)**.** *There exists a hyperbolic group that has a finitely presented subgroup that is not hyperbolic.*

We omit both proofs for these results.

5.3 Hyperbolic boundary

Definition. Let Γ be a hyperbolic graph. A (double) ray R is **geodesic** if

$$d_R(x, y) = d(x, y)$$

for all $x, y \in V(R)$. It is **quasi-geodesic** if there are $\gamma \in \mathbb{R}_{\geq 1}$ and $c \in \mathbb{R}_{\geq 0}$ such that

$$d_R(x, y) \leq \gamma d(x, y) + c$$

holds for all x, y on R. Two quasi-geodesic rays R_1, R_2 are **equivalent** if there exists $m \in \mathbb{N}$ such that the ray R_i has infinitely many vertices of distance at most m to R_j for all $i \neq j \in \{1, 2\}$.

Lemma 5.3.1. *Let* Γ *be a* δ*-hyperbolic graph. If* R_1 *and* R_2 *are two equivalent quasi-geodesic rays, then there exists* $m \in \mathbb{N}$ *such that* $d(x, R_i) \leq m$ *for all* $x \in V(R_j)$ *with* $i \neq j \in \{1, 2\}$.

Proof. Let R_1 and R_2 be (γ, c)-quasi-geodesic rays. Let m be the constant of the equivalence of the rays R_1 and R_2. Let $x_1, x_2 \in V(R_1)$ and $y_1, y_2 \in V(R_2)$ with $d(x_i, y_i) \leq m$ and let P_i be a shortest x_i–y_i path for all $i \in \{1, 2\}$. Then $Q := x_1P_1y_1R_2y_2P_2x_2$ is a $(\gamma, c + 2m)$-quasi-geodesic path and there exists by Proposition 5.1.4 (2) a constant κ, depending only on γ, c and m, such that $x_1R_1x_2$ lies completely in the κ-neighbourhood of Q and vice versa. This implies the assertion. □

Lemma 5.3.2. *Let* Γ *be a hyperbolic graph. Equivalence of quasi-geodesic rays in* Γ *is an equivalence relation.*

Proof. This follows immediately from Lemma 5.3.1. □

Remark 5.3.3. According to Lemma 5.1.5 and Proposition 5.1.4 the definition of the equivalence of quasi-geodesic rays is invariant under quasi-isometries.

Lemma 5.3.2 and Remark 5.3.3 lead us to the following definition.

Definition. Let Γ be a hyperbolic graph and let G be a hyperbolic group. The **hyperbolic boundary** $\partial_h(\Gamma)$ of Γ is the set of equivalence classes of the equivalence relation on the quasi-geodesic rays. The **hyperbolic boundary** of G is the hyperbolic boundary of one of its locally finite Cayley graphs.

We immediately obtain the following.

Proposition 5.3.4. *The cardinality of the hyperbolic boundary of hyperbolic groups is a quasi-isometry invariant.* □

Example 5.3.5. (1) The group $\mathbb{Z}$ has exactly two hyperbolic boundary points.

(2) If F is a free group of finite rank, then $|\partial_h(F)| = e(G)$.

Remark 5.3.6. Let Γ be a locally finite δ-hyperbolic graph.

(1) Similar as in an exercise, where it was shown that every end contains a geodesic ray, we obtain that every hyperbolic boundary points contains a geodesic ray R and that there exists for every vertex x a ray that starts at x and that has a common subrays with R.

(2) Let $R_1 = x_0x_1\dots$ and $R_2 = y_0y_1\dots$ be two geodesic rays that start at the same vertex $x_0 = y_0$ but that are not equivalent (as quasi-geodesic rays). Let η_i be the hyperbolic boundary point that contains R_i. Then there exists $r \in \mathbb{N}$ with $d(x_r, y_r) > 2\delta$ and we have $d(x_r, R_2) > \delta$. Since geodesic triangles are δ-thin, there exists for every geodesic triangle with vertices x_0, x_i, y_i for $i > r$ a vertex of the geodesic x_i–y_i path in $B_\delta(x_r)$. Since there are only finitely many vertices in $B_\delta(x_r)$, on of them, say z, lies on these x_i–y_i paths for infinitely many $i > r$. Thus, we find (similar to an exercise) a geodesic double ray with one subray in η_1 and another subray in η_2.

(3) Let R_1, R_2 be two geodesic double rays such that the hyperbolic boundary points defined by R_1[4] are the same as those defined by R_2. Then there exists $m \in \mathbb{N}$ such that R_1 lies in $B_m(R_2)$ and R_2 lies in $B_m(R_1)$. If we choose vertices x_1, x_2 on R_1 of distance at least $2m + 2\delta$ in Γ and vertices

[4] These are the hyperbolic boundary points that contain subrays of R_1.

y_1, y_2 on R_2 with $d(x_i, y_i) \leq m$, then we can apply the definition of δ-thin geodesic triangles and obtain that for every vertex x on $x_1R_1x_2$ that has distance more than $m+2\delta$ to each x_i there exists a vertex y on $y_1R_2y_2$ with $d(x,y) \leq 2\delta$. Thus, we may choose $m = 2\delta$.

(4) Let R_1, R_2 be two (γ, c)-quasi-geodesic double rays such that the hyperbolic boundary points defined by R_1 are the same as those defined by R_2. Then there exists $m \in \mathbb{N}$ such that R_1 lies in $B_m(R_2)$ and R_2 lies in $B_m(R_1)$. If we choose vertices x_1, x_2 on R_1 of distance at least $2m+2\delta$ in Γ and vertices y_1, y_2 on R_2 with $d(x_i, y_i) \leq m$, then we can apply Proposition 5.1.4 (2) and the definition of δ-thin geodesic triangles and obtain that for every vertex x on $x_1R_1x_2$ that has distance more than $m + 2\delta$ to each x_i there exists a vertex y on $y_1R_2y_2$ with $d(x,y) \leq 2\delta + 2\kappa$, where κ is the constant from Proposition 5.1.4 (2). Thus, we may choose $m = 2\kappa + 2\delta$.

Theorem 5.3.7. *Let G be a hyperbolic group. Then we have $|\partial_h(G)| \in \{0, 2, \infty\}$.*

Proof. If G is finite, then the hyperbolic boundary of G is empty. So let G be infinite. Then G contains an element g of infinite order by Theorem 5.2.4. The quasi-isometric embedding $\psi_g\colon \mathbb{Z} \to G,\ z \mapsto g^z$ (cf. Proposition 5.2.5) defines a quasi-geodesic double ray $\ldots x_{-1}x_0x_1\ldots$ by Lemma 5.1.5. Note that it follows from the definition of a quasi-geodesic double ray that the rays $x_0x_1\ldots$ and $x_0x_{-1}\ldots$ are not equivalent. Thus, we have $|\partial_h(G)| \geq 2$.

Let us now assume that $|\partial_h(G)| \geq 3$. We want to show that the hyperbolic boundary is infinite. Let S be a finite generating set of G and let Γ be the Cayley graph of G and S. Let $\delta \geq 0$ such that Γ is δ-hyperbolic. Let us suppose that the hyperbolic boundary is finite. Note that for every two hyperbolic boundary points there exists a geodesic double ray that defines these two hyperbolic boundary points (Remark 5.3.6 (2)). Since geodesic triangles are δ-thin and because of Remark 5.3.6 (3), there exists a finite subset B of $V(\Gamma)$ such that every geodesic double ray between every two hyperbolic boundary points meets B. Let R be a geodesic double ray. Because of $|\partial_h(G)| \geq 3$, there exists a vertex x on one of the other geodesic double rays that has distance more than $2\mathrm{diam}(B) + 2\delta$ to R. Since G acts transitively on Γ, there exists $g \in G$ with $x \in gB$. But then gB avoids the geodesic double ray R, which contradicts the choice of B: the hyperbolic boundary is G-invariant and thus gB must meet every geodesic double ray. This contradiction show $|\partial_h(G)| = \infty$ in our remaining case. □

Theorem 5.3.8. *Let G be a hyperbolic group.*

(1) *If $|\partial_h(G)| = 2$, then G is virtually $\mathbb{Z}$.*

(2) *If $|\partial_h(G)| = \infty$, then G has a free subgroup of rank 2.*

Proof. Let G be an infinite hyperbolic group with finite generating set S. In order to prove (1) let $|\partial_h(G)| = 2$. Then there exists a geodesic double ray R between the two hyperbolic boundary points of G by Remark 5.3.6 (2) and by (3) of that remark every geodesic double ray (that has to define the same hyperbolic boundary points) lies in $B_{2\delta}(R)$. By the transitivity of Γ, there exists no vertex of distance more than 2δ to R. Thus, we have $e(G) = 2$ and the claim follows from Theorem 3.4.6.

For the proof of (2), we assume $|\partial_h(G)| = \infty$. By Theorem 5.2.4 there exists $g \in G$ of infinite order. We consider the quasi-isometric embedding of $\langle g \rangle$ in G according to Proposition 5.2.5. By Lemma 5.1.5 the image of that embedding defines a (γ, c)-quasi-geodesic double ray R_g and by Remark 5.3.6 there exists a geodesic double ray R that defines the same hyperbolic boundary points. Note that these hyperbolic boundary points are g-invariant. Let g^+ be the hyperbolic boundary points that contains that subray of R which lies close to the g^i with $i \in \mathbb{N}$ and let g^- be the second hyperbolic boundary points defined by R.

Let $f \in G$ such that $d(f, R) > 2\delta$. Then $h := g^f$ has infinite order, too, $f^{-1}R$ is a geodesic double ray and $f^{-1}R$ is a (γ, c)-quasi-geodesic double ray. We set $h^+ := f^{-1}g^+$ and $h^- := f^{-1}g^-$.[5] Then there exists a geodesic double ray between every two hyperbolic boundary points of

$$Y := \{g^+, g^-, h^+, h^-\}.$$

By the choice of f, we have $|Y| \geq 3$, since not all geodesic rays in $f^{-1}R$ can be equivalent to R by Remark 5.3.6 (3). Let us suppose $|Y| = 3$. Then there exists $i_1, i_2, j_1, j_2 \in \mathbb{Z}$ such that $d(g^{i_\ell}, h^{j_\ell}) \leq m$ for $\ell \in \{1, 2\}$ and m as in the definition of equivalent quasi-geodesic rays and such that $d(g^{i_1}, g^{i_2})$ and $d(h^{j_1}, h^{j_2})$ are arbitrarily large. We obtain

$$d(g^{i_1+k|i_1-i_2|}, h^{j_1+k|j_1-j_2|}) \leq m$$

for all $k \in \mathbb{Z}$. But then we have $|Y| = 2$, which we had excluded.

By Remark 5.3.6 and since geodesic triangles are δ-thin, there exists $K \in \mathbb{N}$ such that $B_K(1)$ meets all (γ, c)-quasi-geodesic double rays between elements of Y and for every other hyperbolic boundary point the geodesic double ray from that point to at most one element of Y are not met by $B_K(1)$. Set $B := B_{2K}(1)$. We define:

$$\begin{aligned}
A_1 &:= \{\eta \in \partial_h(G) \mid \exists \text{ geodesic double ray from } \eta \text{ to } g^+ \text{ in } \Gamma \setminus B\},\\
A_2 &:= \{\eta \in \partial_h(G) \mid \exists \text{ geodesic double ray from } \eta \text{ to } g^- \text{ in } \Gamma \setminus B\},\\
B_1 &:= \{\eta \in \partial_h(G) \mid \exists \text{ geodesic double ray from } \eta \text{ to } h^+ \text{ in } \Gamma \setminus B\},\\
B_2 &:= \{\eta \in \partial_h(G) \mid \exists \text{ geodesic double ray from } \eta \text{ to } h^- \text{ in } \Gamma \setminus B\}.
\end{aligned}$$

Now let $n \in \mathbb{N}$ such that

$$d(B, g^n B) > \operatorname{diam}(B) + 2\delta < d(B, h^n B)$$

and let $\eta \in \partial_h(\Gamma) \setminus A_2$. Let Q be a geodesic double ray between $g^n\eta$ and g^+. If this double ray meets B, then it is $\operatorname{diam}(B)$ close to R at that vertex and

[5] This is the canonical extension of the automorphism f from Γ to $\partial_h(\Gamma)$: images of equivalent quasi-geodesic rays are equivalent, too, and thereby we can extend every automorphism to the hyperbolic boundary of Γ.

then it must also meet g^nB on its further way to g^+ (say at the vertex x). On the other side, every geodesic double ray P from g^- to $g^n\eta$ must pass B first and then g^nB. Since geodesic triangles are δ-thin, there exists a vertex y on Q between $g^n\eta$ and B that has distance at most 2δ to $P\cap g^nB$. But then, we have

$$d(x,y) \leq 2\delta + \operatorname{diam}(B) < d(B, g^nB) \leq d(y,B) + d(B,x) \leq d(x,y).$$

This contradiction proves $Q\cap B = \emptyset$ and thus $g^n\eta \in A_1$. Analogously, we obtain the other conditions in order to apply the Ping-Pong-Lemma (Lemma 2.1.12). We then obtain that g^n and h^n freely generate a free subgroup. □

Together with Corollary 3.5.12 we obtain the following result from Theorem 5.3.8.

Corollary 5.3.9. *If a hyperbolic groups is neither finite nor virtually $\mathbb{Z}$, then it has exponential growth.* □

5.4 Quasi-convex subgroups

Definition. Let G be a finitely generated group and let H be a subgroup of G. Let Γ be a locally finite Cayley graph of G and some finite generating set S. Let $k > 0$. Then H is *k-quasi-convex* if every geodesic in Γ with end vertices in H lies in the K-neighbourhood of H. It is *quasi-convex* if it is ℓ-quasi-convex for some $\ell > 0$.

We want to show that quasi-convex subgroups of hyperbolic groups are hyperbolic again.

Lemma 5.4.1. *Let G be a finitely generated group and let H be a quasi-convex subgroup of G. Then H is finitely generated and the canonical map $H \to G$ is a quasi-isometric embedding.*

Proof. Let Γ be a locally finite Cayley graph and let $k > 0$ such that H is k-quasi-convex for that Cayley graph. Let S be the set of all elements of H with distance at most $2k+1$ to 1 in Γ.

Let $P = x_0 \dots x_n$ be a geodesic path between 1 and $h \in H$ in Γ. For every x_i there exists $y_i \in H$ with $d(x_i, y_i) \leq k$. Thus, we have $d(y_i, y_{i+1}) \leq 2k+1$. We may assume $y_0 = 1$ and $y_n = h$. Then $y_0y_1\dots y_n$ is a path in the Cayley graph of H and S, which shows, that S is indeed a generating set of H.

By construction, we have

$$\frac{1}{2k+1} d(a,b) \leq d_S(a,b) \leq d(a,b).$$

Thus, the canonical embedding of H into G is a quasi-isometric embedding. □

Lemma 5.4.2. *Let H be a finitely generated subgroup of a hyperbolic group G. Then H is quasi-convex if and only if the canonical embedding $H \to G$ is a quasi-isometric embedding.*

Proof. If H is quasi-convex in G, then Lemma 5.4.1 implies that the canonical embedding is a quasi-isometric embedding. For the other direction, let us assume that the canonical embedding is a quasi-isometric embedding $\varphi\colon H \to G$. Let P be a geodesic path in a Cayley graph Δ of H with respect to some finite generating set S_H of H. Then its φ-image defines a quasi-geodesic path Q in a Cayley graph Γ of G and some finite generating set S_G of G according to Lemma 5.1.5. So Proposition 5.1.4 implies that there exists a constant κ depending only on the hyperbolicity constant and on the constant for the quasi-isometry such that every geodesic in Γ with the same end vertices as Q lies in the κ-neighbourhood of Q. This shows that H is quasi-convex. □

Corollary 5.4.3. *Every quasi-convex subgroup of a hyperbolic group is hyperbolic.*

Proof. Let H be a quasi-convex subgroup of a hyperbolic group G. Then it is finitely generated by Lemma 5.4.1 and the canonical embedding $H \to G$ is a quasi-isometric embedding. Then Proposition 5.1.6 implies that H is hyperbolic. □

Proposition 5.4.4. *Let G be either a free product with amalgamation or an HNN extension of finitely generated groups over finite subgroups. Then the factors are quasi-convex in G.*

Proof. Obviously, G is finitely generated, too. First, let $G = A *_C B$ with C being finite. Let S be a finite generating set of G that consists of the elements of a finite generating sets for A and of one for B. Let Γ be a Cayley graph of G and S. Let P be a geodesic path in Γ whose end vertices lie in A. Let ℓ be the longest distance in Γ between vertices in C. Then every time P leaves A through a coset of C, it must re-enter A through the same coset. (This follows from the existence of normal forms.) So the last vertex before exiting A and the first vertex after entering A have distance at most ℓ. Thus, every vertex of P lies within distance $\ell/2$ of A.

A similar argument holds in the case of HNN extensions. □

Let us now proof a theorem that is an analogue of Theorem 3.4.6 for free groups of arbitrary finite rank.

Theorem 5.4.5. *A finitely generated group is quasi-isometric to a free group of finite rank, if and only if it has a finitely generated free group as subgroup of finite index.*

Proof. Note that we may assume by Theorem 3.4.6 that the involved free groups have rank at least 2. The backward implication follows from Corollary 3.2.3. So let us assume that G is a finitely generated group that is quasi-isometric to a finitely generated free group. Since free groups of rank at least 2 have infinitely many ends, the same is true for G and we may apply Theorem 4.7.1. Free groups are hyperbolic and so Corollary 5.1.7 implies that G is hyperbolic, too. Since hyperbolic groups are finitely presented (Theorem 5.1.10), they are accessible by Theorem 4.7.6 and we may write G as free products with amalgamation and HNN extensions over finite subgroups such that the factors have at most one end each (Proposition 4.7.4) or, equivalently, as fundamental group of a finite graph of groups with finite edge groups and whose vertex groups have at most one end. By Proposition 5.4.4, the factors are quasi-convex in G and hence they are hyperbolic by Corollary 5.4.3.

Note that the ends of G correspond to its hyperbolic boundary points: there is a canonical bijection between them. Thus, every vertex group has at most one hyperbolic boundary point and hence by Theorem 5.3.7 none at all. Hence, all vertex groups are finite. So we are currently looking at a finite graph of groups with finite vertex groups (and finite edge groups). By an exercise, we know that the fundamental group has a free subgroup of finite index. Corollary 3.2.4 implies that this free subgroup is finitely generated. □

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