Exercises for Geometric Group Theory

Sheet 5

Exercise 1:

(1) Let $n \in \mathbb{N}$ with $n \geq 2$. Show (without using Theorem 3.4.6) that $\mathbb{Z}$ and $\mathbb{Z}^n$ are not quasi-isometric.

(2) Is the rank of free groups a quasi-isometry invariant?
If yes, prove it. If no, partition $\mathbb{N}$ such that this partition induces a quasi-isometry invariant.

Exercise 2: Show the following.

(1) The relation $\sim_{QI}$ is an equivalence relation on the class of metric spaces.

(2) Every quasi-isometry has a quasi-inverse.

Exercise 3*: Let Γ and Δ be two locally finite graphs. Prove that every quasi-isometry $f\colon \Gamma \to \Delta$ induces a bijection on the ends of the graphs.

Exercise 4: Let Γ be a **transitive** connected graph, i. e. the automorphism group acts transitively on Γ. Prove the following.

(1) If Γ has at least three ends, then it has infinitely many ends.

(2) (not mandatory) If Γ is has infinitely many ends, then it has at least $2^{\aleph_0}$ many ends.

(3) (not mandatory) If Γ is locally finite and has infinitely many ends, then it has exactly $2^{\aleph_0}$ many ends.

Exercise 5: Let G, G', H, H' be finitely generated groups such that $G \sim_{QI} G'$ and $H \sim_{QI} H'$. Prove that $G * H$ is quasi-isometric to $G' * H'$

Only four of these five exercises are mandatory.

* This is also a written exercise.

due: 22nd November 2022