Exercises for Geometric Group Theory

Sheet 4

Exercise 1:

(1) Does Theorem 2.5.12 hold for free products of an arbitrary number of factors?

(2) Does Theorem 2.5.12 hold for free products with amalgamations of an arbitrary number of factors?

Exercise 2*:

(1) Define a **cyclically reduced form** for HNN extension similarly to that for free products with amalgamations.

(2) Prove that every element of an HNN extension is conjugated to an element that has a cyclically reduced form.

Exercise 3: For an element g of a group G, its **centraliser** $C_G(g)$ is defined as

$$C_G(g) := \{h \in G \mid gh = hg\}$$

Obviously, the centraliser is a subgroup of G. Show the following two statements for the free product of two groups G and H.

(1) Let $f \in G * H$. For every $g \in G^f$ and $h \in H^f$, we have $C_{G*H}(g) \leq G^f$ and $C_{G*H}(h) \leq H^f$.

(2) If $g \in (G * H) \setminus (\bigcup_{f \in G*H} G^f \cup H^f)$, then C_{G*H} is an infinite cyclic group.

Exercise 4: Let G, H, A be groups and let $\varphi\colon A \to G$ and $\psi\colon A \to H$ be monomorphisms. Assume that $\varphi(A)$ is normal in G and $\psi(A)$ is normal in H. Prove the following.

(1) $\varphi(A)$ is a normal subgroup of $G *_A H$.

(2) There exists a canonical isomorphism from $(G *_A H)/\varphi(A)$ to the free product $(G/\varphi(A)) * (H/\psi(A))$.

Exercise 5: Let G be a finitely presented group, let $A, B \leq G$ and let $\varphi\colon A \to B$ be an isomorphism. Prove that $G*_\varphi$ is finitely presented if and only if A is finitely generated.

Only four of these five exercises are mandatory.

* This is also a written exercise.

due: 15th November 2022