Fachbereich Mathematik
Matthias Hamann

WS 2022/23
1st November 2022

Exercises for Geometric Group Theory

Sheet 3

Exercise 1*:

1. How many elements does the group

$$\langle x, y, z \mid xyx^{-1} = y^2, yxy^{-1} = x^2, xzy = z^{-1} \rangle$$

have?

2. Prove that $D_\infty := \langle s, t \mid t^2, tst^{-1}s \rangle$ is the isometry group of $\mathbb{Z}$ with the standard metric.
(The isometry group is the set of distance-preserving bijections on $\mathbb{Z}$.)

Exercise 2: Let A and G be groups and let $\varphi\colon A \to G$ be a group monomorphism. Determine the free product with amalgamation $G *_A A$ for the monomorphisms φ and id_A.

Exercise 3: Let F be a free group with free generating set $\{a, b\}$ and let $N := \langle a^i b^{-i} \mid i \in \mathbb{Z} \rangle$. Prove the following statements.

(1) The set $\{a^i b^{-i} \mid i \in \mathbb{Z} \setminus \{0\}\}$ is a free generating set of the free group N.

(2) N is a normal subgroup of F.

(3) $F/N \cong \langle a, b \mid a = b \rangle$.

Exercise 4: Let F be a free group of rank at least 2. Show that F has a free subgroup of finite rank at least n for every $n \in \mathbb{N}$.

Exercise 5: Let $G = \langle S \mid R \rangle$, where $S = \{a, b, c_0, c_1, \ldots\}$ and

$$R = \{a^4, b^3, c_0^{-1} b c_0 a^{-2}, c_1^{-1} c_0 c_1 b^{-1}, c_2^{-1} c_1 c_2 c_0^{-1}, c_3^{-1} c_2 c_3 c_1^{-1}, \ldots\}.$$

Prove the following statements.

(1) $G \cong C_2$.

(2) Every finite presentation $\langle S' \mid R' \rangle$ with finite $S' \subseteq S$ and $R' \subseteq R$ defines a group of order either 3 or 4 or an infinite group.

Only four of these five exercises are mandatory.

* This is also a written exercise.

due: 8th November 2022