Fachbereich Mathematik
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WS 2022/23
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Exercises for Geometric Group Theory

Sheet 2

Exercise 1: Let F be a free group and let $f, g, h \in F$. Show the following.

1. If there exists $n \in \mathbb{N}$ with $f^n = g^n$, then $f = g$.
2. If $fg = gf$, $gh = hg$ and $g \neq 1$, then $fh = hf$.

Exercise 2*: Let F be a free group with free generating set $\{g, h\}$ and let $n \in \mathbb{N}$.

1. Determine the rank of the free subgroup $\langle g^{-i}hg^i \mid 1 \leq i \leq n\rangle$.
2. Does there exist a free subgroup of F of infinite rank?

Exercise 3: Prove Theorem 1.3.3.

A connected graph on which some group acts transitively and free is a Cayley graph.

Exercise 4: Prove Lemma 2.1.12.

Let G be a group acting on X. Let $(A_i)_{i \in I}, (B_i)_{i \in I}$ with $|I| \geq 2$ be two families of non-empty subsets of X such that all A_i and B_j are pairwise disjoint. If there are $g_i \in G$ such that $X \setminus B_i \subseteq g_iA_i$ for all $i \in I$, then $\langle g_i \mid i \in I\rangle$ is a free subgroup of G.

Exercise 5:

a) Let G and H be groups and let F be a free group. Let $\varphi\colon F \to H$ be a homomorphism and let $\psi\colon G \to H$ be an epimorphism. Show that there is a homomorphism $\varrho\colon F \to G$ with $\varrho\psi = \varphi$.

   Is ϱ uniquely determined?

b) Let G be a group. Let N be a normal subgroup of G such that G/N is a free group. Show that there is a **complement** for N in G, i. e. there exists $U \leq G$ such that $G = NU$ and $N \cap U = 1$.

Only four of these five exercises are mandatory.

* This is also a written exercise.

due: 1st November 2022