Fachbereich Mathematik
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WS 2022/23
10th January 2023

Exercises for Geometric Group Theory

Sheet 11

Exercise 1: Let G be a group with a parabolic action on a tree T such that all elements are elliptic. Show that T has a non-empty G-invariant proper subtree.

Exercise 2: Let the notations and hypotheses be as in Theorem 4.6.3. Set

$$R := \left\langle \bigcup \{H_{i,x} \mid i \in I, x \in G\} \right\rangle.$$

Show the following statements.

(1) R is the smallest normal subgroup of G that contains the groups $H_{i,x}$ with $i \in I$ and $x \in X_i$.

(2) $H/R \cong F$.

Exercise 3*: Let the notations and hypotheses be as in Theorem 4.6.3. Let $A = 1$, let $G \ncong \mathbb{Z}$ and assume that $G = B * C$ implies either $B = 1$ or $C = 1$. Show that G is contained in a conjugate of one of the G_i.

Exercise 4: Show that a group G has property (FA) if and only if the following statements hold.

(1) There is no infinite sequence $U_0 < U_1 < \ldots$ of subgroups with

$$\bigcup_{i \in \mathbb{N}} U_i = g;$$

(2) G is not a proper free product with amalgamation;

(3) G is no HNN extension.

Exercise 5: Let $\mathbb{G} = (\mathcal{G}, \Gamma, \Lambda)$ be a graph of groups with finite Γ and only finite vertex groups. Show that $\pi_1(\mathbb{G})$ has a free (normal) subgroup of finite index.

(Hint: first, consider the situation that Γ has only 'one' edge and construct an action of the fundamental group on a finite set in order to obtain the normal subgroup.)

Only four of these five exercises are mandatory.

* This is also a written exercise.

due: 17th January 2023