Fachbereich Mathematik
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WS 2022/23
18th October 2022

Exercises for Geometric Group Theory

Sheet 1

Exercise 1*: Prove Lemma 1.1.5:

Let G be a group and let X be a set. G acts on X if and only if there is a group homomorphism $G \to S_X$.

Additionally, G acts faithfully on X if and only if this group homomorphism is injectiv.

Exercise 2: Let T be a tree and let G be a group acting on T. Show or answer the following.

(i) If T is finite, then every $g \in G$ has a *fixed point* in T, i. e., for every $g \in G$ there is a vertex or an edge x with $xg = x$.
(ii) If G is finite, then every $g \in G$ has a fixed point.
(iii) If G and T are infinite, does every $g \in G$ have a fixed point?
(iv) (not mandatory) If T is finite, then there is a vertex or an edge fixed by all elements of G.
(v) (not mandatory) If G is finite, then there is a vertex or an edge fixed by all elements of G.

Exercise 3: Let G be a finite group, K a field and V a K-vector space of dimension $|G|$. Show that G is isomorphic to a subgroup of the general linear group $\mathrm{GL}(V)$.

Exercise 4: (a) Show that the automorphisms of the Petersen graph (Figure 1) act *transitively* on the vertices, i. e. for every two vertices u, v there is an automorphism that maps u to v.

(b) Show that the Petersen graph is not the underlying undirected graph of a Cayley graph.

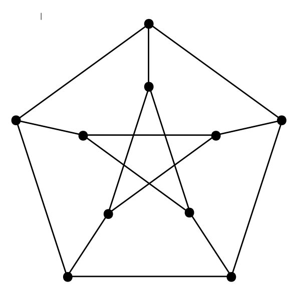

Figure 1: Petersen graph

Exercise 5: Show Theorem 1.2.8:

Every finitely generated group is isomorphic to the automorphism group of some graph.

We may choose this graph to be connected and locally finite.

* This is also a written exercise.

due: 25th Oktober 2022