Calculus – 7. Series, Solution

1. Prove that the following series converge and compute their sums.

$$(a)\ \sum_{k=2}^{\infty}\frac{(-1)^k}{3^{k+1}}\quad (b)\ \sum_{k=1}^{\infty}\frac{100}{4^{k+1}}\quad (c)\ \sum_{k=0}^{\infty}\frac{7^k+2}{8^k}.$$

Solution. We use the formula

$$\sum_{n=n_0}^{\infty} cq^n = \frac{cq^{n_0}}{1-q}\quad \text{if}\quad |\,q\,|<1,$$

which was given in the class.
(a) With $q=-1/3$, $c=1/3$, and $n_0=2$ we obtain

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{3^{k+1}} = \frac{\frac{1}{3}\left(-\frac{1}{3}\right)^2}{1-\left(-\frac{1}{3}\right)} = \frac{\frac{1}{27}}{\frac{4}{3}} = \frac{1}{36}.$$

(b) With $q=1/4$, $c=100/4=25$, and $n_0=1$ we obtain

$$\sum_{k=1}^{\infty}\frac{100}{4^{k+1}} = \frac{25\cdot\frac{1}{4}}{1-\frac{1}{4}} = \frac{25}{3}.$$

(c) The series can be splitted into two converging geometric series with both $n_0=0$; the first one with $q=7/8$ and $c=1$, the second one with $q=1/8$ and $c=2$. Hence,

$$\sum_{k=0}^{\infty}\frac{7^k+2}{8^k} = \frac{1}{1-\frac{7}{8}}+\frac{2}{1-\frac{1}{8}} = 8+\frac{2}{7} = 8\frac{2}{7}.$$

2. Which of the following series are convergent which are divergent. Give reasons!

$$(a)\ \sum_{k=1}^{\infty}k^3 3^{-k}\quad (b)\ \sum_{k=1}^{\infty}\frac{2^k+3^k}{4^k+1}\quad (c)\ \sum_{k=1}^{\infty}\frac{1}{2k-1}.$$

(*Hint.* Use the quotient and the comparison tests)

Solution. (a) Noting that $(5/4)^3<2$, $n\geq 4$ implies

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^3\cdot 3^n}{3^{n+1}n^3} = \frac{\left(1+\frac{1}{n}\right)^3}{3} < \frac{\left(\frac{5}{4}\right)^3}{3} < \frac{2}{3} < 1;$$

hence the ratio test gives convergence.
(b) Since

$$\frac{2^k+3^k}{4^k+1} < \frac{2\cdot 3^k}{4^k} = 2\left(\frac{3}{4}\right)^k$$

the comparison test with the geometric series $2\sum_{k=1}^{\infty}(3/4)^k<\infty$ gives convergence.
(c) Since $1/(2k)<1/(2k-1)$ the comparison test with the harmonic series

$$+\infty = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k} = \sum_{k=1}^{\infty}\frac{1}{2k} < \sum_{k=1}^{\infty}\frac{1}{2k-1}$$

gives divergence.

3. Which of the following series are convergent which are divergent?

(a) $1+\frac{2}{3}+\frac{3}{5}+\cdots+\frac{n}{2n-1}+\cdots$

(b) $\frac{1}{1001}+\frac{1}{2001}+\frac{1}{3001}+\cdots+\frac{1}{1000n+1}+\cdots$

(c) $\frac{1}{\sqrt{2}}+\frac{1}{2\sqrt{3}}+\frac{1}{3\sqrt{4}}+\cdots+\frac{1}{n\sqrt{n+1}}+\cdots$

(d) $\sum_{k=1}^{\infty}\frac{1000^k}{k!}$

(e) $0.001+\sqrt{0.001}+\sqrt[3]{0.001}+\cdots+\sqrt[n]{0.001}+\cdots$

Solution. (a) Since $a_n=\frac{n}{2n-1}\underset{n\to\infty}{\longrightarrow}\frac{1}{2}$ the necessary condition for convergence $a_n\to 0$ does not hold; hence the series diverges.
(b) Since for every positive integer n, $1/(1000n+1)>1/(1001n)$ we have

$$\frac{1}{1001}+\frac{1}{2001}+\frac{1}{3001}+\cdots+\frac{1}{1000n+1}+\cdots>\frac{1}{1001}\sum_{n=1}^{\infty}\frac{1}{n}.$$

Comparing this with the harmonic series gives divergence.
(c) Since

$$\frac{1}{n\sqrt{n+1}}<\frac{1}{n^{\frac{3}{2}}}$$

Example 10 together with the comparison test give convergence.
(d) Since

$$\left|\frac{a_{n+1}}{a_n}\right|=\frac{1000^{n+1}n!}{(n+1)!1000^n}=\frac{1000}{n+1}$$

$n\geq 2000$ implies

$$\left|\frac{a_{n+1}}{a_n}\right|\leq\frac{1}{2}$$

such that the ratio test gives convergence.
(e) Since $\sqrt[n]{p}\underset{n\to\infty}{\longrightarrow}1$ for every positive number $p>0$ (Proposition 5 (b)), Corollary 19 gives divergence.

4. True or false? The series $\sum_{n=1}^{\infty}a_n$ converges if for every positive integer p

$$\lim_{n\to\infty}\left(a_{n+1}+a_{n+2}+\cdots+a_{n+p}\right)=0.$$

Solution. The statement is false. A counterexample is the harmonic series $\sum_{n=1}^{\infty}1/n$. Indeed, for every positive integer p we have

$$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+p}\right)=\lim_{n\to\infty}\frac{1}{n+1}+\cdots+\lim_{n\to\infty}\frac{1}{n+p}=p\cdot 0=0.$$

The condition of the statement is fulfilled; however, the harmonic series diverges.

5. Consider the following series.

$$\sum_{n=1}^{\infty} a_n = \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots .$$

Compute $\underline{\lim} \frac{a_{n+1}}{a_n}$, $\underline{\lim} \sqrt[n]{a_n}$, $\overline{\lim} \sqrt[n]{a_n}$, and $\overline{\lim} \frac{a_{n+1}}{a_n}$. Apply both the root test and the ratio test!

Solution. We have

$$a_{2n-1} = \frac{1}{2^n} \quad \text{and} \quad a_{2n} = \frac{1}{3^n}.$$

Therefore

$$\begin{aligned}
\frac{a_{2n}}{a_{2n-1}} &= \left(\frac{2}{3}\right)^n \underset{n\to\infty}{\longrightarrow} 0; \\
\frac{a_{2n+1}}{a_{2n}} &= \frac{\frac{1}{2^{n+1}}}{\frac{1}{3^n}} = \frac{3^n}{2^{n+1}} \underset{n\to\infty}{\longrightarrow} +\infty; \\
\sqrt[2n-1]{a_{2n-1}} &= \frac{1}{2^{\frac{1}{2-1/n}}} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{2}}; \\
\sqrt[2n]{a_{2n}} &= \frac{1}{\sqrt{3}} \underset{n\to\infty}{\longrightarrow} \frac{1}{\sqrt{3}}.
\end{aligned}$$

Since the odd and the even numbers cover the entire $\mathbb{N}$, we find

$$\begin{aligned}
\underline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} &= 0, & q &:= \overline{\lim}_{n\to\infty} \frac{a_{n+1}}{a_n} = +\infty, \\
\underline{\lim}_{n\to\infty} \sqrt[n]{a_n} &= \frac{1}{\sqrt{3}}, & \alpha &:= \overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}}.
\end{aligned}$$

Since $\alpha < 1$ the root test indicates convergence. The ratio test gives no information, since neither $q < 1$ nor all but finitely many elements of $\left|\frac{a_{n+1}}{a_n}\right|$ are greater than or equal to 1.