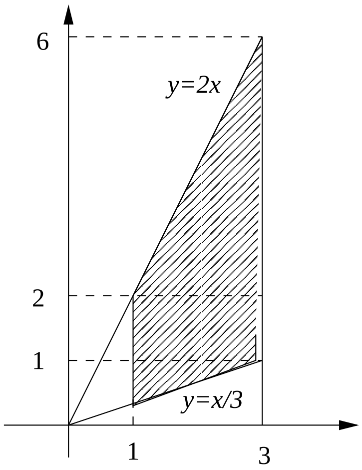
Calculus – 27. Series, Solutions

In what follows all regions are assumed to be *bounded* sets.
Try to draw a picture of the set. What kind of quadratic surfaces are involved (hyperboloids, cones, paraboloids).

1. Find the integration set and change the order of integration.

$$\text{(a)} \quad \int_1^3 \mathrm{d}x \int_{x/3}^{2x} f(x,y)\,\mathrm{d}y,$$

$$\text{(b)} \quad \int_0^3 \mathrm{d}x \int_0^{\sqrt{25-x^2}} f(x,y)\,\mathrm{d}y.$$



Solution. (a) The region is

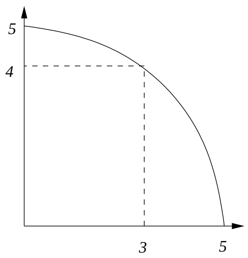
$$D = \{(x,y) \in \mathbb{R}^2 \mid 1 \le x \le 3, x/3 \le y \le 2x\}.$$

It can also be written as follows

$$\begin{aligned} D = \{(x,y) \in \mathbb{R}^2 \mid \frac{1}{3} \le y \le 1, 1 \le x \le 3y\} \cup \\ \cup \{(x,y) \in \mathbb{R}^2 \mid 1 \le y \le 2, 1 \le x \le 3\} \\ \cup \{(x,y) \ni \mathbb{R}^2 \mid 2 \le y \le 6, \frac{y}{2} \le x \le 3\}. \end{aligned}$$

Therefore,

$$I = \int_{1/3}^{1} \left(\int_1^{3y} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y + \int_1^2 \left(\int_1^3 f(x,y)\,\mathrm{d}x \right) \mathrm{d}y + \int_2^6 \left(\int_{y/2}^3 f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$



(b) The domain is

$$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 3, 0 \le y \le \sqrt{25-x^2}\}$$

which can also be written as

$$\begin{aligned} D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 4, 0 \le x \le 3\} \cup \\ \cup \{(x,y) \in \mathbb{R}^2 \mid 4 \le y \le 5, 0 \le x \le \sqrt{25-y^2}\} \end{aligned}$$

Therefore,

$$I = \int_0^4 \left(\int_0^3 f(x,y)\,\mathrm{d}x \right) \mathrm{d}y + \int_4^5 \left(\int_0^{\sqrt{25-y^2}} f(x,y)\,\mathrm{d}x \right) \mathrm{d}y.$$

2. Write $\iint\limits_D f(x,y,z)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$ as an iterated integral

(a) $D = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 \leq 2z,\ x^2 + y^2 + z^2 \leq 3\}$.
(b) D is bounded by the four planes $y = -1$, $y = 1$, $z = 0$, $z = 1$ and the surface $x^2 - y^2 = 1$.

Solution. (a) First we have to compute the intersection of the two surfaces $x^2+y^2 = 2z$ and $x^2 + y^2 + z^2 = 3$. Inserting the first equation into the second one gives $z^2 + 2z - 3 = 0$ which is equivalent to $z = 1$ since $z \geq 0$ in D. The intersection is the circle $x^2 + y^2 = 2$ in the plane $z = 1$. The domain therefore splits into two parts, a segment of a ball ($z > 1$) and a segment of a paraboloid ($0 \leq z \leq 1$):

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid |\,x\,| \leq \sqrt{2}, |\,y\,| \leq \sqrt{2 - x^2}, \frac{x^2 + y^2}{2} \leq z \leq \sqrt{3 - x^2 - y^2}\}.$$

The iterated integral is

$$\iiint_D f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\int_{\frac{x^2+y^2}{2}}^{\sqrt{3-x^2-y^2}} f(x, y, z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x.$$

(b) $x^2 - y^2 = 1$ is a hyperbolical cylinder. The region is

$$D = \{(x, y, z) \in \mathbb{R}^3 \mid 0 \leq z \leq 1,\ -1 \leq y \leq 1,\ -\sqrt{1 + y^2} \leq x \leq \sqrt{1 + y^2}\}.$$

The iterated integral is

$$\iiint_D f(x, y, z)\,\mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \left(\int_{-1}^{1} \left(\int_{-\sqrt{1+y^2}}^{\sqrt{1+y^2}} f(x, y, z)\,\mathrm{d}z \right) \mathrm{d}y \right) \mathrm{d}x.$$

3. Compute the volume of the sets bounded by the following surfaces

(a) $z = x^2 + y^2, \quad y = x^2, \quad y = 1, \quad z = 0.$
(b) $x^2 + y^2 = a^2, \quad x^2 + y^2 - z^2 = -a^2.$

Solution. (a) The region is

$$D = \{(x, y, z) \mid 0 \leq y \leq 1,\ -\sqrt{y} \leq x \leq \sqrt{y},\ 0 \leq z \leq x^2 + y^2\}.$$

Therefore, the volume is

$$\begin{aligned} V &= \iiint_D \mathrm{d}x\mathrm{d}y\mathrm{d}z = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} \mathrm{d}x \int_0^{x^2+y^2} \mathrm{d}z = \int_0^1 \mathrm{d}y \int_{-\sqrt{y}}^{\sqrt{y}} (x^2 + y^2)\,\mathrm{d}x \\ &= \int_0^1 \mathrm{d}y \left[\frac{1}{3}x^3 + y^2 x\right]_{x=-\sqrt{y}}^{x=\sqrt{y}} = \int_0^1 \left(\frac{2}{3} y^{\frac{3}{2}} + 2y^{\frac{5}{2}}\right) \mathrm{d}y \\ &= \frac{4}{15} + \frac{4}{7} = \frac{88}{105}. \end{aligned}$$

(b) We use cylinder coordinates to describe the region;

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}, \quad \frac{\partial(x,y,z)}{\partial(r,\varphi,z)} = r.$$

$$D = \{(x,y,z) \in \mathbb{R}^3 \mid |\,x\,| \le a,\, |\,y\,| \le \sqrt{a^2 - x^2},\, |\,z\,| \le \sqrt{x^2+y^2+a^2}\}$$
$$E = \{(r,\varphi,z) \in \mathbb{R}^3 \mid 0 \le \varphi \le 2\pi,\, 0 \le r \le a,\, |\,z\,| \le \sqrt{a^2+r^2}\}$$

By Theorem 9.7 we have

$$\begin{aligned} V &= \iiint\limits_D \mathrm{d}x\mathrm{d}y\mathrm{d}z = \iiint\limits_E r\,\mathrm{d}r\mathrm{d}\varphi\,\mathrm{d}z = \int_0^a r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\varphi \int_{-\sqrt{a^2+r^2}}^{\sqrt{a^2+r^2}} \mathrm{d}z \\ &= 2\pi \int_0^a \sqrt{a^2+r^2} 2r\,\mathrm{d}r \underset{u=r^2,\mathrm{d}u=2r\mathrm{d}r}{=} 2\pi \int_0^{a^2} (a^2+u)^{\frac{1}{2}} \mathrm{d}u \\ &= 2\pi \left[\frac{2}{3}(a^2+u)^{\frac{3}{2}}\right]_0^{a^2} = \frac{4\pi}{3}\left[\sqrt{8}a^3 - a^3\right] = \frac{4\pi}{3} a^3(\sqrt{8}-1). \end{aligned}$$

4. (a) If $A \subset [0,1]$ is an open set which contains every rational number in $(0,1)$. Then the boundary of A is $[0,1] \setminus A$.

   (b) Give an example of such a set A which is not Jordan measurable.

Proof. (a) *First proof.* By definition, $\partial A = \overline{A} \cap \overline{A^{\mathsf{c}}}$. Since A is open, A^{c} is closed, hence $\overline{A^{\mathsf{c}}} = A^{\mathsf{c}} = [0,1] \setminus A$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ and $\mathbb{Q} \cap [0,1] \subset A \subset [0,1]$, the closure of A is the entire interval $[0,1]$. Hence $\partial A = [0,1] \cap ([0,1] \setminus A) = [0,1] \setminus A$.
Second proof. By the discussion before Theorem 9.5, $[0,1] = A^{\circ} \cup \partial A \cup (A^{\mathsf{c}})^{\circ}$ where the three sets A°, ∂A, and $(A^{\mathsf{c}})^{\circ}$ are pairwise disjoint. Since $A = A^{\circ}$ (A is open), it sufficies to show that $(A^{\mathsf{c}})^{\circ} = \varnothing$. Suppose that $x \in (0,1)$ is an inner point of A^{c}. Then A^{c} contains a whole ε-neighborhood $U_{\varepsilon}(x)$ of x. However, $U_{\varepsilon}(x)$ is an open interval and contains a rational number, say q. Hence $q \in U_{\varepsilon}(x) \subset A^{\mathsf{c}}$. This contradicts the fact that A contains all rationals in $[0,1]$. This proves $(A^{\mathsf{c}})^{\circ}$ is empty; hence $\partial A = A^{\mathsf{c}}$.
(b) Let (x_n) be a sequence of all rational numbers in $(0,1)$ (see Corollary 6.4). Fix some c with $0 < c < 1$ and define

$$A = (0,1) \cap \bigcup_{n\in\mathbb{N}} U_{c/2^{n+1}}(x_n).$$

A is an open subset of $(0,1)$ which contains all rational numbers of $(0,1)$. The length of the interval $U_{c/2^{n+1}}(x_n)$ is $c/2^n$. Hence, the measure (length) of A is at most

$$\sum_{n\in\mathbb{N}} \frac{c}{2^n} = c\left(\frac{1}{2} + \frac{1}{2^2} + \cdots\right) = c.$$

It follows that the measure of the complement set $[0,1] \setminus A$ is at least $1 - c$. In particular, for $c < 1$, $\partial A = [0,1] \setminus A$ does not have measure 0; A is not Jordan measurable. ∎