Calculus – 25. Series, Solutions

1. Find the local extrema of the following functions

   (a) $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$,
   (b) $g(x,y,z) = x^2 + y^2 + z^2 - xy + x - 2z$,
   (c) $h(x,y) = x^3 + y^2$.

   Solution. We have to solve the system of equations $\operatorname{grad} f = 0$; at the solutions we have to compute the Hessians of f and to decide about definiteness.

   (a)

$$f_x = 4x^3 - 4x + 4y = 0, \quad f_y = 4y^3 - 4y + 4x = 0.$$

Taking the sum of both equations we obtain $x^3 + y^3 = 0$ which yields $y = -x$. Inserting this into the first equation we find $4x^3 - 8x = 0$ which has the three solutions $x = 0$ and $x = \pm\sqrt{2}$. There are three possible local extrema $(0,0)$, $(\sqrt{2}, -\sqrt{2})$, and $(-\sqrt{2}, \sqrt{2})$.

The second partial derivatives are

$$f_{xx} = 12x^2 - 4, \quad f_{yy} = 12y^2 - 4, \quad f_{xy} = 4,$$

such that

$$\operatorname{Hess} f(0,0) = \begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix}, \qquad \operatorname{Hess} f(\sqrt{2}, -\sqrt{2}) = \operatorname{Hess} f(-\sqrt{2}, \sqrt{2}) = \begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix},$$

Since $\det \operatorname{Hess} f(0,0) = 0$, one eigenvalue is 0. Since the matrix is nonzero, the second eigenvalue is nonzero; hence the matrix is semidefinit (no information about extrema). However,

$$f(\varepsilon, \varepsilon) = 2\varepsilon^4 > 0, \quad f(-\varepsilon, \varepsilon) = 2\varepsilon^4 - 8\varepsilon^2 = 2\varepsilon^2(\varepsilon^2 - 4) < 0$$

if $\varepsilon < 2$. Hence, $(0,0)$ is not a local extremum.

Since $A_1 = 20 > 0$ and $A_2 = \det \begin{pmatrix} 20 & 4 \\ 4 & 20 \end{pmatrix} = 384 > 0$, the Hessians are positive definite; hence $(\sqrt{21}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$ are local minima of f. (b)

$$\operatorname{grad} g(x,y,z) = (2x - y + 1, 2y - x, 2z - 2) \stackrel{!}{=} (0,0,0)$$

if and only if $x = -2/3$, $y = -1/3$, and $z = 1$. If g has a local extremum then at $(-2/3, -1/3, 1)$. We have for all (x,y,z)

$$A = \operatorname{Hess} g(x,y,z) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

The characteristic polynomial of A is

$$\det(A-\lambda\mathrm{id}\,) = \begin{vmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix} = (2-\lambda)(4-4\lambda+\lambda^2-1) = (2-\lambda)(\lambda-3)(\lambda-1).$$

Hence, the eigenvalues of A are 1, 2, and 3 such that A is positive definit. Hence g has a local minimum at $(-2/3,-1/3,1)$, $g(-2/3,-1/3,1) = -4/3$.

(c) $\operatorname{grad} h(x,y) = (3x^2, 2y) \overset{!}{=} (0,0)$ gives $(x,y) = (0,0)$. The Hessian at $(0,0)$ is

$$\operatorname{Hess} h(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$$

which is positive semidefinit (no information). However $h(x,y) = x^3 + y^2 > 0$ if $x > 0$ whereas $f(x,0) = x^3 < 0$ for $x < 0$. Hence $(0,0)$ is not a local extremum.

2. Find the point or points on the elliptic paraboloid $z = 4x^2 + y^2$ closest to $(0,0,8)$.

Solution. Sketch of a first solution. Insert $z = 4x^2 + y^2$ into the distance function

$$d^2(x,y,z) = (x-0)^2 + (y-0)^2 + (z-8)^2$$

which gives $f(x,y) = 16x^4 + y^4 - 63x^2 - 15y^2 + 8x^2y^2 + 64$. There are 5 critical points

$$(0,0), \qquad \left(0, \pm\frac{\sqrt{15}}{2}\right), \qquad \left(\pm\sqrt{\frac{63}{32}}\right).$$

The origin is a local maximum, the second pair are not local extrema, the third pair gives local minima. We have $z = 4\cdot\frac{63}{32} = \frac{63}{8}$. The corresponding minimal distance is

$$d = \frac{\sqrt{127}}{8}.$$

Second solution. We use the method of Lagrange multiplier; more precisely, we apply Example 8.16 (b). The square of the distance function is

$$f(x,y,z) = x^2 + y^2 + (z-8)^2;$$

while the constraint is

$$\varphi(x,y,z) = 4x^2 + y^2 - z = 0.$$

The minimal distance of $(0,0,8)$ to $M = \{\vec{x} \mid \varphi(\vec{x}) = 0\}$ is attained at some point (x,y,z) with

$$\operatorname{grad} f(x,y,z) = \lambda \operatorname{grad} \varphi(x,y,z),$$

where λ is a real number. This yields the system of equations

$$(2x, 2y, 2(z-8)) = \lambda(8x, 2y, -1), \quad 4x^2 + y^2 - z = 0.$$

This gives the same 5 solutions as in the first approach. However, the computations are much simpler.

3. Find the local maxima and minima of the function

$$f(x,y) = (x^2 + 3y^2)\,\mathrm{e}^{1-x^2-y^2}.$$

Solution. We have

$$f_x(x,y) = (2x - 2x^3 - 6xy^2)\mathrm{e}^{1-x^2-y^2}, \quad f_y(x,y) = (6y - 2x^2y - 6y^3)\mathrm{e}^{1-x^2-y^2}.$$

Hence $\operatorname{grad} f = 0$ is equivalent to

$$x(1 - x^2 - 3y^2) = 0, \quad y(3 - x^2 - 3y^2) = 0.$$

There are exactely 5 solutions $(0,0)$, $(0,1)$, $(0,-1)$, $(1,0)$, and $(-1,0)$. Let us compute the second partial derivatives

$$\begin{aligned} f_{xx}(x,y) &= (4x^4 + 12x^2y^2 - 6y^2 - 10x^2 + 2)\mathrm{e}^{1-x^2-y^2} \\ f_{yy}(x,y) &= (12y^4 + 4x^2y^2 - 30y^2 - 2x^2 + 6)\mathrm{e}^{1-x^2-y^2} \\ f_{xy}(x,y) &= (12xy^3 + 4x^3y - 16xy)\mathrm{e}^{1-x^2-y^2}. \end{aligned}$$

Therefore,

$\operatorname{Hess} f(1,0) = \operatorname{Hess} f(-1,0) = \begin{pmatrix} -4 & 0 \\ 0 & 4 \end{pmatrix}$	indefinite	no local extremum.
$\operatorname{Hess} f(0,1) = \operatorname{Hess} f(0,-1) = \begin{pmatrix} -4 & 0 \\ 0 & -12 \end{pmatrix}$	negative definite	local maxima.
$\operatorname{Hess} f(0,0) = \begin{pmatrix} 2\mathrm{e} & 0 \\ 0 & 6\mathrm{e} \end{pmatrix}$	positive definite	local minimum.

There are two local maxima at $(0,\pm 1)$ with $f(0,\pm 1) = 3$ and one local minimum at $(0,0)$ with $f(0,0) = 0$.

4. Consider the function $f(x,y) = (x-2)^2 y + y^2 - y$ on the triangular region $G = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0,\ y \geq 0,\ x + y \leq 4\}$.
Compute local and global maxima and minima of f on G.

Solution. Local extrema.

$$\operatorname{grad} f(x,y) = (2(x-2)y, (x-2)^2 + 2y - 1) \stackrel{!}{=} (0,0)$$

yields the 3 solutions $(1,0)$, $(3,0)$, and $\left(2, \frac{1}{2}\right)$. Two points are on the boundary, one is in the interior of the region. We have

$$\operatorname{Hess} f(x,y) = \begin{pmatrix} 2y & 2(x-2) \\ 2(x-2) & 2 \end{pmatrix},$$

such that

$$\operatorname{Hess} f(1,0) = \begin{pmatrix} 0 & -2 \\ -2 & 2 \end{pmatrix},$$

$$\operatorname{Hess} f(3,0) = \begin{pmatrix} 0 & 2 \\ 2 & 2 \end{pmatrix},$$

$$\operatorname{Hess} f\left(2, \frac{1}{2}\right) = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}.$$

The first two matrices are indefinite, the last one is positive definite. The only local extremum is at $\left(2, \frac{1}{2}\right)$ where f attains the local minimum $-\frac{1}{2}$.

Global extrema. Since G is compact and f is continuous, f attains its global maximum and minimum at certain points of G. We have to connsider the boundary of G.

Case 1: $x = 0$. Then $f(x, y) = g(y) = y^2 + 3y$. Since f is strictly increasing on $[0, 4]$, the minimum is at $y = 0$, $g(0) = 0$, and the maximum at $y = 4$, $g(4) = 28$.

Case 2: $y = 0$. Here $f(x, 0) = 0$ is constant.

Case 3: $x + y = 4$. Inserting $x = 4 - y$ we have

$$f(x, y) = g(y) = (2 - y)^2 y + y^2 - y = y^3 - 3y^2 + 3y.$$

Since $g'(y) = 3y^2 - 6y + 3 = 3(y - 1)^2 \geq 0$, g is monotonically increasing on $\mathbb{R}$. Therefore, the minimum is $f(4, 0) = 0$ and the maximum is $f(0, 4) = 28$.

Summarizing the three cases we see that the local minimum is also a global minimum, $f\left(2, \frac{1}{2}\right) = -\frac{1}{2}$ and the global maximum 28 is attained at the vertex of G, $(0, 4)$.

5. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = 2x^4 - 4x^2 y + y^2$. Show that f has a local minimum at $(0, 0)$ along every line through $(0, 0)$. Is the origin a local minimum for f as a function on $\mathbb{R}^2$? Is there a curve through $(0, 0)$ such that f has a strict local maximum at $(0, 0)$ along the curve?
   Hint. Try the curve $y = ax^2 + bx$. Look at the graph:
   `http://www.math.uni-leipzig.de/~gunesch/calc1/extrema.eps`
   `http://www.math.uni-leipzig.de/~gunesch/calc1/extrema.pdf`

Solution. A line through the origin has the equation $y = mx$, $m \in \mathbb{R}$ or $x = 0$.
Case $x = 0$. $f(0, y) = y^2$. At $y = 0$ we have a local minimum.
Case $y = mx$. We have

$$f(x, mx) = g(x) = 2x^4 - 4mx^3 + m^2 x^2$$

such that $g'(x) = 8x^3 - 12mx^2 + 2m^2 x$ and $g''(x) = 24x^2 - 24mx + 2m^2$. Since $g'(0) = 0$ and $g''(0) = 2m^2$, $(0, 0)$ is a local minimum on every line with $m > 0$. In case $m = 0$ we have $f(x, 0) = g(x) = 2x^4$; again, $x = 0$ is a local minimum.

Consider the curve $y = x^2$ through $(0, 0)$. Then $f(x, x^2) = g(x) = 2x^4 - 4x^4 + x^4 = -x^4$. This function g has obviously a strict local (and global) maximum at 0 since $-x^4 < 0$ for all nonzero $x \in \mathbb{R}$.