Calculus – 24. Series, Solutions

1. Let $f\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear mapping. Prove that f is differentiable on $\mathbb{R}^n$ and compute $Df(a)$ for all $a \in \mathbb{R}^n$.

   Proof. We will show that $Df(a) = f$ is the derivative of f at a. Indeed, since f is linear we have $f(a+h) = f(a) + f(h)$ and therefore

   $$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - f(h)\|}{\|h\|} = \lim_{h\to 0} \frac{0}{\|h\|} = 0.$$

   This shows that $Df(a)(h) = f(h)$ is the (constant) derivative of f at a. ∎

2. Let $I\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the inner product $I(x,y) = \langle x, y\rangle$.
   (a) Find $DI(a,b)$ and $I'(a,b)$.
   (b) If $f, g\colon \mathbb{R} \to \mathbb{R}^n$ are differentiable and $h\colon \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = \langle f(t), g(t)\rangle$, show that

   $$h'(t) = \langle f'(t), g(t)\rangle + \langle f(t), g'(t)\rangle\,.$$

   Hint. $\mathbb{R}^n \times \mathbb{R}^n = \{(x,y) \mid x, y \in \mathbb{R}^n\}$ is the set of pairs (x,y) of vectors of $\mathbb{R}^n$. It can be identified with the euclidean space $\mathbb{R}^{2n}$. For (b) use (a) and the chain rule.

   Solution. (a) We identify $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ and write

   $$I(x_1, \dots, x_n, y_1, \dots, y_n) = \sum_{k=1}^{n} x_k y_k.$$

   By Example 7 (a), the Jacobian matrix of I is the gradient of I (since we are in case $m = 1$). Since

   $$\begin{aligned} D_i I(x,y) &= \frac{\partial}{\partial x_i}\left(\sum_{k=1}^{n} x_k y_k\right) = \sum_{k=1}^{n} \frac{\partial x_k}{\partial x_i}\, y_k = \sum_{k=1}^{n} \delta_{ik} y_k = y_i, \\ D_{n+i} I(x,y) &= \frac{\partial}{\partial y_i}\left(\sum_{k=1}^{n} x_k y_k\right) = \sum_{k=1}^{n} x_k \frac{\partial y_k}{\partial y_i} = \sum_{k=1}^{n} x_k \delta_{ki} = x_i, \end{aligned}$$

   we obtain the gradient of I at (a,b) to be

   $$I'(a_1, \dots, a_n, b_1, \dots, b_n) = (b_1, \dots, b_n, a_1, \dots, a_n).$$

   The associated linear mapping $DI(a,b)$ is a linear functional on $\mathbb{R}^{2n}$ and can be written using the inner product:

   $$DI(a,b)(x,y) = \langle (b,a), (x,y)\rangle = \langle b, x\rangle + \langle a, y\rangle\,. \qquad (1)$$

We will give now an alternative proof of (1) using the definition of the derivative directly. Let $(h,k) \in \mathbb{R}^{2n}$ then

$$\begin{aligned}\varphi_{(a,b)}(h,k) &= I(a+h,b+k) - I(a,b) - DI(a,b)(h,k) \\ &= \langle a+h, b+k\rangle - \langle a,b\rangle - \langle b,h\rangle - \langle a,k\rangle = \langle h,k\rangle\,.\end{aligned}$$

Since $\|(h,k)\|^2 = \sum(h_i^2+k_i^2) \geq \sum h_i^2 = \|h\|^2$ we obtain using Cauchy–Schwarz inequality

$$\frac{|\,\varphi_{(a,b)}(h,k)\,|}{\|(h,k)\|} = \frac{|\,\langle h,k\rangle\,|}{\|h\|} \leq \frac{\|h\|\;\|k\|}{\|h\|} \leq \|k\| \underset{(h,k)\to 0}{\longrightarrow} 0.$$

This proves (1) directly without using Proposition 5.

(b) Since $h(t) = I(f(t),g(t))$ the chain rule (Theorem 7) gives

$$h'(t) = DI(f(t),g(t)) \circ D(f(t),g(t)).$$

Since $DI(f(t),g(t)) = (g(t),f(t))$ by (a) and

$$D(f(t),g(t)) = \begin{pmatrix} f'(t) \\ g'(t)\end{pmatrix}$$

is a column vector (see Example 7 (a), case $n=1$), we get

$$h'(t) = \begin{pmatrix} g(t) & f(t)\end{pmatrix}\begin{pmatrix} f'(t) \\ g'(t)\end{pmatrix} = \langle g(t), f'(t)\rangle + \langle f(t), g'(t)\rangle\,.$$

3. Let $v = v(x,y)$, $v \in \mathrm{C}(\mathbb{R}^2)$, be given and $x = x(r,\varphi) = r\cos\varphi$, $y = y(r,\varphi) = r\sin\varphi$ for all $r,\varphi \in \mathbb{R}$. Define a new function $u(r,\varphi) = v(x,y) = v(r\cos\varphi, r\sin\varphi)$. Compute

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi}$$

in terms of x, y and the partial derivatives of v with respect to x and y.

Solution. We first compute the partial derivatives of x and y with respect to r and φ:

$$x_r = \cos\varphi, \qquad x_\varphi = -r\sin\varphi, \tag{2}$$
$$y_r = \sin\varphi, \qquad y_\varphi = r\cos\varphi, \tag{3}$$
$$x_{r\varphi} = -\sin\varphi, \qquad x_{\varphi\varphi} = -r\cos\varphi, \quad x_{rr} = 0, \tag{4}$$
$$y_{r\varphi} = \cos\varphi, \qquad y_{\varphi\varphi} = -r\sin\varphi, \quad y_{rr} = 0. \tag{5}$$

Using the chain rule we find

$$\begin{aligned} u_r &= v_x\, x_r + v_y\, y_r, \\ u_\varphi &= v_x\, x_\varphi + v_y\, y_\varphi, \\ u_{rr} &= (v_{xx}\, x_r + v_{xy}\, y_r)x_r + v_x x_{rr} + (v_{yx}\, x_r + v_{yy}\, y_r)y_r + v_y\, y_{rr} \\ &= v_{xx}\, x_r^2 + 2v_{xy}\, x_r y_r + v_{yy}\, y_r^2 + v_x\, x_{rr} + v_y\, y_{rr} \\ u_{\varphi\varphi} &= v_{xx}\, x_\varphi^2 + 2v_{xy}\, x_\varphi y_\varphi + v_{yy}\, y_\varphi^2 + v_x\, x_{\varphi\varphi} + v_y\, y_{\varphi\varphi}\end{aligned}$$

Inserting (2) to (5) into the above equations we find

$$\begin{aligned} u_r &= v_x \cos\varphi + v_y \sin\varphi, \\ u_{rr} &= v_{xx}\cos^2\varphi + 2v_{xy}\cos\varphi\sin\varphi + v_{yy}\sin^2\varphi \\ u_{\varphi\varphi} &= v_{xx}r^2\sin^2\varphi - 2v_{xy}r^2\sin\varphi\cos\varphi + v_{yy}r^2\cos^2\varphi - rv_x\cos\varphi - rv_y\sin\varphi \end{aligned}$$

Finally we have

$$\begin{aligned} u_{rr} + \frac{1}{r^2}u_{\varphi\varphi} + \frac{1}{r}u_r &= v_{xx}\cos^2\varphi + 2v_{xy}\cos\varphi\sin\varphi + v_{yy}\sin^2\varphi + \\ &+ \left(v_{xx}\sin^2\varphi - 2v_{xy}\sin\varphi\cos\varphi + v_{yy}\cos^2\varphi - v_x\frac{1}{r}\cos\varphi - v_y\frac{1}{r}\sin\varphi\right) \\ &+ \frac{1}{r}\left(v_x\cos\varphi + v_y\sin\varphi\right) \\ &= v_{xx} + v_{yy} = \Delta(v). \end{aligned}$$

Remark: We have shown that $u_{rr} + \frac{u_r}{r} + \frac{u_{\varphi\varphi}}{r^2}$ is the Laplacian in polar coordinates.

4. (a) Compute the Taylor polynomial of degree 2 of $f(x, y) = x^y$ at $(2, 1)$.
(b) Compute the Taylor polynomial of degree 3 of $f(x, y) = 1/(1 - x - y^2)$ at $(0, 0)$.
Solution. (a) We have

$$\begin{aligned} &f_x = yx^{y-1}, && f_{xx} = y(y-1)x^{y-2}, && f_{xy} = x^{y-1} + yx^{y-1}\log x, \\ &f_y = x^y\log x, && f_{yy} = x^y(\log x)^2. \end{aligned}$$

Therefore $f(2, 1) = 2$ and

$$\begin{aligned} &f_x(2,1) = 1, && f_{xx}(2,1) = 0, && f_{xy}(2,1) = 1 + \log 2, \\ &f_y(2,1) = 2\log 2, && f_{yy}(2,1) = 2(\log 2)^2. \end{aligned}$$

Thus, the Taylor polynomial of degree 2 of f at $(2, 1)$ is (use the second formula after the proof of Theorem 11 (Taylor's theorem))

$$x^y = 2 + (x-2) + 2\log 2(y-1) + (1+\log 2)(x-2)(y-1) + (\log 2)^2(y-1)^2 + R_3(x, y).$$

MAPLE gives

```
> mtaylor(x^y, [x=2,y=1],3 );

                                                                    2        2
 x + 2 (y - 1) ln(2) + (1 + ln(2)) (y - 1) (x - 2) + (y - 1)  ln(2)
```

which is obviously the same.

(b) Instead computing all partial derivatives up to order 3 of f we can use the geometric series. If $|\, x + y^2 \,| < 1$ we have

$$\begin{aligned} \frac{1}{1 - x - y^2} = \frac{1}{1-(x+y^2)} &= \sum_{n=0}^{\infty}(x+y^2)^n \\ &= 1 + (x+y^2) + (x+y^2)^2 + (x+y^2)^3 + \text{higher terms} \\ &= 1 + x + x^2 + y^2 + x^3 + 2xy^2 + \text{higher terms}. \end{aligned}$$

MAPLE gives

```
> mtaylor(1/(1-x-y^2), [x,y],4 );
                                     2     2     3         2
                          1 + x + y  + x  + x  + 2 x y
```

5. Compute the equation of the tangent plane to the graph of the function f at point $(x_0, y_0, f(x_0, y_0))$.
(a) $f(x, y) = x^3 + y^3 - 3xy$, $(x_0, y_0) = (2, 1)$.
(b) $f(x, y) = \sqrt{9 - x^2 - y^2}$, $(x_0, y_0) = (1, 2)$.

Solution. By Remark 8.5, the tangent hyperplane (in $\mathbb{R}^{n+1}$) to the graph of $f(x)$ at point $(a, f(a))$ is

$$x_{n+1} - f(a) = \sum_{i=1}^{n} f_{x_i}(a)(x_i - a_i) = \langle \operatorname{grad} f(a), x - a \rangle .$$

In both questions (a) and (b), $n = 2$; the hyperplanes in $\mathbb{R}^3$ are actually planes.
(a) $\operatorname{grad} f(2, 1) = (3x^2 - 3y, 3y^2 - 3x)|_{(2,1)} = (9, -3)$. Hence, the equation of the plane is (with $x_3 = z$) since $f(2, 1) = 3$:

$$z - 3 = 2(x - 2) - 3(y - 1).$$

(b)

$$\operatorname{grad} f(1, 2) = \left(\frac{-x}{\sqrt{9 - x^2 - y^2}}, \frac{-y}{\sqrt{9 - x^2 - y^2}} \right)\Bigg|_{(1,2)} = \left(-\frac{1}{2}, -1 \right).$$

Since $f(1, 2) = \sqrt{9 - 1^2 - 2^2} = 2$, the equation of the tangent plane through $(1, 2, 2)$ is

$$z - 2 = -\frac{1}{2}(x - 1) - (y - 2).$$