Calculus – 23. Series, Solutions

1. Let $T \in \mathrm{L}(\mathbb{R}^n)$ be a linear mapping.

   (a) Prove: If λ is an eigenvalue of T, then $\|T\| \geq |\,\lambda\,|$.

   (b) Compute the norm of $A, B \in \mathrm{L}(\mathbb{R}^2)$, where

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

   Which are the eigenvalues of A and B?

   Proof. (a) We use the characterization of $\|T\|$ given after Definition 1:

$$\|T\| = \sup \left\{ \frac{\|Tx\|}{\|x\|} \,\middle|\, x \in \mathbb{R}^n, x \neq 0 \right\}. \tag{1}$$

   Since λ is an eigenvalue of T there is a non-zero eigenvector $v \in \mathbb{R}^n$ with $Tv = \lambda v$. Applying the norm to this equation yields

$$\|Tv\| = \|\lambda\, v\| = |\,\lambda\,|\ \|v\| \underset{v \neq 0}{\Longrightarrow} \frac{\|Tv\|}{\|v\|} = |\,\lambda\,|\,.$$

   Now (1) yields $\|T\| \geq |\,\lambda\,|$. (b) Inserting the special vector $x = (0,1)^\top$ into (1) we obtain na lower bound for the norm:

$$\|A\| \geq \frac{\|Ax\|}{\|x\|} = \frac{\left\| \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \binom{0}{1} \right\|}{\left\| \binom{0}{1} \right\|} = \frac{\left\| \binom{1}{0} \right\|}{\sqrt{0^2 + 1^2}} = 1.$$

   On the other hand by Proposition 1 and Definition 1 we have

$$\|A\| \leq \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}^2}$$

   Hence, $\|A\| \leq \sqrt{1^2 + 0 + 0 + 0} = 1$. We conclude $\|A\| = 1$. Since

$$B \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

   we have by (1)

$$\|Bx\| = \sqrt{(-x_2)^2 + x_1^2} = \|x\| \implies \|B\| = 1.$$

   Note that the eigenvalues of A are $\lambda_1 = \lambda_2 = 0$ whereas the eigenvalues of B are $\pm\mathrm{i}$. ∎

2. Find the partial derivatives of the following functions

(a) $f(x,y,z) = \sin(x\sin y)^{\cos z}$.
(b) $f(x,y,z) = x^{y+z}$.
(c) $f(x,y,z) = (y+z)^x$.
(d) $f(x,y) = g(x)^{h(y)}$ (in terms of g, h, g', and h'.)
(e) Compute $D_2 f(1,y)$ for

$$f(x,y) = y^{x^{x^{x^x}}} + \arctan(\arctan(\arcsin(x^2+y^2)))\log x.$$

Hint. There is an easy way to do this.

Solution. (a) Sorry, it was not quite clear how to read the function, $f(x,y,z) = (\sin(x\sin y))^{\cos z}$ or $\sin((x\sin y)^{\cos z})$. In the first case we have

$$\begin{aligned} f_x(x,y,z) &= \cos z\ \sin(x\sin y)^{\cos z - 1}\cos(x\sin y)\sin y, \\ f_y(x,y,z) &= \cos z\ \sin(x\sin y)^{\cos z - 1}\cos(x\sin y)x\cos y, \\ f_z(x,y,z) &= (\sin(x\sin y))^{\cos z}\log(\sin(x\sin y))(-\sin z). \end{aligned}$$

In the second reading we have

$$\begin{aligned} f_x(x,y,z) &= \cos((x\sin y)^{\cos z})\cos z(x\sin y)^{\cos z-1}\sin y, \\ f_y(x,y,z) &= \cos((x\sin y)^{\cos z})\cos z(x\sin y)^{\cos z-1}x\cos y, \\ f_z(x,y,z) &= \cos((x\sin y)^{\cos z})(x\sin y)^{\cos z}\log(x\sin y)(-\sin z). \end{aligned}$$

(b)

$$f_x = (y+z)x^{y+z-1}, \quad f_y = f_z = x^{y+z}\log x.$$

(c)

$$f_x = (y+z)^x\log(y+z), \quad f_y = f_z = x(y+z)^{x-1}.$$

(d)

$$f_x(x,y) = h(y)g(x)^{h(y)-1}g'(x), \quad f_y(x,y) = g(x)^{h(y)}\log(g(x))h'(y).$$

(e) Since we have to compute the partial derivative with respect to y, we can first insert $x = 1$ and then compute the derivative:

$$f(1,y) = y^{1^{1^{1^1}}} + \arctan(\arctan(\arcsin(1+y^2)))\log 1 = y.$$

Hence $D_2 f(1,y) = \frac{\mathrm{d}}{\mathrm{d}y}(y) = 1$.

3. Let

$$f(x,y) = \begin{cases} xy\frac{x^2-y^2}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y) = (0,0). \end{cases}$$

Prove that f is twice partial differentiable on $\mathbb{R}^2$, however $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Proof. It is obvious that $f \in \mathrm{C}^\infty(\mathbb{R}^2 \setminus \{(0,0)\})$, i.e. f is infinitely often continuously partial differentiable on $\mathbb{R}^2 \setminus 0$. This follows from Proposition 4.3. First we compute the partial derivatives of first order for $(x,y) \neq (0,0)$ and we show that $f_x(0,0) = f_y(0,0) = 0$ exist. For $(x,y) \neq (0,0)$ we have

$$D_1 f(x,y) = y\frac{x^2-y^2}{r^2} + xy\frac{2x(x^2+y^2)-(x^2-y^2)2x}{r^4} = \frac{yx^2-y^3}{r^2} + \frac{4x^2y^3}{r^4}.$$

Similarly,

$$D_2 f(x,y) = \frac{x^3-xy^2}{r^2} - \frac{4x^3y^2}{r^4}.$$

By definition,

$$D_1 f(0,0) = \lim_{h\to 0} \frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Similarly,

$$D_2 f(0,0) = \lim_{h\to 0} \frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

This shows that f is partial differentiable at $(0,0)$, too.
By definintion

$$D_2 D_1 f(0,0) = \frac{\mathrm{d}}{\mathrm{d}y}\left(f_x(0,y)\right)\big|_{y=0} = \lim_{h\to 0} \frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{-\frac{h^3}{h^2}}{h} = -1.$$

On the other hand

$$D_1 D_2 f(0,0) = \lim_{h\to 0} \frac{f_y(h,0)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1.$$

Similarly,

$$D_1 D_1 f(0,0) = \lim_{h\to 0} \frac{f_x(h,0)-f_x(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0$$

and

$$D_2 D_2 f(0,0) = \lim_{h\to 0} \frac{f_y(0,h)-f_y(0,0)}{h} = \lim_{h\to 0} \frac{0}{h} = 0.$$

Hence, all partial derivatives of f up to order 2 exist on $\mathbb{R}^2$; however, $f_{yx}(0,0) = -1 \neq 1 = f_{xy}(0,0)$. ∎