Calculus – 22. Series, Solutions

1. Prove: If f is an even or odd periodic function, i. e. $f(-x) = f(x)$ or $f(-x) = -f(x)$ for all x, respectively, then the Fourier series of f takes the form

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos kx \quad \text{and} \quad \sum_{k=1}^{\infty} b_k \sin kx, \quad \text{respectively.}$$

Hint. Show first that $\int_0^{2\pi} f(x)\,\mathrm{d}x = \int_{-\pi}^{\pi} f(x)\,\mathrm{d}x$.

Proof. Suppose f is periodic and integrable on $[0, 2\pi]$. Then we have for every $a \in \mathbb{R}$

$$\begin{aligned}\int_0^{2\pi} f(x)\,\mathrm{d}x &= \int_a^{a+2\pi} f(x-a)\,\mathrm{d}x = \int_a^{2\pi} f(x-a)\,\mathrm{d}x + \int_{2\pi}^{a+2\pi} f(x-a)\,\mathrm{d}x \\ &= \int_a^{2\pi} f(x-a)\,\mathrm{d}x + \int_0^a f(x+2\pi-a)\,\mathrm{d}x \\ &= \int_a^{2\pi} f(x-a)\,\mathrm{d}x + \int_0^a f(x-a)\,\mathrm{d}x \\ &= \int_0^{2\pi} f(x-a)\,\mathrm{d}x = \int_a^{2\pi+a} f(x)\,\mathrm{d}x. \end{aligned} \tag{1}$$

This shows that all integrals over an intervall of length 2π coincide; in particular $\int_0^{2\pi} f\,\mathrm{d}x = \int_{-\pi}^{\pi} f\,\mathrm{d}x$.

It is easy to see that for any odd function $g \in \mathcal{R}$ on $[-a, a]$ and every for every $a \in \mathbb{R}_+$, $\int_{-a}^{a} g(x)\,\mathrm{d}x = 0$:

$$\begin{aligned}\int_{-a}^{a} g(x)\,\mathrm{d}x &= \int_{-a}^{0} g(x)\,\mathrm{d}x + \int_0^a g(x)\,\mathrm{d}x \underset{t=-x}{=} \int_a^0 g(-t)(-\,\mathrm{d}t) + \int_0^a g(x)\,\mathrm{d}x \\ &= \int_0^a -g(t)\,\mathrm{d}t + \int_0^a g(x)\,\mathrm{d}x = 0. \end{aligned} \tag{2}$$

Suppose now f is even; then $f(x)\sin kx$ is odd since

$$f(-x)\sin(k(-x)) = -f(x)\sin kx,$$

and we have by (1) and (2)

$$b_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin kx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{odd function})\,\mathrm{d}x = 0.$$

Hence $f \sim a_0/2 + \sum_{k=1}^{\infty} a_k \cos kx$.

Suppose now, f is odd. Then $f(x)\cos kx$, $k \in \mathbb{Z}_+$, is odd since $\cos kx$ is even, and we obtain

$$a_k = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos kx\,\mathrm{d}x = \frac{1}{\pi}\int_{-\pi}^{\pi} (\text{odd function})\,\mathrm{d}x = 0.$$

Hence, $f \sim \sum_{k=1}^{\infty} b_k \sin kx$. ∎

2. Compute the Fourier series of the periodic function $f\colon \mathbb{R} \to \mathbb{R}$

$$f(x) = |\,x\,| \quad \text{if} \quad x \in [-\pi, \pi].$$

Apply Parseval's formula. Compare this result with Homework 22.4 (a).

Solution. By the result of the previous homework, $b_k = 0$ since $|\,x\,|$ is even. We have

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} |\,x\,|\,\mathrm{d}x = \frac{2}{\pi}\int_0^{\pi} x\,\mathrm{d}x = \frac{2}{\pi}\frac{1}{2}\pi^2 = \pi.$$

Using integration by parts with $u = x$, $v' = \cos kx$, $u' = 1$, $v = 1/k \sin kx$ we have for $k \geq 1$

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^{\pi} x\cos kx\,\mathrm{d}x = \frac{2x}{k\pi}\sin kx\Big|_{x=0}^{\pi} - \frac{2}{k\pi}\int_0^{\pi}\sin kx\,\mathrm{d}x \\ &= 0 + \frac{2}{k\pi}\left(\frac{1}{k}\cos kx\right)\Big|_0^{\pi} = \frac{2(\cos k\pi - 1)}{\pi k^2} = \frac{2((-1)^k - 1)}{\pi k^2} \\ &= \begin{cases} 0, & \text{if } k \text{ is even,} \\ -\frac{4}{\pi k^2}, & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

The Fourier series of f reads

$$f(x) \sim \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)x}{(2n+1)^2}.$$

Since

$$\sum_{k\in\mathbb{Z}} |\,c_k\,|^2 = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n\in\mathbb{N}}(a_n^2 + b_n^2)$$

Parseval's formula gives on the left

$$\|f\|_2^2 = \frac{2}{2\pi}\int_0^{\pi} x^2\,\mathrm{d}x = \frac{2\pi^3}{2\cdot 3\pi} = \frac{\pi^2}{3}.$$

and on the right

$$\frac{\pi^2}{2} + \frac{1}{2}\cdot\frac{16}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4}.$$

Hence

$$\frac{\pi^2}{3} = \frac{\pi^2}{4} + \frac{8}{\pi^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} \Longrightarrow \sum_{n=0}^{\infty}\frac{1}{(2n+1)^4} = \frac{\pi^4}{96}.$$

This coincides with the result of Question 22.4 (a) since

$$\begin{aligned} s &= s_{\text{odd}} + s_{\text{even}} = s_{\text{odd}} + \frac{1}{2^4}\left(\frac{1}{1} + \frac{1}{2^4} + \cdots\right) = s_{\text{odd}} + \frac{1}{16}s \\ s_{\text{odd}} &= \frac{15}{16}s = \frac{15}{16}\cdot\frac{\pi^4}{90} = \frac{\pi^4}{96}. \end{aligned}$$

3. Compute the Fourier series of $f(x) = |\sin x|$. Does the Fourier series converge uniformly to f? What happens at $x = 0$?

Solution. Since $|\sin x|$ is an even function, $b_k = 0$ for all k and we have

$$a_k = \frac{2}{\pi}\int_0^\pi \sin x \cos kx \,\mathrm{d}x = \frac{2}{\pi}\left(\frac{(-1)^{k+1}-1}{k^2-1}\right), \quad k \in \mathbb{Z}_+. \tag{3}$$

This formula is easily obtained using integration by parts twice. Indeed with $u = \sin x$, $v' = \cos kx$ we have $u' = \cos x$, $v = (\sin kx)/k$

$$I_k = \int_0^\pi \sin x \cos kx \,\mathrm{d}x = -\frac{1}{k}\int_0^\pi \cos x \sin kx \,\mathrm{d}x$$
$$I_k = -\frac{1}{k}\left(\frac{-\cos x \cos kx}{k}\bigg|_0^\pi - \frac{1}{k} I_k\right)$$

in the second line we used $u = \cos x$, $v' = \sin kx$, $u' = -\sin x$, $v = -(\cos kx)/k$. Since $\cos k\pi = (-1)^k$, this implies

$$I_k = \frac{(-1)^{k+1}-1}{k^2-1},$$

which gives (3). Hence,

$$a_k = \begin{cases} 0, & \text{if } k \text{ is odd,} \\ -\frac{4}{\pi(k^2-1)}, & \text{if } k \text{ is even.} \end{cases}$$

The Fourier series reads

$$|\sin x| \sim \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=1}^\infty \frac{\cos 2kx}{4k^2-1}.$$

Since $f(x) = |\sin x|$ is continuous and piecewise continuously differentiable, the Fourier series converges uniformly to f on $\mathbb{R}$ by Theorem 20. In particular we have pointwise convergence. At $x = 0$ we obtain

$$|\sin 0| = 0 = \frac{2}{\pi} - \frac{4}{\pi}\sum_{k=0}^\infty \frac{1}{(2k+1)(2k-1)}$$

which is equivalent to

$$\frac{1}{2} = \sum_{k=1}^\infty \frac{1}{(2k-1)(2k+1)} = \frac{1}{2}\sum_{k=1}^\infty \left(\frac{1}{2k-1} - \frac{1}{2k+1}\right).$$

The last equation is obvious since $1 - 1/3 + 1/3 - 1/5 + 1/5 - + \cdots$ converges to 1.

4. Use Example 6 (b), the Fourier series of

$$f(x) = \frac{(x-\pi)^2}{4} - \frac{\pi^2}{12} \sim \sum_{k=1}^\infty \frac{\cos kx}{k^2}$$

to prove that

$$\text{(a)} \quad \sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \qquad \text{(b)} \quad \sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^6}{945}.$$

Hint. Use Parseval's formula for f and for $\int f(x)\,\mathrm{d}x$.

Solution. (a) On the one hand we have

$$\|f\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi} f(x)^2\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{x^2}{4} - \frac{\pi}{2}x + \frac{\pi^2}{6}\right)^2 \mathrm{d}x$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{16}x^4 - \frac{\pi}{4}x^3 + \frac{\pi^2}{3}x^2 - \frac{\pi^3}{6}x + \frac{\pi^4}{36}\right)\mathrm{d}x = \frac{1}{2\pi}\left(\frac{2^5\pi^5}{16\cdot 5} - \frac{2^4\pi^5}{4\cdot 4} + \frac{8\pi^5}{9} - \frac{4\pi^5}{12} + \frac{2\pi^5}{36}\right)$$
$$= \frac{\pi^5}{2\pi}\left(\frac{2}{5} - 1 + \frac{8}{9} - \frac{1}{3} + \frac{1}{18}\right) = \frac{1}{2\pi}\cdot\frac{\pi^5}{180}(72 - 180 + 160 - 60 + 10) = \frac{\pi^4}{180}.$$

On the other hand

$$\frac{1}{2}\sum_{k=0}^{\infty} a_k^2 = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^4}.$$

Comparing this with the previous line, Parseval's formula yields $\displaystyle\sum_{k=1}^{\infty}\frac{1}{k^4} = \frac{\pi^4}{90}$.

(b) Since the Fourier series of $f(x) = \frac{1}{4}x^2 - \frac{\pi}{2}x + \frac{\pi^2}{6}$ converges uniformly on $\mathbb{R}$ to f, we can integrate the Fourier series term by term

$$\int_0^x f(t)\,\mathrm{d}t = \int_0^x \sum_{k=1}^{\infty}\frac{\cos kt}{k^2}\,\mathrm{d}t = \sum_{k=1}^{\infty}\int_0^x \frac{\cos kt}{k^2}\,\mathrm{d}t$$
$$g(x) = \frac{x^3}{12} - \frac{\pi}{4}x^2 + \frac{\pi^2}{6}x = \sum_{k=1}^{\infty}\frac{\sin kx}{k^3}.$$

Since the right series converges uniformly (by Theorem 3 (Weierstraß)): $\left|\dfrac{\sin kx}{k^3}\right| \le \dfrac{1}{k^3}$ and $\sum \dfrac{1}{k^3} < \infty$), $\displaystyle\sum_{k=1}^{\infty}\frac{\sin kx}{k^3}$ is the Fourier series of g. We compute the L^2-norm of g to apply Parseval's formula:

$$\|g\|_2^2 = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{x^3}{12} - \frac{\pi}{4}x^2 + \frac{\pi^2}{6}x\right)^2\mathrm{d}x$$
$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1}{144}x^6 - \frac{\pi}{24}x^5 + \frac{13\pi^2}{144}x^4 - \frac{\pi^3}{12}x^3 + \frac{\pi^4}{36}x^2\right)\mathrm{d}x$$
$$= \frac{1}{2\pi}\left(\frac{x^7}{1008} - \frac{\pi x^6}{144} + \frac{13\pi^2 x^4}{720} - \frac{\pi^3 x^3}{48} + \frac{\pi^4 x^3}{108}\right)\Bigg|_0^{2\pi} = \frac{\pi^6}{2\cdot 945}.$$

By Parseval's formula,

$$\|g\|_2^2 = \frac{\pi^6}{2\cdot 945} = \frac{1}{2}\sum_{k=1}^{\infty}\frac{1}{k^6};$$

the claim follows.