Calculus – 21. Series, Solutions

(turn in: May 21, 2003)

1. Compute the radius of convergence R and the sum of the series on $(-R, R)$, respectively.

$$\sum_{n=1}^{\infty} n^2 x^n, \qquad \sum_{n=1}^{\infty} n^3 x^n.$$

Solution. In both cases we have $R = 1/\lim_{n\to\infty} \sqrt[n]{n^2} = 1/\lim_{n\to\infty} \sqrt[n]{n^3} = 1$. Since power series can be differentiated term by term (Corollary 11), Example 4 shows

$$\begin{aligned}
\left(\frac{x}{(1-x)^2}\right)' &= \sum_{n=0}^{\infty} (nx^n)' = \sum_{n=0}^{\infty} n^2 x^{n-1} \\
\frac{1+x}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2 x^{n-1} \qquad\qquad | \cdot x \\
\frac{x(1+x)}{(1-x)^3} &= \sum_{n=1}^{\infty} n^2 x^n.
\end{aligned}$$

Differentiating the preceding equation we get

$$\begin{aligned}
\frac{x^4+4x+1}{(x-1)^4} &= \sum_{n=1}^{\infty} n^3 x^{n-1} \qquad\qquad | \cdot x \\
\frac{x(x^4+4x+1)}{(x-1)^4} &= \sum_{n=1}^{\infty} n^3 x^n.
\end{aligned}$$

2. Prove that the power series

$$\sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{2m+n}}{m!(m+n)!} \qquad (n \in \mathbb{N}_0)$$

converges for all $z \in \mathbb{C}$. This function is called the *Bessel function* of order n and it is denoted by $J_n(z)$.
Prove that $J_n(x)$ satisfies the differential equation

$$x^2 f''(x) + x f'(x) + (x^2 - n^2) f(x) = 0.$$

Proof. We compute the radius of convergence R_n of the series $J_n(z)$.

$$\begin{aligned}
R_n &= \lim_{m\to\infty} \left| \frac{a_m}{a_{m+1}} \right| = \lim_{m\to\infty} \frac{(m+1)!2^{2(m+1)+n}(m+n+1)!}{m!2^{2m+n}(m+n)!} \\
&= \lim_{m\to\infty} 4(m+1)(m+n+1) = +\infty.
\end{aligned}$$

Hence the series converges on the whole complex plane and defines an infinitely often differentiable function $J_n(z)$ (Corollary 12).
Since power series can be differentiated term by term we obtain

$$
\begin{aligned}
xJ_n'(x) &= \sum_{m=0}^{\infty} \boldsymbol{(2m+n)} \frac{(-1)^m (x/2)^{2m+n}}{m!(m+n)!}, \\
x^2 J_n''(x) &= \sum_{m=0}^{\infty} \boldsymbol{(2m+n)(2m+n-1)} \frac{(-1)^m (x/2)^{2m+n}}{m!(m+n)!}.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&x^2 J_n''(x) + xJ_n'(x) - n^2 J_n(x) = \\
&\qquad = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!(m+n)!} \big((2m+n)(2m+n-1) + (2m+n) - n^2\big). \qquad (1)
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
x^2 J_n(x) &= \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n} x^2}{m!(m+n)!} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^{m+1} (x/2)^{2(m+1)+n}}{(m+1)!(m+n+1)!} \left(-4(m+1)(m+n+1)\right) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!(m+n)!} \left(-4m(m+n)\right) \qquad \text{(index shift } m := m-1) \quad (2)
\end{aligned}
$$

Since in the last series there is a factor m, summation can start with $m = 0$. Since $(2m+n)(2m+n-1) + (2m+n) - n^2 = 4m^2 + 4mn = 4m(m+n)$ the sum of (1) and (2) gives 0; this completes the proof. ■

3. Using Stirling's formula prove that

$$
\left| n^{\frac{3}{2}} \binom{\frac{1}{2}}{n} \right| \underset{n\to\infty}{\longrightarrow} \frac{1}{2\sqrt{\pi}}.
$$

Proof. Since (see Section 7.3)

$$
a_n = \binom{\frac{1}{2}}{n} = \frac{\frac{1}{2} \cdot \left(\frac{1}{2} - 1\right) \cdots \left(\frac{1}{2} - n + 1\right)}{n!} = (-1)^{n-1} \frac{1}{2^{2n-1}(2n-1)} \binom{2n-1}{n},
$$

we have

$$
\begin{aligned}
\left| n^{\frac{3}{2}} \binom{\frac{1}{2}}{n} \right| &= \frac{n^{\frac{3}{2}}(2n-1)!}{2^{2n-1}(2n-1)(n-1)!n!} = \frac{n^{\frac{3}{2}} n (2n)!}{2^{2n-1} 2n(2n-1)(n!)^2} \\
&= \frac{n^{\frac{3}{2}}(2n)!}{2^{2n}(2n-1)(n!)^2}. \qquad (3)
\end{aligned}
$$

Now Stirling's formula (Proposition 5.31) shows

$$\frac{(2n)!}{(n!)^2} \sim \frac{\left(\frac{2n}{\mathrm{e}}\right)^{2n}\sqrt{2\pi 2n}}{\left(\frac{n}{\mathrm{e}}\right) 2\pi n} = \frac{2^{2n} n^{-\frac{1}{2}}}{\sqrt{\pi}}.$$

Inserting this into (3) we have

$$\left| n^{\frac{3}{2}} a_n \right| \sim \frac{n^{\frac{3}{2}} 2^{2n} n^{-\frac{1}{2}}}{2^{2n}(2n-1)\sqrt{\pi}} = \frac{1}{\sqrt{\pi}\left(2 - \frac{1}{n}\right)} \sim \frac{1}{2\sqrt{\pi}}.$$

■

4. Prove that for positive integers $k, m \in \mathbb{N}$ we have

$$\int_0^{2\pi} \cos kx \, \sin mx \, \mathrm{d}x = 0,$$
$$\int_0^{2\pi} \cos kx \, \cos mx \, \mathrm{d}x = \pi \delta_{km}, \quad k, m \in \mathbb{N},$$
$$\int_0^{2\pi} \sin kx \, \sin mx \, \mathrm{d}x = \pi \delta_{km},$$

where $\delta_{km} = 1$ if $k = m$ and $\delta_{km} = 0$ if $k \neq m$ is the so called *Kronecker symbol.*
Hint. Use $\int_0^{2\pi} \mathrm{e}^{inx}\,\mathrm{d}x = 0$ if $n \in \mathbb{Z} \setminus 0$.

Proof. Since $\mathrm{e}^{2\pi n \mathrm{i}} = 1$, $n \in \mathbb{Z}$, (Proposition 3.23) the fundamental theorem of calculus for complex valued functions (Section 5.5) gives for $n \in \mathbb{Z} \setminus \{0\}$ gives

$$\int_0^{2\pi} \mathrm{e}^{inx}\,\mathrm{d}x = \frac{1}{in} \mathrm{e}^{inx} \Big|_0^{2\pi} = \frac{1}{in}(1-1) = 0.$$

In case $n = 0$ we have $\int_0^{2\pi} \mathrm{e}^{inx}\,\mathrm{d}x = \int_0^{2\pi} \mathrm{d}x = 2\pi$; hence for all $n \in \mathbb{Z}$ we have

$$\int_0^{2\pi} \mathrm{e}^{inx}\,\mathrm{d}x = 2\pi \delta_{n0} \quad \text{(Kronecker symbol).} \tag{4}$$

Using the definition of the cosine function $\cos x = \frac{1}{2}(\mathrm{e}^{\mathrm{i}x} + \mathrm{e}^{-\mathrm{i}x})$ we obtain

$$\begin{aligned}\int_0^{2\pi} \cos kx \cos mx \,\mathrm{d}x &= \frac{1}{4}\int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}kx} + \mathrm{e}^{-\mathrm{i}kx}\right)\left(\mathrm{e}^{\mathrm{i}mx} + \mathrm{e}^{-\mathrm{i}mx}\right)\,\mathrm{d}x \\ &= \frac{1}{4}\int_0^{2\pi} \left(\mathrm{e}^{\mathrm{i}(k+m)x} + \mathrm{e}^{-\mathrm{i}(k+m)x} + \mathrm{e}^{\mathrm{i}(k-m)x} + \mathrm{e}^{-\mathrm{i}(k-m)}\right)\,\mathrm{d}x\end{aligned}$$

Since k, m are positive integers the first two summands vanish by (4). The third and the fourth summands are 2π if anf only if $k = m$, hence

$$\int_0^{2\pi} \cos kx \cos mx \,\mathrm{d}x = \frac{1}{4}(2\pi + 2\pi)\,\delta_{km} = \pi \delta_{km}.$$

The other two equations are proved in the same way. ■

5. Put $p_0 = 0$ and define for all positive integers $n \in \mathbb{N}$

$$p_{n+1}(x) = p_n(x) + \frac{1}{2}\left(x^2 - p_n(x)^2\right).$$

Prove that $p_n(x)$ uniformly converges to $|\,x\,|$ on $[-1, 1]$.
Hint. Use the identity

$$|\,x\,| - p_{n+1}(x) = (|\,x\,| - p_n(x))\left(1 - \frac{1}{2}(|\,x\,| + p_n(x))\right) \tag{5}$$

to prove that $0 \le p_n(x) \le p_{n+1}(x) \le |\,x\,|$ if $|\,x\,| \le 1$, and that

$$|\,x\,| - p_n(x) \le |\,x\,|\left(1 - \frac{|\,x\,|}{2}\right)^n < \frac{2}{n+1}$$

if $|\,x\,| \le 1$. For the last inequality consider two cases and induction over n.

Proof. We show (5). The distributive law and the binomial formula give

$$(|\,x\,| - p_n(x))\left(1 - \frac{1}{2}(|\,x\,| + p_n(x))\right) = |\,x\,| - p_n(x) - \frac{1}{2}(x^2 - p_n(x)^2) \underset{\text{by def.}}{=} |\,x\,| - p_{n+1}(x).$$

We use induction on n to prove $0 \le p_n(x) \le p_{n+1}(x) \le |\,x\,|$ for all positive integers n. Since $p_1(x) = \frac{1}{2}x^2$ the induction start $0 \le p_0(x) \le p_1(x) \le |\,x\,|$ is obvious. Suppose the inequality holds for some fixed n. We will show that it also holds for $n + 1$. Since

$$1 \ge 1 - \frac{1}{2}(|\,x\,| + p_{n+1}(x)) \ge 1 - \frac{1}{2}(1 + 1) \ge 0$$

(5) shows that both factors on the right are positive, the first one by induction hypothesis, such that $|\,x\,| \ge p_{n+2}(x)$. Moreover,

$$|\,x\,| - p_{n+2}(x) \le |\,x\,| - p_{n+1}(x)$$

since the right factor is less than 1. This shows $p_{n+2}(x) \ge p_{n+1}(x)$. Finally, since $p_{n+2}(x) \ge p_{n+1}(x)$ and the later is nonnegative, so is $p_{n+2}(x)$. This completes the induction proof. Since $(|\,x\,| - p_n(x))p_n(x) \ge 0$ from (5) it follows

$$|\,x\,| - p_{n+1}(x) \le (|\,x\,| - p_n(x))\left(1 - \frac{1}{2}|\,x\,|\right). \tag{6}$$

Iteration of (6) yields

$$\begin{aligned} |\,x\,| - p_{n+1}(x) &\le (|\,x\,| - p_{n-1}(x))\left(1 - \frac{1}{2}|\,x\,|\right)^2 \le \cdots \\ &\le (|\,x\,| - p_0(x))\left(1 - \frac{1}{2}|\,x\,|\right)^{n+1} = |\,x\,|\left(1 - \frac{1}{2}|\,x\,|\right)^{n+1}. \end{aligned}$$

This shows for that for every positive integer $n \in \mathbb{N}$ we have

$$|\,x\,| - p_n(x) \le |\,x\,|\left(1 - \frac{|\,x\,|}{2}\right)^n.$$

We prove for $|x| \leq 1$ and all $n \in \mathbb{N}$ by induction on n

$$|x|\left(1-\frac{|x|}{2}\right)^n < \frac{2}{n+1}.$$

The case $n=1$ is trivial since at least one factor is less than 1. Suppose the statement is true for some n; we will show the statement for $n+1$. If $|x| < 2/(n+2)$ this is obvious since the second factor is less than 1. Suppose now $\frac{2}{n+2} \leq |x| \leq 1$. This implies

$$1-\frac{1}{2}|x| \leq 1-\frac{1}{2}\frac{2}{n+2} = \frac{n+1}{n+2}.$$

Then

$$|x|\left(1-\frac{1}{2}|x|\right)^{n+1} \underset{\text{Ind.hyp.}}{<} |x|\frac{2}{n+1}\left(1-\frac{1}{2}|x|\right) \leq \frac{2}{n+1}\cdot\frac{n+1}{n+2} = \frac{2}{n+2}.$$

This proves the induction assertion. The inequality

$$0 \leq |x| - p_n(x) < \frac{2}{n+1}$$

shows that $p_n(x)$ uniformly converges to $|x|$ on $[-1,1]$.

■