Calculus – 20. Series, Solutions

1. Investigate uniform convergence of the following sequence and series on the interval $[a, b]$, $a, b \in \mathbb{R}$.

   (a) $f_n(x) = \dfrac{1}{1 + n^2 x^2}$;

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{x^n} \quad (a > 0)$.

Solution. (a) Since $f_n(0) = 1$ for all n and $\displaystyle\lim_{n\to\infty} \frac{1}{1 + cn^2} = 0$ for all nonzero numbers c, the pointwise limit of (f_n) is

$$f(x) = \begin{cases} 1, & x = 0, \\ 0, & x \neq 0. \end{cases}$$

We will show that the convergence is uniform on $[a, b]$ if and only if 0 is not in $[a, b]$. Suppose first that $a \leq 0 < b$. To $\varepsilon = \frac{1}{2}$ we find elements $x_n = 1/n$, $x_n \in [a, b]$ for sufficiently large n, such that

$$|\, f_n(x_n) - f(x_n) \,| = \frac{1}{1 + n^2/n^2} - 0 = \frac{1}{2}.$$

Hence, (f_n) does not converge uniformly to f. The same argument works for intervals with $a < 0 \leq b$ and $x_n = -1/n$.

Suppose now $0 < a < b$. Given ε with $1 > \varepsilon > 0$ choose $n_0 = \dfrac{1}{a\varepsilon}$. Then $n \geq n_0$ implies

$$n^2 > \frac{1}{a^2\varepsilon^2} > \frac{1}{a^2\varepsilon} = \frac{\frac{1}{\varepsilon}}{a^2} > \frac{\frac{1}{\varepsilon} - 1}{a^2}.$$

This implies

$$\frac{1}{\varepsilon} < 1 + n^2 a^2 \Longrightarrow \varepsilon > \frac{1}{1 + n^2 a^2} \geq \frac{1}{1 + n^2 x^2} = f_n(x).$$

The last inequality is due to $a \leq x$. This shows $f_n \rightrightarrows 0$ on $[a, b]$. The proof for $b < 0$ is quite the same.

(b) The quotient test (Corollary 2.26) shows that the series converges if and only in $|\, x \,| > 1$:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)x^n}{n x^{n+1}} \right| \overset{!}{<} 1 \Longrightarrow \frac{1}{|\, x \,|} \overset{!}{<} 1 \Longrightarrow |\, x \,| > 1.$$

Hence, in case $0 < a \leq 1$ the series does not converge for all x in $[a, b]$, in particular, the series does not converge uniformly. We will show that the series converges uniformly on $[a, +\infty)$ for $a > 1$.

Proof. Choose y with $1 < y < a \leq x$. Since $y/a < 1$ there is an n_0 (Proposition 2.5 (d)) such that

$$n\left(\frac{y}{a}\right)^n < 1 \quad \text{if} \quad n \geq n_0.$$

Putting $q = 1/y < 1$, we conclude that $x \geq a$ and $n \geq n_0$ implies

$$\frac{n}{x^n} \leq \frac{n}{a^n} < \left(\frac{1}{y}\right)^n = q^n.$$

Since $\sum q^n$ converges, Theorem 3 implies that $\sum \frac{n}{x^n}$ converges uniformly. ∎

2. Consider the sequences $f_n(x) = (1-x)x^n$ and $g_n(x) = \dfrac{1}{1-x}$ on $(0,1)$. Show that both f_n and g_n converge uniformly on $(0,1)$ to some functions f and g, respectively. Show that $(f_n \cdot g_n)$ does not converge uniformly to fg on $(0,1)$.

   Hint. For f_n split the interval $(0,1)$ into $(0, 1-\varepsilon)$ and $(1-\varepsilon, 1)$ and show that f_n becomes small on both segments.

   Proof. (a) We will show that $f_n \rightrightarrows 0$ on $(0,1)$ (so, $f = 0$). For, let $1 > \varepsilon > 0$ be given and fix $x \in (0, 1-\varepsilon)$. Since the geometric sequence $(1-x)x^n$ converges to 0 there is an n_0 such that $n \geq n_0$ implies

$$(1-x)x^n < x^n < \varepsilon.$$

   Fix y with $1-\varepsilon \leq y < 1$. Then

$$(1-y)y^n < 1 - y \leq \varepsilon$$

   for all $n \in \mathbb{N}$. Hence, $x \in (0,1)$ and $n \geq n_0$ implies $f_n(x) \leq \varepsilon$ which shows $f_n \rightrightarrows 0$ on $(0,1)$.
   It is trivial that the constant sequence $1/(1-x)$ uniformly converges to $g(x) = 1/(1-x)$ on $(0,1)$.

   (b) We will show that $h_n(x) = f_n(x)g_n(x) = x^n$ does not uniformly converge to $f(x)g(x) = 0$ on $(0,1)$. For let $\varepsilon = 1/4$ and $x_n = 1 - \dfrac{1}{n}$. Then for $n \geq 2$,

$$h_n(x_n) = x_n^n = \left(1 - \frac{1}{n}\right)^n \geq \frac{1}{4}$$

   since the sequence (x_n) is a (monotonically increasing) converging to $1/\mathrm{e}$ sequence. The point is that (x_n^n) *does not* approach 0 as n tends to ∞. This shows that there is no n_0 such that $|\, h_n(x) \,| \leq \frac{1}{4}$ for all $n \geq n_0$ and $x \in (0,1)$. ∎

3. Consider

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1+n^2 x}.$$

For what values of x does the series converge absolutely? On what intervals does it converge uniformly? On what intervals does it fail to converge uniformly? Is f continuous whenever the series converges? Is f bounded?

Solution. Obviously, the series diverges for $x = 0$. The series converges absolutely for all nonzero x. Namely,

$$\left| 1 + xn^2 \right| \geq 1 + \left| xn^2 \right| \geq |x| n^2$$

implies

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{1 + n^2 x} \leq \frac{1}{|x|} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

and the comparing test with $\sum 1/n^2$ shows convergence.

The series converges uniformly on all closed (finite or infinite) intervals not containing 0 (and hence on all open intervals (a, b) with $a > 0$). It does not converge uniformly on closed intervals containing 0.
Suppose first $0 < a$. The above argument shows that for $x \geq a$

$$\frac{1}{1 + xn^2} \leq \frac{1}{xn^2} \leq \frac{1}{an^2};$$

and Theorem 3 implies uniform convergence. A similar argument works for $x \leq b < 0$.

If f converges at x_0, it is continuous at x_0: We can find a closed interval $[a, b]$ with $x_0 \in [a, b]$ and $0 \notin [a, b]$. Since the series of continuous functions converges uniformly on $[a, b]$, Theorem 4 implies continuity of f on $[a, b]$.

Suppose 0 is a limit point of the (closed or open) interval from a to b. Suppose first that $x_n = 1/n^2$ belongs to the interval for sufficiently large n. Then we have

$$f(x_n) = \sum_{k=1}^{\infty} \frac{1}{1 + k^2/n^2} \geq \sum_{k=1}^{n} \frac{1}{1 + k^2/n^2} \geq \sum_{k=1}^{n} \frac{1}{1 + 1} = \frac{n}{2}.$$

This shows that f is unbounded at 0.

In a similar way we will show that the series does not converge uniformly when 0 is a limit point of the interval. For, let $\varepsilon = 1$ be given and suppose to the contrary there is an n_0 such that the Cauchy criterion, Proposition 1 (b), is satisfied, that is, for all $m, n \in \mathbb{N}$ with $m, n \geq n_0$ and for all x we have

$$\sum_{k=m}^{n} \frac{1}{1 + k^2 x} \leq 1. \tag{1}$$

However, since the sequence $\left(\frac{1}{1+k^2 x}\right)_{k \in \mathbb{N}}$ is decreasing we have

$$\sum_{k=m}^{n} \frac{1}{1 + k^2 x} \geq \sum_{k=m}^{n} \frac{1}{1 + n^2 x} = \frac{n - m + 1}{1 + xn^2}.$$

Inserting $x = 1/n^2$ we find

$$\sum_{k=m}^{n} \frac{1}{1+k^2 x} \geq \frac{n-m+1}{1+1} \geq 1$$

which contradicts (1). The series does not converge uniformly. A similar argument works if $x_n = -1/(2n^2)$ is in the interval from a to b for sufficiently large n.

4. For $n \in \mathbb{N}$ and real x put

$$f_n(x) = \frac{x}{1+nx^2}.$$

Show that (f_n) converges uniformly to a function f, and that the equation $f'(x) = \lim_{n\to\infty} f_n'(x)$ is correct if $x \neq 0$ but false if $x = 0$.

Proof. It is obvious that $f(x) = 0$ and so $f'(x) = 0$. Differentiation of f_n gives

$$f_n'(x) = \frac{1+nx^2 - x \cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}.$$

For nonzero x, $f_n'(x)$ is the quotient of a linear polynomial in n, $1 - nx^2$, and a quadratic polynomial in n, $(1+nx^2)^2$, such that

$$\lim_{n\to\infty} f_n'(x) = 0$$

for nonzero x. Hence, $\lim_{n\to\infty} f_n'(x) = f'(x) = 0$, $x \neq 0$. In case $x = 0$ we have $f_n'(0) = 1$ not converging to $f'(0) = 0$. ∎

5. Compute the radius of convergence R and the sum of the series on $(-R, R)$, respectively.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n}$;

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)}$.

Hint. Apply Theorem 7 to the geometric series.

Solution. (a) The root test gives $R = \lim_{n\to\infty} \sqrt[n]{n} = 1$. Let $x \in (-1,1)$. By Proposition 5, the geometric series

$$\sum_{n=0}^{\infty} t^n = \frac{1}{1-t}, \quad |\,t\,| < 1$$

converges uniformly on every compact subinterval of $(-1,1)$. Integrating the series from 0 to x we obtain by Theorem 7

$$\begin{aligned}\int_0^x \sum_{n=0}^{\infty} t^n \,\mathrm{d}t = \sum_{n=0}^{\infty} \int_0^x t^n \,\mathrm{d}t &= \int_0^x \frac{\mathrm{d}t}{1-t} \\ \sum_{n=0}^{\infty} \frac{x^{n+1}}{n+1} &= -\log(1-t)\big|_0^x = -\log(1-x) \\ \sum_{n=1}^{\infty} \frac{x^n}{n} &= -\log(1-x), \quad -1 < x < 1.\end{aligned}$$

(b) It is clear that $f(0) = 0$ since the series starts with x^1. Let us assume now $x \neq 0$. As in (a), the radius of convergence is $R = 1$ and the power series converges uniformly on every compact subinterval of $(-1,1)$. By (a) we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n} = -\log(1-t), \quad |\,t\,| < 1.$$

Integrating this from 0 to x, which can be done by Theorem 7 elementwise on the left, we have

$$\begin{aligned}\int_0^x \sum_{n=1}^{\infty} \frac{t^n}{n} \,\mathrm{d}t = \sum_{n=1}^{\infty} \int_0^x \frac{x^n}{n} \,\mathrm{d}t &= -\int_0^x \log(1-t)\,\mathrm{d}t \\ \sum_{n=1}^{\infty} \frac{x^{n+1}}{n(n+1)} &= (1-t)\left(\log(1-t) - 1\right)\big|_0^x \\ x\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} &= (1-x)\log(1-x) + x \\ \sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} &\underset{x\neq 0}{=} 1 + \frac{1-x}{x}\log(1-x), \quad |\,x\,| < 1,\, x \neq 0.\end{aligned}$$