Calculus – 19. Series, Solutions

1. Let X be a metric space and A_α, A, B subsets of X.
   (a) Prove that $\bigcup_\alpha \overline{A_\alpha} \subset \overline{\bigcup_\alpha A_\alpha}$
   (b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.
   (c) Give an example of subsets $F_n \subset \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\bigcup_{n\in\mathbb{N}} \overline{F_n} \neq \overline{\bigcup_{n\in\mathbb{N}} F_n}.$$

Proof. (a) This is an easy consequence of the fact that the closure is a *monotonic* set function, i. e. $M \subset N$ implies $\overline{M} \subset \overline{N}$. For, let $x \in \overline{M}$, i. e. every neighborhood U of x has a nonempty intersection with M. Since $M \subset N$, U has a nonempty intersection with N. Thus $x \in \overline{N}$; hence $\overline{M} \subset \overline{N}$.
Let us denote $N = \bigcup_\alpha A_\alpha$. Since by definition of the union $A_\alpha \subset N$ for all α, the above argument shows $\overline{A_\alpha} \subset \overline{N}$ for all α. Therefore,

$$\bigcup_\alpha \overline{A_\alpha} \subset \overline{N}.$$

(b) By (a) it suffices to show that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. By definition F is closed if and only if $F = \overline{F}$. This together with (a) proves that $N \subset F$, F closed, implies $\overline{N} \subset F$. Since $A \cup B \subset \overline{A} \cup \overline{B}$ and the latter set is closed by Proposition 10 (d) the assertion follows.

(c) Let $(x_n) \subset \mathbb{R}$ be the set of all rational numbers of $[0, 1)$ arranged in a sequence, see Corollary 4. Define $F_n = \{x_n\}$, $n \in \mathbb{N}$ be the one-point-set consisting of x_n only which is closed by Remark 1 (b). We find

$$\bigcup_n \overline{F_n} = \bigcup_n \{x_n\} = \mathbb{Q} \cap [0, 1) \subsetneq \overline{\mathbb{Q} \cap [0, 1)} = [0, 1],$$

where we used that $\mathbb{Q}$ is dense in $\mathbb{R}$, see Example 4 (a). ∎

2. (a) Define

$$f(x, y) = \begin{cases} \frac{xy}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove that f is not continuous at $(0, 0)$.
(b) Define

$$g(x, y) = \begin{cases} \frac{x^3+y^3}{x^2+y^2}, & (x, y) \neq (0, 0), \\ 0, & (x, y) = (0, 0). \end{cases}$$

Prove that g is continuous on $\mathbb{R}^2$.

Proof. (a) The sequence $(x_n, y_n) := (1/n, 1/n)$ converges converges to $(0,0)$ by Proposition 12. However, the sequence of images

$$f(x_n, y_n) = \frac{\frac{1}{n^2}}{\frac{1}{n^2} + \frac{1}{n^2}} = \frac{1}{2}$$

does not converge to 0. Hence, f is not continuous at the origin.

(b) By Proposition 17 it is clear by Proposition 17 that g is continuous on $\mathbb{R}^2 \setminus 0$. We have to prove the continuity of g at $(0,0)$ only. Given $\varepsilon > 0$ choose $\delta = \varepsilon/2$, then $(x, y) \in U_\delta((0,0)) \setminus (0,0)$, i.e. $x^2 + y^2 < \varepsilon^2/4$, $(x, y) \neq (0,0)$, implies

$$|\, f(x,y)\,| = \left| \frac{x^3 + y^3}{x^2 + y^2} \right| \le |\, x\,| \frac{x^2}{x^2 + y^2} + |\, y\,| \frac{y^2}{x^2 + y^2} \le |\, x\,| + |\, y\,| \le \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence, f is continuous at $(0,0)$. ■

3. Let ℓ_2 be the normed vector space $\ell_2 = \{(x_n) \mid x_n \in \mathbb{R}, \quad \sum_{n=1}^{\infty} x_n^2 < \infty\}$ with the norm $\|(x_n)\|_2 = \sqrt{\sum_{n=0}^{\infty} x_n^2}$. For $k \in \mathbb{N}$ put $\mathrm{e}_k = (0, \dots, 0, 1, 0, \dots)$ where the 1 is at the kth place.
(a) Is the sequence $(\mathrm{e}_k)_{k\in\mathbb{N}}$ bounded in ℓ_2?
(b) Does there exist a Cauchy subsequence of $(\mathrm{e}_k)_{k\in\mathbb{N}}$?
(c) Prove that the closed unit ball $B_1 = \{x \in \ell_2 \mid \|x\|_2 \le 1\} \subset \ell_2$ is not compact.

Hint. For (c) use Proposition 19.

Solution. (a) The sequence is bounded since

$$d(\mathrm{e}_k, \mathrm{e}_n) = \|\mathrm{e}_k - \mathrm{e}_n\|_2 = \|(0, \dots, 0, 1, 0, \dots, 0, 1, 0, \dots)\|_2 = \sqrt{2}.$$

(b) The above equation shows that there is no Cauchy subsequence of (e_k).

(c) Suppose to the contrary that B_1 is compact. By Proposition 19 any sequence in B_1 contains a converging in B_1 subsequence. Since $\mathrm{e}_k \in B_1$, (e_k) has a convergent subsequence which is, of course, a Cauchy sequence. This contradicts (b).

4. Give an example of an open cover of $(0,1)$ which has no finite subcover.

Solution. For a positive integer $n \in \mathbb{N}$ put

$$I_n = (0,1) \cap U_{\frac{1}{n(n+1)}}\left(\frac{1}{n}\right)$$

Then $\{I_n \mid n \in \mathbb{N}\}$ is an open cover of $(0,1)$ since

$$\frac{1}{n+1} < x \le \frac{1}{n} \quad \text{implies} \quad x \in I_n.$$

Suppose to the contrary that $\{I_n \mid n \in \mathbb{N}\}$ has a finite subcover, say, $\{I_1, I_2, \dots, I_m\}$. Then $1/(m+1)$ is not covered; a contradiction.

5. (a) Let $\mathrm{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ be the unit circle and consider the map $f\colon [0, 2\pi) \to \mathrm{S}^1$ given by $f(x) = \mathrm{e}^{\mathrm{i}x}$.
Is f continuous? Is f injective? Is f surjective? Is f a homeomorphism?
(b) Let $f\colon X \to Y$ be a bijective continuous mapping of the *compact* metric space X onto Y. Prove that f is a homeomorphism.

Hint. For (a) use the properties of the sine and cosine functions (monotony, existence of inverses). For (b) use Propositions 18 and 22.

Solution. (a) Since $\mathrm{e}^{\mathrm{i}x} = \cos x + \mathrm{i} \sin x$ and the sine and cosine functions are continuous on $\mathbb{R}$ (Proposition 3.20), $\mathrm{e}^{\mathrm{i}x}$ is continuous on $\mathbb{R}$ and in particular on the subset $[0, 2\pi)$.
f is injective. Suppose to the contrary that f is not injective, i. e. that there exist x and y with $x, y \in [0, 2\pi)$, $x < y$, and $\mathrm{e}^{\mathrm{i}x} = \mathrm{e}^{\mathrm{i}y}$. The Euler formula gives $\cos x = \cos y$ and $\sin x = \sin y$. Since $\cos x$ is strictly decreasing on $[0, \pi]$ and strictly increasing on $[\pi, 2\pi]$, the equation $\cos x = \cos y$ yields $\pi - x = \pi + y$. The equation $\sin x = \sin y$ then implies $\sin y = \sin(-x) = -\sin x = \sin x$; hence $\sin x = \sin y = 0$, such that $x = y = \pi$. A contradiction.
f is surjective. Suppose $z = a + b\mathrm{i} \in \mathrm{S}^1$, $a, b \in \mathbb{R}$ is given. Then $a^2 + b^2 = 1$ and $|a|, |b| \leq 1$. Put $\phi = \arcsin(a) \in [-\pi/2, \pi/2]$ and

$$x = \begin{cases} \phi, & \text{if } \quad 0 \leq \phi \leq \pi/2 \quad \text{and} \quad b \geq 0, \\ \phi + \pi, & \text{if } \quad 0 \leq \phi \leq \pi/2 \quad \text{and} \quad b < 0, \\ 2\pi + \phi, & \text{if } \quad -\pi/2 \leq \phi \leq 0 \quad \text{and} \quad b \leq 0, \\ \pi + \phi, & \text{if } \quad -\pi/2 \leq \phi \leq 0 \quad \text{and} \quad b \geq 0. \end{cases}$$

The addition formulas for sine and cosine show that $\mathrm{e}^{\mathrm{i}x} = a + b\mathrm{i}$.
f is not a homeomorphism since the inverse map $g\colon \mathrm{S}^1 \to [0, 2\pi)$, $g(\mathrm{e}^{\mathrm{i}x}) = x$, is not continuous at 1. Indeed, $z_n = \mathrm{e}^{(2\pi - 1/n)\mathrm{i}}$ converges to 1 as n tends to ∞. However $g(z_n) = 2\pi - 1/n$ converges to 2π which is different from $g(1) = 0$.
We can also use the result of (b), see below. The inverse mapping g cannot be continuous since S^1 is compact and the continuous image of a compact set is compact; however, $[0, 2\pi)$ is not a compact subset of $\mathbb{R}$.
Proof. (b) Let $g\colon Y \to X$ be the inverse mapping to f. We have to show that g is continuous. By Proposition 18 it is sufficient to show that the preimage $g^{-1}(F)$ of every closed set $F \subset X$ is closed. Since X is compact, F is compact by Proposition 21 (b). Since f and g are inverse to each other, $g^{-1}(F) = f(F)$. By Proposition 22 (a), $f(F) \subset Y$ is compact. By Proposition 21 (a), $f(F)$ is closed in X. This completes the proof. ∎