Calculus – 17. Series, Solutions

1. A point is moving in $\mathbb{R}^3$ according to

$$s(t) = (vt, l\cos\omega t, l\sin\omega t),$$

where v is a given speed, ω is a given angular speed, l is a given length.
(a) Compute velocity and acceleration vectors. Show that they are orthogonal.
(b) Compute the distance travelled by the point in a time interval of length t.

Solution. (a) The velocity and acceleration vectors are given by the first and second derivative of $s(t)$, respectively.

$$\begin{aligned} v(t) &= s'(t) = (v, -l\omega\sin\omega t, l\omega\cos\omega t), \\ a(t) &= s''(t) = (0, -l\omega^2\cos\omega t, -l\omega^2\sin\omega t), \\ \|v(t)\| &= \sqrt{v^2 + l\omega^2\sin^2\omega t + l^2\omega^2\cos^2\omega t} = \sqrt{v^2 + l^2\omega^2}. \end{aligned}$$

We show that the scalar product $v(t)$ by $a(t)$ vanishes.

$$\langle v(t), a(t)\rangle = v\cdot 0 + l^2\omega^3\sin\omega t\cos\omega t - l^2\omega^3\sin\omega t\cos\omega t = 0.$$

Hence the vectors are orthogonal.

(b) The instantaneous velocity is constant. The length of the path is

$$\ell(\gamma) = \int_{t_0}^{t_0+t} \|v(\tau)\|\,\mathrm{d}\tau = t\sqrt{v^2 + l^2\omega^2}.$$

2. Prove that the length of the arc given by the graph of the function

$$y = a\cosh\frac{x}{a}, \quad a > 0,$$

from point $A(0, a)$ to point $B(b, h)$ is $\sqrt{h^2 - a^2}$.

Proof. We have

$$1 + (y')^2 = 1 + \left(a\frac{1}{a}\sinh\frac{x}{a}\right)^2 = 1 + \sinh^2\frac{x}{a} = \cosh^2\frac{x}{a},$$

where we used $\cosh^2 z - \sinh^2 z = 1$ (cf. Homework 11.2). Hence

$$\ell(\gamma) = \int_0^b \cosh\frac{x}{a}\,\mathrm{d}x = a\sinh\frac{x}{a}\Big|_0^b = a\sinh\frac{b}{a}.$$

Since $h = a\cosh\dfrac{b}{a}$ we find

$$\ell(\gamma) = a\sqrt{\cosh^2\frac{b}{a} - 1} = \sqrt{a^2\cosh^2\frac{b}{a} - a^2} = \sqrt{h^2 - a^2}.$$

$\blacksquare$

3. Find the volume of the torus obtained by rotating the circle $(x-a)^2+y^2=b^2$, $a>b$, around the y-axis.

Hint. You can use both the slice and the shell methods.

Solution. We use the following general facts to simplify integrals of odd and even functions. If f is even and g is odd then

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x, \quad \int_{-a}^{a} g(x)\,\mathrm{d}x = 0.$$

Indeed, since $f(x)=f(-x)$, the substitution $z=-x$, $\mathrm{d}z=-\,\mathrm{d}x$ yields

$$\int_{-a}^{a} f\,\mathrm{d}x = \int_{-a}^{0} f\,\mathrm{d}x + \int_0^a f\,\mathrm{d}x = -\int_a^0 f(-z)\,\mathrm{d}z + \int_0^a f\,\mathrm{d}x = 2\int_0^a f\,\mathrm{d}x.$$

The same substitution in case of the odd function $g(-z)=-g(z)$ gives

$$\begin{aligned}\int_{-a}^{a} g\,\mathrm{d}x = \int_{-a}^{0} g\,\mathrm{d}x + \int_0^a g\,\mathrm{d}x = -\int_a^0 g(-z)\,\mathrm{d}z + \int_0^a g\,\mathrm{d}x =\\ = \int_0^a (-g(z))\,\mathrm{d}z + \int_0^a g\,\mathrm{d}x = 0.\end{aligned}$$

(a) The shell method. We compute the volume $V/2$ of the upper half of the given torus. Since $f(x)=y=\sqrt{b^2-(x-a)^2}$ and the zeros of f are at $a-b$ and $a+b$ we have using the substitution $z=x-a$, $\mathrm{d}z=\mathrm{d}x$,

$$\begin{aligned} V/2 &= 2\pi\int_{a-b}^{a+b} x\sqrt{b^2-(x-a)^2}\,\mathrm{d}x \\ &= 2\pi\int_{-b}^{b}(z+a)\sqrt{b^2-z^2}\,\mathrm{d}z \\ &= 4\pi a\int_0^b \sqrt{b^2-z^2}\,\mathrm{d}z. \end{aligned}$$

In the last line we used the fact that $z\sqrt{b^2-z^2}$ is an odd function and $\sqrt{b^2-z^2}$ is an even function. Using the result of *Test, April 14, Question 4* or integration by parts, we have $\int\sqrt{b^2-z^2}\,\mathrm{d}z = \frac{z}{2}\sqrt{b^2-z^2}+\frac{b^2}{2}\arcsin\frac{z}{b}+C$. Hence,

$$\begin{aligned} V &= 8\pi a\left(\frac{z}{2}\sqrt{b^2-z^2}+\frac{b^2}{2}\arcsin\frac{z}{b}\right)\Bigg|_0^b = 8\pi a\frac{\pi}{2}\frac{b^2}{2} \\ V &= 2\pi^2 ab^2. \end{aligned}$$

(b) The disc method. Rotating the circle $(x-a)^2+y^2=b^2$ in the x-y plane by 90° around the origin, we obtain the circle $x^2+(y-a)^2=b^2$. The volume of the torus can be obtained by taking the difference of the two volumes V_+-V_- where $V_\pm$ are obtained by revolving the graphs of

$$f_+(x)=a+\sqrt{b^2-x^2} \quad f_-(x)=a-\sqrt{b^2-x^2}$$

around the x-axis. Since $V_\pm = \pi \int_{-b}^{b} f_\pm^2 \,\mathrm{d}x$ we have

$$\begin{aligned} V = V_+ - V_- &= \pi \int_{-b}^{b} (f_+^2 - f_-^2)\,\mathrm{d}x = 2\pi \int_{-b}^{b} a\sqrt{b^2 - x^2}\,\mathrm{d}x \\ &= 4\pi a \int_0^b \sqrt{b^2 - x^2}\,\mathrm{d}x \underset{\text{cf. (a)}}{=} 2\pi^2 ab^2. \end{aligned}$$

4. Find the surface area of the above torus.

   Solution. Rotation around the y-axis. The same method as given in the lecture shows that the surface area of a solid obtained by revolution of the graph of $y = f(x)$, $x \in [a, b]$, around the y-axis is

$$A = 2\pi \int_a^b x\,\mathrm{d}s = 2\pi \int_a^b x\sqrt{1 + (f'(x))^2}\,\mathrm{d}x.$$

   We compute the area of the upper half; since $f(x) = \sqrt{b^2 - (x-a)^2}$ we find

$$1 + (f'(x))^2 = 1 + \frac{(x-a)^2}{b^2 - (x-a)^2} = \frac{b^2}{b^2 - (x-a)^2}.$$

   Therefore, using the substitution $z = x - a$, $x = z + a$, $\mathrm{d}z = \mathrm{d}x$,

$$\begin{aligned} A/2 &= 2\pi \int_{a-b}^{a+b} \frac{xb\,\mathrm{d}x}{\sqrt{b^2 - (x-a)^2}} \\ &= 2\pi b \int_{-b}^{b} \frac{(z+a)\,\mathrm{d}z}{\sqrt{b^2 - z^2}} \\ &= 4\pi ab \int_0^b \frac{\mathrm{d}z}{\sqrt{b^2 - z^2}} = 4\pi ab \left(\arcsin\frac{z}{b}\right)\Big|_0^b \\ &= 4\pi ab\frac{\pi}{2} = 2\pi^2 ab \\ A &= 4\pi^2 ab. \end{aligned}$$

   Rotation around the x-axis. As in the previous homework we rotate the circle in the plane by 90° and compute the area A_+ of the outer shell given by f_+ and the area of the inner shell given by f_-. Since

$$1 + (f_\pm')^2 = \frac{b^2}{b^2 - x^2}$$

   we obtain using $A = 2\pi \int f(x)\,\mathrm{d}s$

$$\begin{aligned} A_\pm &= 2\pi \int_{-b}^{b} f_\pm(x)\frac{b}{\sqrt{b^2 - x^2}}\,\mathrm{d}x = 4\pi b \int_0^b \frac{a \pm \sqrt{b^2 - x^2}}{\sqrt{b^2 - x^2}}\,\mathrm{d}x \\ &= 4\pi b\left(a\frac{\pi}{2} \pm b\right) = 2\pi^2 ab \pm 4\pi b^2. \end{aligned}$$

   Taking the sum of both areas we have $A = A_+ + A_- = 4\pi^2 ab$.