Calculus – 16. Series, Solutions

1. Use integration by parts to show that
$$\int_0^\infty \frac{\cos x}{1+x}\,\mathrm{d}x = \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.$$
Show that one of these integrals converges absolutely.

Proof. The left boundary 0 is not a singularity, the only singularity is ∞. The integral on the right converges absolutely. Indeed, the following integral is bounded (with the common bound 1 for all R),
$$\int_0^R \left|\frac{\sin x}{(1+x)^2}\right|\mathrm{d}x \le \int_0^R \frac{1}{(1+x)^2}\,\mathrm{d}x = \int_1^{R+1}\frac{\mathrm{d}x}{x^2} = -\left.-\frac{1}{x}\right|_1^{R+1} = 1 - \frac{1}{R+1} \le 1.$$
By Proposition **??**, the integral converges absolutely. Similarly as in Example **??**, the integral on the left does *not* converge absolutely.

We use integration by parts with $u = 1/(1+x)$, $u' = -1/(1+x)^2$, $v' = \cos x$, and $v = \sin x$ and obtain
$$\begin{aligned}\int_0^R \frac{\cos x}{1+x}\,\mathrm{d}x &= \left.\frac{\sin x}{1+x}\right|_0^R + \int_0^R \frac{\sin x}{(1+x)^2}\,\mathrm{d}x \\ &= \frac{\sin R}{R+1} + \int_0^R \frac{\sin x}{(1+x)^2}\,\mathrm{d}x \underset{R\to\infty}{\longrightarrow} \int_0^\infty \frac{\sin x}{(1+x)^2}\,\mathrm{d}x.\end{aligned}$$
Since the integral on the right converges, the integral on the left also converges, and they coincide. ∎

2. Show that for $a > 0$,
$$\int_0^\infty \mathrm{e}^{-x^a}\,\mathrm{d}x = \frac{1}{a}\Gamma\left(\frac{1}{a}\right).$$

Proof. The Gamma function is defined as an improper integral with singularities at 0 and ∞
$$\Gamma\left(\frac{1}{a}\right) = \int_0^\infty t^{\frac{1}{a}-1}\,\mathrm{e}^{-t}\,\mathrm{d}t.$$
Choosing $\varepsilon > 0$ and $R > 0$ we avoid the singularities. Using the change of variable $t = s^a$, $s = t^{\frac{1}{a}}$, $\mathrm{d}t = as^{a-1}$, and
$$t^{\frac{1}{a}-1} = t^{\frac{1-a}{a}} = s^{1-a},$$
we have
$$\begin{aligned}\int_\varepsilon^R t^{\frac{1}{a}-1}\,\mathrm{e}^{-t}\,\mathrm{d}t &= \int_{\varepsilon^{1/a}}^{R^{1/a}} s^{1-a}\,\mathrm{e}^{-s^a} as^{a-1}\,\mathrm{d}s = \\ &= a\int_{\varepsilon^{1/a}}^{R^{1/a}} \mathrm{e}^{-s^a}\,\mathrm{d}s \underset{\substack{\varepsilon\to 0\\ R\to\infty}}{\longrightarrow} a\int_0^\infty \mathrm{e}^{-s^a}\,\mathrm{d}s\end{aligned}$$

On the other hand, this limit equals $\Gamma(1/a)$, which proves the claim. ∎

3. Show that

$$\int_0^\infty \frac{\mathrm{d}x}{x^3+1} = \frac{2\sqrt{3}\pi}{9}.$$

Proof. First note that the integral converges by the integral criterion for series. One easily sees that $\alpha = -1$ is a zero of x^3+1. Long division yields $x^3+1 = (x+1)(x^2-x+1)$, where the quadratic factor does not factorize into linear factors. The partial fraction decomposition ansatz

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$$

gives $A = 1/3$, $B = -1/3$, and $C = 2/3$ such that

$$I = \int_0^R \frac{\mathrm{d}x}{x^3+1} = \frac{1}{3}\int_0^R \frac{\mathrm{d}x}{x+1} + \frac{1}{3}\int_0^R \frac{-x+2}{x^2-x+1}\,\mathrm{d}x.$$

Using $(x^2-x+1)' = 2x-1$ we find

$$\begin{aligned}
I &= \frac{1}{3}\log(R+1) + \frac{1}{3}\int_0^R \frac{-x+\frac{1}{2}}{x^2-x+1}\,\mathrm{d}x + \frac{1}{2}\int_0^R \frac{\mathrm{d}x}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}} \\
&= \frac{1}{3}\log(R+1) - \frac{1}{6}\log(R^2-R+1) + \frac{1}{2}\int_{-\frac{1}{2}}^{R-\frac{1}{2}} \frac{\mathrm{d}t}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2} \\
&= \frac{1}{6}\log\frac{(R+1)^2}{R^2-R+1} + \frac{1}{\sqrt{3}}\arctan\frac{2t}{\sqrt{3}}\Bigg|_{-\frac{1}{2}}^{R-\frac{1}{2}} \\
&= \frac{1}{6}\log\frac{R^2+2R+1}{R^2-R+1} + \frac{1}{\sqrt{3}}\left(\arctan\frac{2R-1}{\sqrt{3}} + \arctan\frac{1}{\sqrt{3}}\right) \\
&\xrightarrow[R\to\infty]{} \frac{1}{3}\log 1 + \frac{1}{\sqrt{3}}\left(\frac{\pi}{2}+\frac{\pi}{6}\right) \\
&= \frac{2\pi}{3\sqrt{3}} = \frac{2\sqrt{3}\pi}{9}.
\end{aligned}$$

∎

4. Show that for $n \in \mathbb{N}$, $a > 0$ and $b \in \mathbb{R}$

$$J_n = \int_0^\infty x^n \cos bx\, \mathrm{e}^{-ax}\,\mathrm{d}x = n!\,\mathrm{Re}\left(\frac{a+b\mathrm{i}}{a^2+b^2}\right)^{n+1}.$$

Compute J_0, J_1, and J_2 explicitly.

Hint. Using integration by parts find a recurrence relation for the complex integral $I_n = \int_0^\infty x^n\, \mathrm{e}^{(-a+b\mathrm{i})x}\,\mathrm{d}x$.

Solution. Set $\alpha = -a + b\mathrm{i}$, $u = x^n$, and $v' = \mathrm{e}^{(-a+b\mathrm{i})x}$; then we have $u' = nx^{n-1}$ and

$$v = \frac{1}{\alpha}\mathrm{e}^{\alpha x}.$$

Integration by parts gives

$$\begin{aligned} I_n(R) := \int_0^R x^n\, \mathrm{e}^{\alpha x}\, \mathrm{d}x &= \frac{1}{\alpha} x^n\, \mathrm{e}^{\alpha x}\Big|_0^R - \frac{n}{\alpha}\int_0^R x^{n-1}\mathrm{e}^{\alpha x}\, \mathrm{d}x \\ &= \frac{R^n}{\alpha \mathrm{e}^{aR}}\, \mathrm{e}^{b\mathrm{i}R} - \frac{n}{\alpha} I_{n-1}(R). \end{aligned}$$

Taking the limt $R \to \infty$ we obtain since $\left|\, \mathrm{e}^{b\mathrm{i}R} \,\right| = 1$ and $a > 0$

$$I_n = 0 - \frac{n}{\alpha}\, I_{n-1} = \cdots \frac{n!}{(-\alpha)^n} I_0,$$

where

$$I_0 = \int_0^\infty \mathrm{e}^{\alpha x}\, \mathrm{d}x = \frac{1}{\alpha}\mathrm{e}^{\alpha x}\Big|_0^\infty = -\frac{1}{\alpha}.$$

Hence,

$$I_n = \frac{n!}{(a - b\mathrm{i})^{n+1}}.$$

Since $1/(a - b\mathrm{i}) = (a + b\mathrm{i})/(a^2 + b^2)$, the claim follows.
Inserting $n = 0, 1, 2$ we have

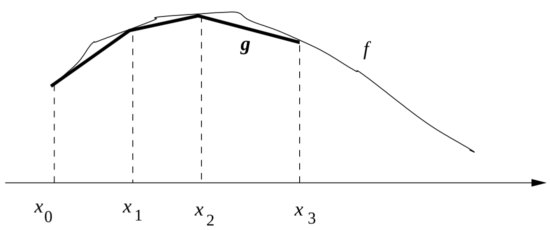
$$J_0 = \frac{a}{a^2 + b^2}, \quad J_1 = \frac{a^2 - b^2}{(a^2 + b^2)^2}, \quad J_2 = \frac{2(a^3 - 3ab^2)}{(a^2 + b^2)^3}.$$

5. Suppose $f \in \mathcal{R}(\alpha)$ on $[a, b]$, $\varepsilon > 0$, and define the L^2-norm for $h \in \mathcal{R}(\alpha)$ by

$$\|h\|_2 = \left(\int_a^b |\, h\,|^2\, \mathrm{d}\alpha\right)^{\frac{1}{2}}.$$

Prove that there exists a continuous function g on $[a, b]$ such that $\|f - g\|_2 < \varepsilon$.

Hint. Let $\{x_0, \dots, x_n\}$ be a suitable partition of $[a, b]$, define $g(t)$ to be the piecewise linear continuous function with $g(x_i) = f(x_i)$ for $i = 0, \dots, n$. Use Lemma 5.4 (c). What happens with the sum on the left if you insert $|\, f - g\,|^2$ in place of f.



Proof. First, consider an arbitrary partition P of $[a, b]$. Later, we will specify the choice of P. Since the linear function y through the two points $(x_{i-1}, f(x_{i-1}))$ and $(x_i, f(x_i))$ is given by

$$\frac{y - f(x_{i-1})}{x - x_{i-1}} = \frac{f(x_i) - f(x_{i-1})}{x_{i-1} - x_i},$$

we find the piecewise linear function $g(x)$

$$g(x) = \frac{x_i - x}{\Delta x_i} f(x_{i-1}) + \frac{x - x_{i-1}}{\Delta x_i} f(x_i), \quad \text{if} \quad x \in [x_{i-1}, x_i].$$

In particular, g is continuous on $[a, b]$ and therefore $g \in \mathcal{R}(\alpha)$, by Theorem 5. By Propositionsi 9 and 10, $\left| f - g \right|^2 \in \mathcal{R}(\alpha)$. By the Riemann criterion (Proposition 3) we find a partition $P = \{x_0, \dots, x_n\}$ of $[a, b]$ such that

$$U(\left| f - g \right|^2, P, \alpha) - L(\left| f - g \right|^2, P, \alpha) < \varepsilon^2$$

By Lemma 4 (c) with x_i in place of t_i and $\left| f - g \right|^2$ in place of f we have

$$\left| \sum_{i=1}^{n} ((f - g)(x_i))^2 \Delta\alpha_i - \int_a^b \left| f - g \right|^2 \, \mathrm{d}\alpha \right| < \varepsilon^2.$$

Since $f(x_i) = g(x_i)$ by construction, the sum on the left side is zero and we have

$$\int_a^b \left| f - g \right|^2 \, \mathrm{d}\alpha < \varepsilon^2 \Longrightarrow \left\| f - g \right\|_2 < \varepsilon.$$

In other words, every integrable function $f \in \mathcal{R}(\alpha)$ can be approximated in the L^2-norm by a continuous function g. We say $\mathrm{C}([a, b]) \subset \mathcal{R}(\alpha)$ is dense in the L^2-topology. ∎