Calculus – 15. Series, Solutions

1. Compute $\int_a^b f\mathrm{d}\alpha$ for

   (a) $f(x) = x^7$, $\alpha(x) = x^5$, $a = 0$, $b = 1$.

   (b) $f(x) = x$, $a = 0$, $b = 3$,

$$\alpha(x) = \begin{cases} 0, & \text{for} \quad 0 \le x < 1 \\ 4, & \text{for} \quad x = 1 \\ x^3, & \text{for} \quad 1 < x < 2 \\ \mathrm{e}^x, & \text{for} \quad 2 \le x < 3 \\ 30, & \text{for} \quad x = 3. \end{cases}$$

Solution. (a) Since $\alpha(x)$ is differentiable with $\alpha'(x) = 5x^4 \in \mathcal{R}$, by Proposition 12 we have

$$\int_0 11x^7 \,\mathrm{d}x^5 = \int_0^1 x^7\, 5x^4 \,\mathrm{d}x = 5\int_0^1 x^{11}\,\mathrm{d}x.$$

By the Fundamental Theorem of Calculus,

$$5\int_0^1 x^{11}\,\mathrm{d}x = \frac{5}{12}x^{12}\Big|_0^1 = \frac{5}{12}.$$

(b) Since f is continuous and α is piecewise differentiable, Proposition 13 applies with $(c_0, \dots, c_3) = (0, 1, 2, 3)$:

$$\begin{aligned}
\int_0^3 x\,\mathrm{d}\alpha &= \int_0^1 x\,\mathrm{d}0 + \int_1^2 x\,\mathrm{d}x^3 + \int_2^3 x\,\mathrm{d}\mathrm{e}^x + \\
&\quad + f(0)(\alpha(0+0) - \alpha(0)) + f(1)(\alpha(1+0) - \alpha(1-0)) + \\
&\quad + f(2)(\alpha(2+0) - \alpha(2-0)) + f(3)(\alpha(3) - \alpha(3-0)) \\
&= \int_0^1 0\,\mathrm{d}x + \int_0^1 3x^3\,\mathrm{d}x + \int_0^1 x\mathrm{e}^x\,\mathrm{d}x + 0 + 1(1-0) + 2(\mathrm{e}^2 - 2^3) + 3(30 - \mathrm{e}^3) \\
&= \frac{3}{4}x^4\Big|_1^2 + \mathrm{e}^x(x-1)|_2^3 + 1 + 2\mathrm{e}^2 - 16 + 90 - 3\mathrm{e}^3 \\
&= \frac{3}{4}(16-1) + 2\mathrm{e}^3 - \mathrm{e}^2 - 3\mathrm{e}^3 + 2\mathrm{e}^2 + 75 \\
&= \frac{345}{4} - \mathrm{e}^3 + \mathrm{e}^2 = 86\frac{1}{4} + \mathrm{e}^2 - \mathrm{e}^3 \approx 73.55
\end{aligned}$$

2. Using the table of antiderivatives compute

$$\text{(a)}\quad \int (3-x^2)^3\,\mathrm{d}x$$
$$\text{(b)}\quad \int \tan^2 x\,\mathrm{d}x$$
$$\text{(c)}\quad \int \frac{\mathrm{e}^{3x}+1}{\mathrm{e}^x+1}\,\mathrm{d}x$$
$$\text{(d)}\quad \int \left(\frac{1-x}{x}\right)^2\,\mathrm{d}x$$

Solution. (a)

$$\int (3-x^2)^3\,\mathrm{d}x = \int (27-27x^2+9x^4-x^6)\,\mathrm{d}x = -\frac{1}{7}x^7+\frac{9}{5}x^5-9x^3+27x+C.$$

(b) Using

$$\tan^2 x = \frac{\sin^2 x}{\cos^2 x} = \frac{\sin^2 x+\cos^2 x-\cos^2 x}{\cos^2 x} = \frac{1}{\cos^x}-1,$$

we have

$$\int \tan^2 x\,\mathrm{d}x = \int \left(\frac{1}{\cos^2 x}-1\right)\,\mathrm{d}x = \tan x - x + C.$$

(c)

$$\int \frac{\mathrm{e}^{3x}+1}{\mathrm{e}^x+1}\,\mathrm{d}x = \int \frac{(\mathrm{e}^x+1)(\mathrm{e}^{2x}-\mathrm{e}^x+1)}{\mathrm{e}^x+1}\,\mathrm{d}x = \frac{1}{2}\mathrm{e}^{2x}-\mathrm{e}^x+x+C.$$

(d)

$$\int \left(\frac{1-x}{x}\right)^2\,\mathrm{d}x = \int \frac{1-2x+x^2}{x^2}\,\mathrm{d}x = \int \left(\frac{1}{x^2}-\frac{2}{x}+1\right)\,\mathrm{d}x$$
$$= -\frac{1}{x}-2\log|\,x\,|+x+C.$$

3. Using an appropriate change of variables compute the following Riemann integrals

$$\text{(a)}\quad \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$$
$$\text{(b)}\quad \int_0^{\frac{\pi}{2}} \sin^5 x\cos x\,\mathrm{d}x$$
$$\text{(c)}\quad \int_0^{4} x\mathrm{e}^{-x^2}\,\mathrm{d}x$$

Solution. (a) Let $u = 1-x^2$, $\mathrm{d}u = -2x\,\mathrm{d}x$. The interval changes into $u_0 = 1$, $u_1 = 3/4$. We have

$$\int_0^{\frac{1}{2}} \frac{x\,\mathrm{d}x}{\sqrt{1-x^2}} = -\frac{1}{2}\int_1^{\frac{3}{4}} \frac{\mathrm{d}u}{\sqrt{u}} = \frac{1}{2}\int_{\frac{3}{4}}^{1} \frac{\mathrm{d}u}{\sqrt{u}}\ \sqrt{u}\Big|_{\frac{3}{4}}^{1} = 1-\frac{\sqrt{3}}{2}$$

(b) The change of variables $u = \sin x,\ \mathrm{d}u = \cos x\,\mathrm{d}x$ gives

$$\int_0^{\frac{\pi}{2}} \sin^5 x \cos x\,\mathrm{d}x = \int_0^1 u^5\,\mathrm{d}u = \frac{1}{6}$$

(c) Setting $u = x^2,\ \mathrm{d}u = 2x\,\mathrm{d}x$ we obtain $u_0 = 0,\ u_1 = 16$ and

$$\int_0^4 x\mathrm{e}^{-x^2}\,\mathrm{d}x = \frac{1}{2}\int_0^{16} \mathrm{e}^{-u}\,\mathrm{d}u = -\frac{1}{2}\mathrm{e}^{-u}\Big|_0^{16} = \frac{1}{2}\left(1 - \mathrm{e}^{-16}\right).$$

4. Using integration by parts compute the following Riemann integrals

$$\begin{aligned}
&\text{(a)} \quad \int_0^x \arctan t\,\mathrm{d}t, \quad x > 0,\\
&\text{(b)} \quad \int_0^1 x^3\mathrm{e}^{-x^2}\,\mathrm{d}x,\\
&\text{(c)} \quad \int_0^4 \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x
\end{aligned}$$

Solution. (a) $u = \arctan t,\ v' = 1$ yields $u' = 1/(1+t^2)$ and $v = t$. Hence,

$$\begin{aligned}
\int_0^x \arctan t\,\mathrm{d}t &= t\arctan t|_0^x - \int_0^x t\frac{1}{1+t^2}\,\mathrm{d}t\\
&= x\arctan x - \frac{1}{2}\int_0^x \frac{\mathrm{d}t^2}{1+t^2}\\
&= x\arctan x - \frac{1}{2}\log(1+x^2).
\end{aligned}$$

(b) $u = x^2\ v' = x\mathrm{e}^{-x^2}$ yields $u' = 2x,\ v = -\frac{1}{2}\mathrm{e}^{-x^2}$. This gives

$$\begin{aligned}
\int_0^1 x^3\mathrm{e}^{-x^2}\,\mathrm{d}x &= -\frac{1}{2}x^2\mathrm{e}^{-x^2}\Big|_0^1 - \int_0^1 -\frac{1}{2}\mathrm{e}^{-x^2}2x\,\mathrm{d}x = -\frac{1}{2\mathrm{e}} + \frac{1}{2}\int_0^1 \mathrm{e}^{-x^2}\,\mathrm{d}x^2\\
&= -\frac{1}{2\mathrm{e}} + \left(-\frac{1}{2}\mathrm{e}^{-x^2}\right)|_0^1 = -\frac{1}{2\mathrm{e}} + \left(-\frac{1}{2\mathrm{e}} + \frac{1}{2}\right) = \frac{1}{2} - \frac{1}{\mathrm{e}}.
\end{aligned}$$

(c) $u = \sqrt{x}$ gives $x = u^2,\ \mathrm{d}x = 2u\,\mathrm{d}u$ and

$$\int_0^4 \mathrm{e}^{\sqrt{x}}\,\mathrm{d}x = \int_0^2 \mathrm{e}^u 2u\,\mathrm{d}u = 2\mathrm{e}^u(u-1)|_0^2 = 2\mathrm{e}^2 - (-2) = 2(\mathrm{e}^2 + 1).$$

5. Find a recurrence relation for the antiderivatives

$$I_n = \int x^{2n}\sin x\,\mathrm{d}x, \quad n = 0, 1, \ldots$$

and compute I_1 and I_2.

Solution. We apply integration by parts twice. Setting first $u = x^{2n}$, $v' = \sin x$ we have $u' = 2nx^{2n-1}$ and $v = -\cos x$ and then $r = x^{2n-1}$, $s' = \cos x$ we find $r' = (2n-1)x^{2n-2}$ and $s = \sin x$. Hence

$$\begin{aligned}
I_n &= -x^{2n}\cos x + 2n\int x^{2n-1}\cos x\,\mathrm{d}x \\
I_n &= -x^{2n}\cos x + 2n\left(x^{2n-1}\sin x + (2n-1)\int x^{2n-2}\sin x\,\mathrm{d}x\right) \\
I_n &= -x^{2n}\cos x + 2nx^{2n-1}\sin x - 2n(2n-1)I_{n-1}.
\end{aligned}$$

Since $I_0 = \int \sin x\,\mathrm{d}x = -\cos x + C_0$ we find

$$\begin{aligned}
I_1 &= -x^2\cos x + 2x\sin x - 2I_0 = (2-x^2)\cos x + 2x\sin x + C_1 \\
I_2 &= -x^4\cos x + 4x^3\sin x - 12I_1 \\
&= -x^4\cos x + 4x^3\sin x - 24x\sin x - (24-12x^2)\cos x + C_2 \\
&= (4x^3 - 24x)\sin x + (-x^4 + 12x^2 - 24)\cos x + C_2,
\end{aligned}$$

where C_i, $i = 0, 1, 2, 3$, are integration constants.