Calculus – 13. Series, Solutions

1. Let $f(x) = x^x$ be defined for positive real numbers $x > 0$. Compute $\lim_{x\to 0+0} f(x)$. Determine the local extrema of f. On which parts of its domain is f convex; on which parts is f concave?

   Solution. By Homework 11.4 $\lim_{x\to 0+0} x \log x = 0$. This implies $\lim_{x\to 0+0} \log x^x = 0$ and since e^x is continuous, $\lim_{x\to 0+0} x^x = 1$.
   By Homework 12.1, $f'(x) = x^x(\log x + 1)$. We compute the second derivative.

   $$f''(x) = x^x(\log x + 1)^2 + x^x \frac{1}{x}.$$

   Since both summands are nonnegative, $f''(x) > 0$ for all positive x. By Proposition 13, f is convex on its domain. For the local extrema consider $f'(x) = 0$. This implies $\log x = -1$ and finally $x = 1/\mathrm{e}$. Since $f''(x) > 0$, by Proposition 12, f attains its local (and global) minimum at $1/\mathrm{e}$.

2. Prove that for every $x > 0$

   $$\frac{x}{1+x} \le \log(1+x) \le x.$$

   Hint. Apply the mean value theorem to $f(x) = \log(1+x)$.

   Proof. Since f is continuous on the closed interval $[0, x]$ and differentiable on the open interval $(0, x)$, by the mean value theorem there exists $\xi \in (0, x)$ such that

   $$f'(\xi) = \frac{f(x) - f(0)}{x - 0} = \frac{\log(1+x)}{x}$$
   $$\frac{1}{1+\xi} = \frac{\log(1+x)}{x}.$$

   Solving this equation for ξ and noting $0 < \xi < x$ gives

   $$0 < \xi = \frac{x}{\log(1+x)} - 1 < x.$$

   Since $\log(1+x) > 0$ for $x > 0$ this yields on the one hand $\log(1+x) < x$ and on the other hand $x < (x+1)\log x$. The assertion follows. ■

3. Compute the limits where $a > 0$ and $b > 0$ denote fixed positive real numbers.

   (a) $\displaystyle \lim_{x\to 0} \frac{\log\cosh(ax)}{\log\cos(bx)}$

   (b) $\displaystyle \lim_{x\to \pi/2} \frac{1 - \sin^{a+b} x}{\sqrt{(1 - \sin^a x)(1 - \sin^b x)}}$

   (c) $\displaystyle \lim_{x\to +\infty} \arccos(\sqrt{x^2 + x} - x)$

   (d) $\displaystyle \lim_{x\to +\infty} x\left(\frac{\pi}{4} - \arctan\frac{x}{x+1}\right)$

   (e) $\displaystyle \lim_{x\to 0} \frac{\cos\frac{1}{x}\sin^2 x}{x}$

Solution. (a) The limit in question is of the form $0/0$. Using l'Hospital's rule twice we find

$$\lim_{x\to 0}\frac{\log\cosh(ax)}{\log\cos(bx)}=\lim_{x\to 0}\frac{\dfrac{a\sinh(ax)}{\cosh(ax)}}{\dfrac{-b\sin(bx)}{\cos(bx)}}=-\frac{a}{b}\lim_{x\to 0}\frac{\sinh(ax)}{\sin(bx)}$$
$$=-\frac{a}{b}\lim_{x\to 0}\frac{a\cosh(ax)}{b\cos(bx)}=-\frac{a^2}{b^2}.$$

(b) The limit in question is of the form $0/0$. To simplify computations we first consider the square of the function and take the root of the limit afterwards. This is justified by the continuity of the square root. After first application of l'Hospital's rule we can cancel the term $\cos x$, apply l'Hospital's rule a second time and just cancel $\cos x$ in all terms.

$$\lim_{x\to\pi/2}\frac{\left(1-\sin^{a+b}x\right)^2}{1-\sin^a x-\sin^b x+\sin^{a+b}x}$$
$$=\lim_{x\to\pi/2}\frac{2\left(1-\sin^{a+b}x\right)\left(-(a+b)\sin^{a+b-1}x\cos x\right)}{-a\sin^{a-1}x\cos x-b\sin^{b-1}x\cos x+(a+b)\sin^{a+b-1}x\cos x}$$
$$=-2(a+b)\lim_{x\to\pi/2}\frac{\sin^{a+b-1}x(1-\sin^{a+b}x)}{-a\sin^{a-1}x-b\sin^{b-1}x+(a+b)\sin^{a+b-1}x}$$
$$=-2(a+b)\lim_{x\to\pi/2}\frac{(a+b-1)\sin^{a+b-2}x(1-\sin^{a+b}x)+\sin^{a+b-1}x(-(a+b)\sin^{a+b-1}x)}{-a(a-1)\sin^{a-2}x-b(b-1)\sin^{b-2}+(a+b)(a+b-1)\sin^{a+b-2}x}$$
$$=-2(a+b)\frac{0-(a+b)}{-a^2+a-b^2+b+a^2+2ab+b^2-a-b}=\frac{(a+b)^2}{ab}.$$

Taking the square root we obtain the final result $\dfrac{a+b}{\sqrt{ab}}$.

(c) Since $\arccos x$ is continuous on its domain,

$$\lim_{x\to+\infty}\arccos(\sqrt{x^2+x}-x)=\arccos\left(\lim_{x\to\infty}\sqrt{x^2+x}-x\right).$$

The limit is now of the form $\infty-\infty$. We give two possible solutions.

First solution. We multiply both the numerator and the denominator (which is 1) by $\sqrt{x^2+x}+x$ and obtain

$$\lim_{x\to+\infty}\left(\sqrt{x^2+x}-x\right)=\lim_{x\to+\infty}\frac{\sqrt{x^2+x}^2-x^2}{\sqrt{x^2+x}+x}=\lim_{x\to+\infty}\frac{x}{\sqrt{x^2+x}+x}$$
$$=\lim_{x\to+\infty}\frac{1}{\dfrac{\sqrt{x^2+x}}{x}+1}=\frac{1}{2}.$$

In the last step we used continuity of the square root:

$$\lim_{x\to+\infty}\frac{\sqrt{x^2+x}}{x}=\sqrt{\lim_{x\to+\infty}\frac{x^2+x}{x^2}}=1.$$

Second solution. We substitute $h = 1/x$ and consider the limit

$$\lim_{h\to 0+0}\left(\sqrt{\frac{1}{h^2}+\frac{1}{h}}-\frac{1}{h}\right) = \lim_{h\to 0+0}\frac{\sqrt{h+1}-1}{h} = \left(\sqrt{x+1}\right)'\,|_{x=0}= \frac{1}{2\sqrt{x+1}}\,|_{x=0}= \frac{1}{2}.$$

Hence the limit in question equals $\arccos\frac{1}{2} = \frac{\pi}{3}$.
(d) Since $\arctan 1 = \pi/4$, the limit is of the form $\infty\cdot 0$. We substitute $h = 1/x$ and compute the limit

$$\begin{aligned}
&\lim_{h\to 0+0}\frac{\frac{\pi}{4}-\arctan\frac{\frac{1}{h}}{1+\frac{1}{h}}}{h} = \lim_{h\to 0+0}\frac{\frac{\pi}{4}-\arctan\frac{1}{1+h}}{h}\\
&= -\left(\arctan\frac{1}{1+x}\right)'\,|_{x=0}\\
&= -\frac{1}{1+\left(\frac{1}{1+x}\right)^2}\left(-\frac{1}{(1+x)^2}\right)\,|_{x=0}= \frac{1}{2}.
\end{aligned}$$

(e) The limit is of the form $0/0$ but l'Hospital's rule does not apply since the derivative of the numerator has no limit as x approaches 0. We show directly that the limit is 0 using the fact that

$$\left|\frac{\sin x}{x}\right| \le 2$$

for small x (Corollary 3.25) and $|\cos y| \le 1$ for every y. We have for $|x| < \delta$

$$\left|\frac{\cos\frac{1}{x}\sin^2 x}{x}\right| \le 1\cdot|\sin x|\left|\frac{\sin x}{x}\right| \le 2\,|\sin x|\,.$$

Since the right hand side tends to 0 as x approaches 0, the limit in question is also 0.

4. Compute the Taylor polynomial of degree 3 of the function

$$f(x) = \mathrm{e}^{-x}\cos x$$

at $x_0 = 0$. Give an estimate for the remainder if $|x| \le \frac{1}{2}$.

Solution. We need the first three derivatives of f at point 0 and the 4th derivative to estimate the remainder term. We have

$$\begin{aligned}
f'(x) &= -\mathrm{e}^{-x}\cos x - \mathrm{e}^{-x}\sin x = -\mathrm{e}^{-x}(\cos x + \sin x),\\
f''(x) &= \mathrm{e}^{-x}(\cos x + \sin x + \sin x - \cos x) = 2\mathrm{e}^{-x}\sin x,\\
f'''(x) &= -2\mathrm{e}^{-x}(\sin x - \cos x),\\
f^{(4)}(x) &= 2\mathrm{e}^{-x}(\sin x - \cos x - \cos x - \sin x) = -4\mathrm{e}^{-x}\cos x.
\end{aligned}$$

Therefore, $f(0) = 1$, $f'(0) = -1$, $f''(0) = 0$, and $f'''(0) = 2$. Hence the Taylor polynomial of degree 3 reads

$$p_3(x) = 1 - x + \frac{1}{3}x^3.$$

The Lagrange form of the remainder term gives

$$|\, r_3(x) \,| = |\, f(x) - p_3(x) \,| = \left| \frac{f^{(4)}(\xi)}{4!} x^4 \right| \le \frac{4\mathrm{e}^{-\xi}}{24} \cdot \frac{1}{16} \le \frac{\sqrt{\mathrm{e}}}{96} < 0.1717.$$

In the last estimate we used the fact that e^{-x} is strictly decreasing and therefore it attains its maximum at $\xi = -\frac{1}{2}$; $\mathrm{e}^{-\xi} < \mathrm{e}^{\frac{1}{2}}$.

5. (a) Compute the Taylor series T_f and T_g of $f(x) = \cos x$ and $g(x) = \log(1 + x)$ at $x_0 = 0$, respectively. Compute their radii of convergence.
(b) Show that $T_f(x)$ converges to $f(x)$ for all $x \in \mathbb{R}$. Show that $T_g(x)$ converges to $g(x)$ for all $x \in (0, 1)$.

Solution. (a) Since the sequence of derivatives for the cosine function is of period 4 with

$$(f^{(n)}(x))_{n=0,1,\dots} = (\cos x, -\sin x, -\cos x, \sin x, \cos x, \dots)$$

we have at $x_0 = 0$ the sequence $(1, 0, -1, 0, 1, \dots)$. Hence the Taylor series at $x_0 = 0$ is

$$T_f(x) = 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + - \cdots .$$

The Taylor series obviously coincides with the definition of the cosine function given in Proposition 3.20. The radius of convergence is also $R = +\infty$.
Since $g'(x) = \frac{1}{1+x}$, $g''(x) = -\frac{1}{(1+x)^2}$ it is easy to prove using induction on n that

$$g^{(n)}(x) = (-1)^{n+1} \frac{(n-1)!}{(1+x)^n}.$$

Hence $g^{(n)}(0) = (-1)^{n+1}(n-1)!$ and the Taylor series of $g(x)$ at $x_0 = 0$ reads

$$T_g(x) = 0 + x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} x^n.$$

Since

$$\alpha = \overline{\lim} \sqrt[n]{\frac{1}{n}} = \frac{1}{\lim_{n\to\infty} \sqrt[n]{n}} = 1,$$

the radius of convergence of T_g is $R = 1/\alpha = 1$.
(b) For f there is nothing to show. To show convergence for g we must prove that for every $x \in (0, 1)$ the remainder term $r_n(x)$ tends to 0 as n goes to ∞. The Lagrange remainder term allows the following estimate since $|\, x \,| \le 1$ and $\dfrac{1}{1+\xi} < 1$

$$|\, r_n(x) \,| = \left| \frac{1}{n} \frac{1}{(1+\xi)^n} x^n \right| \le \frac{1}{n},$$

which proves the claim. Note that this estimate also holds for $x = 1$, hence

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots .$$