Calculus – 10. Series, Solutions

1. Determine all values of $c \in \mathbb{R}$ such that

$$f(x) = \begin{cases} (x-2)^2 & \text{if } \; x \le 4, \\ cx^2 - 8 & \text{if } \; x > 4 \end{cases}$$

is continuous on $\mathbb{R}$.

Solution. The function f is continuous on $\mathbb{R} \setminus \{4\}$ for every c since polynomials are continuous on $\mathbb{R}$. Suppose f is continuous at $x = 4$ then $\lim_{x\to 4+0} f(x) = f(4)$. That is

$$\lim_{x\to 4+0} f(x) = (cx^2 - 8)\mid_{x=4} = 16c - 8 \overset{!}{=} f(4) = 4.$$

This implies $c = 12/16 = 3/4$.
On the other hand, if $c = 3/4$, then $\lim_{x\to 4-0} f(x) = \lim_{x\to 4+0} f(x) = f(4) = 4$ and f is continuous on $\mathbb{R}$.

2. (a) Prove the following fixed point theorem. Let $D = [a, b]$ be a finite closed interval. Every continuous function $f\colon D \to D$ has a fixed point, i. e. there exists $c \in [a, b]$ such that $f(c) = c$.
Give examples of functions $f\colon D \to D$ such that the fixed point theorem fails if
(b) D is a closed infinite interval.
(c) $D = [a, b)$.
(d) $D = [0, 1] \cup [2, 3]$.
(e) f is not continuous.
Hint. Use the intermediate value theorem for (a).

Proof. (a) The function $g(x) = f(x) - x$ is continuous on $[a, b]$ by Proposition 2. Further, since $f(a) \ge a$ and $f(b) \le b$,

$$g(a) = f(a) - a \ge 0 \quad \text{and} \quad g(b) = f(b) - b \le 0.$$

By the intermediate value theorem there exists $c \in [a, b]$ such that $g(c) = 0$; that is $f(c) = c$.
(b) Let $D = \mathbb{R}_+$. Then $f(x) = x + 1$ maps D continuously into D. However, f has no fixed point since $x = x + 1$ has no solution.
(c) Let $D = [0, 2)$. Then $f(x) = x/2 + 1$ maps D continuously into D. However, the only fixed point of $x/2 + 1$, $c = 2$, is not in D.
(d) Let f be 2 on $[0, 1]$ and 1 on $[2, 3]$. Then f is continuous on D (since it is locally constant at every point of D) and f maps D into D. However, f has no fixed point.
(e) Let $D = [0, 1]$ and $f(x) = \frac{1}{2}$ for every $x \ne \frac{1}{2}$ and $f(\frac{1}{2}) = 1$. Then f has no fixed point.
Remark. The more general statement is Brower's fixed point theorem: A continuous mapping $f\colon D \to D$ of the *compact and convex* set $D \subset \mathbb{R}^n$ into itself has a fixed

point. ∎

3. Let a and b be real numbers with $1 < a < b$. Prove that the equation

$$\frac{x^7+1}{x-a} + \frac{x^3-1}{x-b} = 0$$

has a solution $x \in (a, b)$.

Hint. Define an appropriate function f and apply the intermediate value theorem.

Proof. Consider the function

$$f(x) = \frac{x^7+1}{x-a} + \frac{x^3-1}{x-b}$$

on the open interval (a, b). Since $x^7 + 1 > 0$, $\lim_{x\to a+0}(x-a) = 0$, and $\frac{x^3-1}{x-b} \geq C$ in a neighborhood of $x = a$, Homework 9.3 (a) and (b) shows that

$$\lim_{x\to a+0} f(x) = +\infty. \tag{1}$$

Similarly, $\frac{x^7+1}{x-a}$ is bounded above in a neighborhood of b, $x^3 - 1 > 0$ since $b > 1$, and $\lim_{x\to b-0}(x-b) = 0$, $x - b < 0$. Hence,

$$\lim_{x\to b-0} f(x) = -\infty. \tag{2}$$

Using (1) and (2), the intermediate value theorem applied to $\gamma = 0$ shows that f has a zero in (a, b). ∎

4. Prove. If $f\colon [a, b] \to \mathbb{R}$ is continuous at $x_0 \in (a, b)$ and $f(x_0) = A > 0$ then there exists a real number $\delta > 0$ such that for every $x \in [a, b]$ the inequality $|\, x - x_0 \,| < \delta$ implies $f(x) > A/2$.
(In other words: If a continuous function is nonzero at a point x_0, then f is nonzero on a whole neighborhood $U_\delta(x_0)$)

Proof. Since f is continuous at x_0 to $\varepsilon = A/2 > 0$ one can find $\delta > 0$ such that for every $x \in [a, b]$

$$|\, x - x_0 \,| < \delta \quad \text{implies} \quad |\, f(x) - A \,| < \frac{A}{2}.$$

For those x we have

$$-\frac{A}{2} < f(x) - A < \frac{A}{2}$$
$$\frac{A}{2} < f(x) < \frac{3A}{2}$$

which proves the assertion. ∎

5. Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $\mathbb{R}_+$, whereas $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}_+$.

Proof. (a) Since $f(x) = \sqrt{x}$ is continuous (by Proposition 11), it is uniformly continuous on the compact set $[0,1]$ (by Proposition 12). That is, given $\varepsilon > 0$ there exists $\delta_1 > 0$ such that for every $x, y \in [0,1]$

$$|x - y| < \delta_1 \Longrightarrow |f(x) - f(y)| < \varepsilon. \tag{3}$$

Assume now that $x \geq 1$ or $y \geq 1$ (or both). Choose $\delta_2 = \varepsilon$. Noting that $\left|\sqrt{x} - \sqrt{y}\right| \left|\sqrt{x} + \sqrt{y}\right| = |x - y|$,

$$|x - y| < \delta_2 = \varepsilon \quad \text{implies} \quad \left|\sqrt{x} - \sqrt{y}\right| = \frac{|x - y|}{\sqrt{x} + \sqrt{y}} < \varepsilon.$$

The last inequality follows from $\sqrt{x} + \sqrt{y} \geq 1$.
Choosing $\delta = \min\{\delta_1, \delta_2\}$ one can see that

$$|x - y| < \delta \Longrightarrow |f(x) - f(y)| < \varepsilon$$

for all $x, y \in \mathbb{R}_+$.
(b) Consider $f(x) = x^2$. Choose $\varepsilon = 2$, $\delta_n = \frac{1}{n}$,

$$x_n = n + \frac{1}{2n} \quad \text{and} \quad y_n = n - \frac{1}{2n}.$$

Then $x_n - y_n = 1/n$ but

$$f(x_n) - f(y_n) = x_n^2 - y_n^2 = (x_n - y_n)(x_n + y_n) = \frac{1}{n} \cdot 2n = 2.$$

f is not uniformly continuous. ■