1 Introduction

We want to give a short overview on the basic concept of fixed point index theory and especially the index theory for periodic orbits. This last index was first invented by Fuller in [F67] and then lay a long time dormant. We are not dealing with Fullers constructions here but explain two different approaches. The first chapter is only a summary of the classical analytical approach to fixed point index theory, as can be found e.g. in [D85], where one has to note that the degree defined there counts zeros and not fixed points. The concept nevertheless is the same. The second chapter is a condensation of material in [CM78], as is the theorem which we prove in chapter 3. See also the references in [CM78]. The fourth chapter deals with a topological approach to index theory. What has to be done was already evident in Fullers original paper [F67], but the first satisying treatment from a modern point of view is, as far as we know, [Fr90]. The last chapter gives an outlook on equivariant index theory, see [BKS06, IV03]. For equivariant homology theories, [tD87] might be the best reference.

2 Classical Fixed Point Index

The classical idea of index theory is to assign to each continuous self map of an n-dimensional manifold with boundary (for example an open subset of $\mathbb{R}^n$ with smooth boundary) a number which should count the number of fixed points. Of course one can not assume that the assignment of the actual number of fixed points is a quantity that has nice properties to work with. Instead, a more conceptual approach should be taken.

Firstly, we know that having only hyperbolic fixed points is a generic property of a map f. Secondly, if f has a hyperbolic isolated fixed point, then all maps sufficiently close to f also have a fixed point. So it is reasonable to demand of an index to be invariant under small perturbations. In fact, we can even allow large perturbations, as long as no fixed points appear on the boundary. This is the most important property of the index which makes it computable for a large class of maps (granted that we have some kind of nontriviality).

Now the idea how one should count fixed points is to take the orientation behaviour into account. We work in charts first, since the degree should be a local quantity. If $f : \overline{\Omega} \to \mathbb{R}^n$ is a smooth map and x a hyperbolic fixed point of f, then we assign an index to f by defining it to be 1, if $\mathbb{1} - f$ preserves orientation, and -1, if $\mathbb{1} - f$ reverses orientation. This property can be expressed by the Jacobian of $\mathbb{1} - f$ at x: If the sign of its determinant is positive, then $\mathbb{1} - f$ preserves orientation, if it is negative, it reverses orientation. So, to say the same thing in other words, the index of the isolated fixed point is the number

$$\operatorname{ind}(f, x) = (-1)^{\sigma_x},$$

where σ_x is the number of eigenvalues of the Jacobian of f at x that lie in $(1, \infty)$.

Now it is clear that the index of f with respect to Ω should be just the sum over the indices of its fixed points, if all of those are hyperbolic.

As we already mentioned above, smooth maps having only hyperbolic fixed points are dense in the space of continuous maps, so it is reasonable, since the index should be invariant under small perturbations, to define the index for arbitrary continuous maps by an approximation argument. Of course it is not obvious that this is well defined. But we have the following

Proposition 2.1 *Let $f_0, f_1 : \overline{\Omega} \to \mathbb{R}^n$ be two smooth maps having only hyperbolic fixed points and no fixed points on the boundary. If $H : \Omega \times I$ is a homotopy between f_0 and f_1 such that no H_t has fixed points on the boundary, then the indices of f_0 and f_1 are equal.*

In particular, if $f : \overline{\Omega} \to \mathbb{R}^n$ has no fixed points on the boundary, then there is a neighbourhood of f in the space of continuous functions such that no map in this neighbourhood has fixed points on the boundary. So any two smooth maps in this neighbourhood having only hyperbolic fixed points have the same index, since we can join them by the standard convex homotopy. Thus, the index of f is well defined by approximating it with generic maps. Furthermore it is clear that the definition extends to maps of manifolds, since all definitions are local in nature.

We summarize the most important properties of the index.

Proposition 2.2 *There is a map $\operatorname{ind}(\cdot, \Omega) : \mathcal{C} \to \mathbb{Z}$, where $\mathcal{C} \subseteq C(\overline{\Omega}, \mathbb{R}^n)$ is the subset of maps without fixed points on $\partial\Omega$, with the following properties.*

(i) ind *is homotopy invariant under admissible homotopies, i.e.*

$$\operatorname{ind}(H_t, \Omega) = \operatorname{ind}(H_0, \Omega)$$

for a homotopy $H : \overline{\Omega} \times I \to \mathbb{R}^n$ in $\mathcal{C}$ and all $t \in I$.

(ii) ind *is additive. If $\Omega_1, \Omega_2 \subseteq \Omega$ are open and disjoint such that f has no fixed points in $\Omega \backslash (\Omega_1 \cup \Omega_2)$, then*

$$\operatorname{ind}(f, \Omega) = \operatorname{ind}(f\big|_{\Omega_1}, \Omega_1) + \operatorname{ind}(f\big|_{\Omega_2}, \Omega_2).$$

(iii) ind *has the solving property, i.e. if $\operatorname{ind}(f; \Omega) \neq 0$, then f has a fixed point in Ω.*

If Ω is understood, we sometimes write $\operatorname{ind}(f)$ instead of $\operatorname{ind}(f, \Omega)$. Applications of index theory are clear. On the one hand, one can just solve a fixed point problem $f(x) = x$ by either computing $\operatorname{ind}(f)$ directly and noticing that it is not zero, or by computing $\operatorname{ind}(g) \neq 0$ for some map where the index is easy to calculate and then showing that there is an admissible homotopy between f and g.

But there are many more possibilities. For example, if one can join two maps by a homotopy but the indices are not equal, then the homotopy could not have been admissible. But then, there must have been a fixed point on the boundary of the set for some parameter value. This is important for applications in bifurcation theory by finding nontrivial solutions, because they lie on the boundary of, say, a ball, in an arbitrary neighbourhood of some possible bifurcation point.

3 The Index for Periodic Orbits

Periodic orbits of flows are closely connected to fixed points by considering a Poincaré map of the orbit. If the orbit is isolated, then the Poincaré map has an isolated fixed point, namely the point lying on the orbit. So a naive approach to defining an index for periodic orbits could be to just take the fixed point index of a Poincaré map.

Unfortunately, this approach doesn't work, because the periods of the orbit have to be taken into account. Periodic orbits can vanish by expanding to infinity or by merging into a fixed point. But it can also happen that the least period of the orbit goes to infinity. So instead of working in $\Omega \subseteq \mathbb{R}^n$, we should add a factor $\mathbb{R}^+$, standing for the periods of the orbits. Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be smooth with flow given by $\varphi(x, t)$ and denote with

$$\Pi(f) = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}^+ \mid \varphi(x, t) = x\}$$

the set of periodic points. Then a subset $\Omega \subseteq \mathbb{R}^n \times \mathbb{R}^+$ is called admissible for f, if it is open, bounded, bounded away from $\mathbb{R}^n \times \{0\}$ and there are no periodic points on its boundary.

We want to define an index $\operatorname{ind}(f, \Omega)$ that enjoys the same properties as the fixed point index: Homotopic vector fields should have equal index, it should be additive and nontriviality should imply existence of periodic orbits.

We take up the idea of using the Poincaré map, but a bit more subtle. As in the case of the fixed point index we start with considering a hyperbolic periodic orbit γ with least period $T > 0$. Recall that a periodic orbit is called hyperbolic, if the nontrivial eigenvalues $\lambda_1, \ldots, \lambda_{n-1}$ of

$$D_x\varphi(x, T)$$

have absolute value different from 1 for some $x \in \gamma$. In the above situation, $(x, kT) \in \Pi(f)$ for all $k \in \mathbb{Z}$. So we have to assign an index to γ, considered as an orbit with period kT. We do this by defining

$$\operatorname{ind}(f, (x, kT)) = \frac{1}{k}(-1)^\sigma,$$

where σ is the number of eigenvalues λ_i such that $\lambda_i^k \in (1, \infty)$. In other words, the index is the fixed point index of the k-th iterate of the Poincaré map, multiplied with $\frac{1}{k}$.

The rest of the definition is canonical: For a map with a finite number of hyperbolic periodic orbits in Ω we define the index to be the sum over all orbit indices. Then, for an arbitrary map, we can approximate with a map that has only hyperbolic periodic orbits by a theorem of Kupka and Smale. Homotopy invariance can now be proven using bifurcation theory. It has been shown that, if two hyprbolic maps are homotopic, the homotopy can be chosen that for all parameters except for a finite number, the map H_t is hyperbolic. Furthermore, for the finitely many exceptional parameters, the situation is easy to handle: The derivative of H_{t_j} for an exceptional value t_j at the trivial solution has either exactly two multipliers on the unit circle, and these are of the form $\exp(\pm 2\pi i\vartheta$ with ϑ irrational. Or it has a simple eigenvalue 1 and no other eigenvalues on the unit circle. Or it has a simple eigenvalue -1 and no other eigenvalues on the unit circle.

Now one only has to investigate the bifurcations that can occur here. In the first case, an invariant torus bifurcates which contains all possible periodic solutions. But since ϑ is irrational, the periods of periodic orbits on this torus become large as we approach the bifurcation parameter, hence they can not be contained in Ω. So neither are orbits destroyed nor generated inside Ω, which implies that the index remains the same.

In the second case two branches of periodic orbits emerge from t_j in one or both directions, whose minimal period approaches a common value at t_j. The continuation of the eigenvalue 1 is lesser than 1 on one branch and larger than 1 on the other. Using this, one can easily calculate the degree directly and find that it remains unchanged.

The most interesting case is the third case. The eigenvalue -1 indicates that a period doubling bifurcation occurs, that is a branch of periodic orbits runs through t_j and in addition, a branch with minimal period approximately twice the minimal period of the other branch exists on one side of t_j. The multipliers of the doubled orbit are approximately the squares of the multipliers of the single orbit. So one calculates the following indices of the orbits:

$$\operatorname{ind}(\gamma_1^k) = \begin{cases} \frac{1}{k}(-1)^{\sigma} & k \text{ odd} \\ \frac{1}{k}(-1)^{\sigma+\tau+1} & k \text{ even}, t < t_j \\ \frac{1}{k}(-1)^{\sigma+\tau} & k \text{ even}, t > t_j, \end{cases}$$

where σ is the number of multipliers of γ_1 in $(1,\infty)$ corresponding to the minimal period, whereas τ is the number of multipliers in $(-\infty,-1)$, and similarly by the arguments above,

$$\operatorname{ind}(\gamma_2^k) = \frac{1}{k}(-1)^{\sigma+\tau+1}.$$

Since the period doubles, the orbit γ_2^k branches from γ_1^{2k}. So for k odd, there is no contribution of the doubled branch and we have the same index on both sides. For k even, on the left side of t_j the index is $\frac{1}{k}(-1)^{\sigma+\tau+1}$ and on the right we have to add up:

$$\frac{1}{k}(-1)^{\sigma+\tau} + \frac{2}{k}(-1)^{\sigma+\tau+1} = \frac{1}{k}(-1)^{\sigma+\tau+1}.$$

Hence, the index remains unchanged in all cases (modulo sign changes), proving homotopy invariance.

We again summarize what we have achieved.

Proposition 3.1 *There is a map* $\operatorname{ind}(\cdot,\Omega) : \mathcal{C} \to \mathbb{Q}$*, where* $\mathcal{C} \subseteq C(\mathbb{R}^n,\mathbb{R}^n)$ *is the subset of vector fields without periodic orbits on* $\partial\Omega$*, with the following properties.*

(i) ind *is homotopy invariant under admissible homotopies, i.e.*

$$\operatorname{ind}(H_t,\Omega) = \operatorname{ind}(H_0,\Omega)$$

for a homotopy $H : \mathbb{R}^n \times I \to \mathbb{R}^n$ *in* $\mathcal{C}$ *and all* $t \in I$*.*

(ii) ind *is additive. If* $\Omega_1,\Omega_2 \subseteq \Omega$ *are open and disjoint such that there are no periodic orbits in* $\Omega\backslash(\Omega_1 \cup \Omega_2)$*, then*

$$\operatorname{ind}(f,\Omega) = \operatorname{ind}(f,\Omega_1) + \operatorname{ind}(f,\Omega_2).$$

(iii) ind *has the solving property, i.e. if* $\operatorname{ind}(f,\Omega) \neq 0$*, then there exists a periodic orbit in* Ω.

Once one has established the existence of an index for periodic orbits, many classical theorems on fixed points extend to the case of periodic orbits. Note, however, that it can be very difficult to find appropriate sets Ω, since we must have a priori bounds for the periods of periodic orbits in Ω.

4 A Global Bifurcation Result

As an application of the orbit index we give a version of a standard global bifurcation theorem of Rabinowitz on the global behaviour of solution branches, but in this case of course for branches of periodic orbits.

We consider the ODE

$$\dot{x} = f(\lambda, x),$$

where f is assumed to be $\mathcal{C}^2$ and λ is a real parameter in some open intervall J. Furthermore assume $f(\lambda, 0) = 0$ for all λ. We have, by Taylors theorem,

$$f(\lambda, x) = A(\lambda)x + O(\|x\|^2),$$

where $A(\lambda)$ is some $n \times n$-matrix, varying smoothly with λ. Now assume there is a discrete subset $P \subseteq J$ of parameter values such that $A(\lambda)$ is hyperbolic, i.e. has no eigenvalues on the imaginary axis for $\lambda \in J - P$ and for $\lambda_j \in P$ there is exactly one pair of eigenvalues $\pm i\omega_j$ which cross the imaginary axis with nonzero velocity. It is well known that periodic orbits can only branch from a parameter $\lambda_j \in P$ and only with period $\frac{2\pi k}{\omega_j}$. So the set of possible bifurcation points is given by

$$B = \bigcup_j \{(\lambda_j, 0, \frac{2\pi k}{\omega_j}) \mid k \in \mathbb{N}\}.$$

Let

$$\Lambda = \{(\lambda, x, t) \in J \times \mathbb{R}^n \times [0, \infty) \ \mid \varphi_\lambda(x,t) = x\},$$

the union of the set of periodic orbits (and fixed points) for all parameter values. Then the set of nontrivial periodic orbits is just $\Lambda - (J \times \{0\} \times \mathbb{R}^+)$. Thus we set

$$K = \Lambda - \big(J \times \{0\} \times \mathbb{R}^+\big) \cup B.$$

Then a bifurcating branch is contained in a connected component of K which intersects B. Now fix a possible bifurcation parameter $p = (\lambda_j, 0, \frac{2\pi k}{\omega_j})$. We want to describe the connected component K_0 of K which contains p. There are the following possibilities.

(1) K_0 contains a nontrivial fixed point of the flow, i.e. there is an $x \neq 0$, $f(\lambda, x) = 0$ and $(\lambda, x, t) \in K_0$ for some λ, t. That is, a branch of periodic orbits bifurcates from p and runs into a stationary point. Then K_0 is clearly unbounded since $(\lambda, x, t) \in K_0$ for all $t \in \mathbb{R}^+$.

(2) Otherwise, K_0 is disjoint from the set $J \times \mathbb{R}^n \times \{0\}$, i.e. it consists of proper nontrivial periodic orbits (and possible bifurcation points). Then there are again two possibilities.
   (a) K_0 is unbounded in the space/period component for λ varying in a compact subset of J.
   (b) K_0 is bounded in the space/period component for λ varying in any compact subset of J. Then for any $\varepsilon > 0$ there is an open neighbourhood Ω of K_0 in $J \times \mathbb{R}^n \times \mathbb{R}^+$ that is bounded for every fixed λ, whose boundary is ε-close to K_0 and whose boundary has empty intersection with K.

The main ingredient in the bifurcation theorem is the following

Proposition 4.1 *Suppose in the above alternative we have $(2)(b)$. Take two parameters λ_-, λ_+ such that λ_j is the only critical parameter between λ_- and λ_+ and the distance from λ_- and λ_+ to the critical parameter set P is greater than ε. Let $\Omega = \Omega_\varepsilon$ by the set provided by the above statement for such an ε. Let $\Omega(\lambda)$ be the λ-fibre of Ω. Then we have*

$$\mathrm{ind}(f(\lambda_+, \cdot), \Omega(\lambda_+)) = \mathrm{ind}(f(\lambda_-, \cdot), \Omega(\lambda_-)) \pm \frac{1}{k}(-1)^\rho,$$

where ρ is the number of eigenvalues of $A(\lambda_j)$ with positive real part and the sign is $+$ or $-$ according to the eigenvalues moving from left to right or from right to left over the imaginary axis.

The idea of the proof is the following. One constructs the set Ω by standard analysis with addition of a nontrivial theorem giving a lower bound for the periods. Approximate f by a sequence of smooth functions which have the same derivative at $x = 0, \lambda = \lambda_j$ as f, have the same set of possible bifurcation parameters, have 0 as trivial fixed point for all λ and such that the support of $f - f_n$ converges to $\{\lambda_j, 0\}$. For n large enough, the set Ω, constructed for some $\varepsilon > 0$ and the map f, has the same properties for the map f_n. Furthermore, since we have some control over the higher order terms of f_n, we can require that λ_j is a generic Hopf bifurcation parameter for f_n, thus there is a unique periodic solution bifurcating from λ_j of standard form. To be precise, let v_1, v_2 be a basis for the subspace of $\mathbb{R}^n$ corresponding to the pair $\pm i\omega_j$ of eigenvalues of $A(\lambda_j)$. Then one can write

$$\lambda = \lambda(\delta) = \lambda_j + c \cdot \delta^2 + O(\delta^3),$$

where $c \neq 0$ and its sign depends on the direction in which the orbit bifurcates. The periodic orbit is then given by

$$x_n(t, \lambda) = \delta(\cos(\omega_j t)v_1 + \sin(\omega_j t)v_2) + O(\delta^2)$$

with least period $\frac{2\pi}{\omega_j} + O(\delta)$.

Choose an $\varepsilon > 0$ as in the proposition and take the according set Ω. By the properties of the index, $\mathrm{ind}(f_{\lambda_\pm}, \Omega(\lambda_\pm)) = \mathrm{ind}(f^n_{\lambda_\pm}, \Omega(\lambda_\pm))$ for n large enough. But since the bifurcation of f_n is generic, the index changes when crossing λ_j just by the orbit index of the bifurcating nontrivial branch. This is certainly nonzero. The concrete form is a somewhat tedious computation of the eigenvalues of the Poincaré map for the standard orbit above. But for our applications, the only thing one needs to know is that the index changes when crossing λ_j, because this is the ingredient to prove

Theorem 4.2 *For the component K_0 there holds one of the following statements.*

(i) K_0 contains another bifurcation point $q = (\lambda_\ell, 0, \frac{2\pi m}{\omega_\ell})$ which is distinct from p.

(ii) K_0 is unbounded, meaning that either the periods blow up or the orbits approach infinity or the parameter approaches a boundary value of $J \subseteq \mathbb{R}$.

PROOF. Suppose neither (1) nor (2) holds. Then, since K_0 is bounded in parameter space, there is a compact subset $J_1 \subset J$ such that $p_1(K_0) \subseteq J_1$. Since by assumption K_0 is also bounded in period and spatial dimension, we are in case (2)(b) of the above proposition, so for any $\varepsilon > 0$ we get a set $\Omega = \Omega_\varepsilon$. If we take ε sufficiently small, the parameters λ_+, λ_- can be chosen independently of ε, so fix these parameters accordingly. Now take $\mu_- < \inf J_1$, $\mu_+ > \sup J_1$ in J. Since K_0 contains no bifurcation point different form p and certainly no point of the form $(\lambda, 0, t)$ for a non-bifurcation parameter λ, there is an $\varepsilon > 0$ such that the distance of K_0 and the set $J \times \{0\} \times \mathbb{R}^+$ is greater than ε. But since $K \cap \partial\Omega = \emptyset$, we have $\partial\Omega \cap \Lambda \subseteq J \times \{0\} \times \mathbb{R}^+$, so, since Ω is an ε neighbourhood of K_0, there are no points in $\partial\Omega \cap \Lambda$ with parameter in $[\mu_-, \lambda_-] \cup [\lambda_+, \mu_+]$, i.e. the set $\Omega(\lambda)$ is admissible for f_λ. So by homotopy invariance,

$$\mathrm{ind}(f_{\mu_\pm}, \Omega(\mu_\pm)) = \mathrm{ind}(f_{\lambda_\pm}, \Omega(\lambda_\pm)),$$

and by the formula for the index change when crossing λ_j,

$$\mathrm{ind}(f_{\lambda_-}, \Omega(\lambda_-)) = \mathrm{ind}(f_{\lambda_+}, \Omega(\lambda_+)) + c$$

for some $c \in \mathbb{Q} - \{0\}$. The formula states that $c = \pm\frac{1}{k}$, but since we didn't go into the calculation, it is enough to know $c \neq 0$. Because now we have

$$\mathrm{ind}(f_{\mu_-}, \Omega(\mu_-)) = \mathrm{ind}(f_{\mu_+}, \Omega(\mu_+)) + c,$$

but $\Omega(\mu_\pm) = \emptyset$ for ε sufficiently small, so both indices are zero, giving the intended contradiction. □

5 The Topological Approach

So far we defined the index analytically, counting signs of Jacobians and approximating in the non-generic case. There is also a topological approach which is conceptually much easier, but maybe a bit less accessible for intuition. We assume that M is an orientable, n-dimensional compact manifold. The orientability assumption can be dropped when considering $\mathbb{Z}_2$-orientability, but we will not do so. The idea how to count fixed points is now the following. Let $f : M \to M$ and let $U \subseteq M$ be some open set such that there are no fixed points of f on the boundary. Then f induces a map of pairs

$$(\mathbb{1}, f) : (\overline{U}, \partial U) \to (M \times M, M \times M \backslash \Delta)\,, x \mapsto (x, f(x))$$

where Δ denotes the diagonal. Since M is compact, there is a small neighbourhood $V \subseteq M \times M$ of the diagonal such that Δ is a deformation retract of V and where $\overline{V}$ is disjoint from the image of ∂U under f. We have canonical isomorphisms in homology

$$H_n(V) \cong H_n(\Delta),\ H^n(M \times M, M \times M - \overline{V}) \cong H^n(M \times M, M \times M - \overline{V}).$$

Thus, we can take the orientation class $\mathcal{O}_M \in H_n(M)$ and send it through the following sequence of maps

$$\begin{array}{ccccccccccc}
H_n(M) & \xrightarrow[\cong]{i_*} & H_n(\Delta) & \longrightarrow & H^n(M \times M, M \times M - \Delta) & \xrightarrow{(\mathbb{1},f)^*} & H^n(\overline{U}, \partial U) & \xrightarrow[\cong]{PD} & H_0(U) & \xrightarrow{i_*} & H_0(M)\ . \\
& \searrow & \cong \downarrow i_* & & \cong \downarrow j^* & & & & & & \\
& & H_n(\overline{V}) & \xrightarrow[\cong]{PD} & H^n(M \times M, M \times M - \overline{V}) & & & & & &
\end{array}$$

We end up in $H_0(M)$, which, if M is connected, is naturally isomorphic to $\mathbb{Z}$, i.e. we get an integer. Now, if U is contained in a coordinate chart and if f is fixed point free, then the map $(1, f)$ can be deformed into a constant map by a homotopy without fixed points on the boundary of U. But then, we will end up with $0 \in H_0(M)$. So, our element in fact has the property that nontriviality implies existence of fixed points. Homotopy invariance and additivity follows immediately from the properties of homology. Hence it is justified to call the image of $\mathcal{O}_M$ under this sequence the index of f. In fact, this number is called the Lefschetz number of f and it equals the fixed point index defined analytically.

To generalize this definition to the periodic orbit setting, we again take a factor $\mathbb{R}^+$ for the periods. Then let $U \subseteq M \times \mathbb{R}^+$ be an open subset such that there are no periodic orbits on its boundary. We get the map

$$(p_1, \varphi) : (U, \partial U) \to (M \times M, M \times M - \Delta),\ (x, t) \mapsto (x, \varphi(x, t)).$$

As above we find a neighbourhood V of Δ and so we can imitate the construction above:

$$\begin{array}{ccccccccccc}
H_n(M) & \xrightarrow[\cong]{i_*} & H_n(\Delta) & \longrightarrow & H^n(M \times M, M \times M - \Delta) & \xrightarrow{(p_1,\varphi)^*} & H^n(\overline{U}, \partial U) & \xrightarrow[\cong]{PD} & H_1(U) & \xrightarrow{i_*} & H_1(M)\ . \\
& \searrow & \cong \downarrow i_* & & \cong \downarrow j^* & & & & & & \\
& & H_n(\overline{V}) & \xrightarrow[\cong]{PD} & H^n(M \times M, M \times M - \overline{V}) & & & & & &
\end{array}$$

There is one significant difference: We end up in $H_1(M)$, so we do not get (in general) an integer but a 1-homology class as index. This can be interpreted as follows. If γ_k is an isolated periodic orbit, considered with period kT, then the homology class of γ, considered with its least period T, is a generator of $H_1(U)$ for a tubular neighbourhood of γ. Thus, the index is of the form $c \cdot k \cdot [\gamma]$, where c is some integer and $[\gamma]$ the generator. It is not difficult to show that $c = \frac{\mathrm{ind}(P)}{k}$, where $\mathrm{ind}(P)$ is the classical fixed point index of a Poincaré map for the orbit, so this agrees with the analytical definition. The problem that the first homology may vanish can also be solved to give the analytical index as a rational number, but we will not go into detail here. It is also worth noting that the homological index is a finer invariant (after solving the triviality problem), than the analytical one.

6 Equivariant Indices

When considering equivariant problems, the indices constructed above are not the appropriate tools in some sense. We state some of the defects that might occur.

- It may well happen that two G-maps are homotopic, so would give the same index, but are not G-homotopic. So a degree that would respect only G-homotopies might give different degrees of those maps, which would provide finer results than the non-equivariant method.
- The index counts the number of periodic orbits, or fixed points. So for a G-map, a nontrivial index implies existence of periodic orbits or fixed points in the ordinary sense. But we know nothing about the symmetry properties. It is desirable to conclude from looking at the index, that a solution which at least a given symmetry occurs.
- Assume we have a $\mathbb{Z}_2$-action on some euclidean space and an equivariant map which has only two fixed points, which are assumed to be hyperbolic, so they are necessarily connected by symmetry. Assume further more that the action of $\mathbb{Z}_2$ reverses orientation. Then by definition, one of these fixed points is calculated as 1 and the other as -1, so the index of our map is zero. But of course, from an equivariant viewpoint, the index of an isolated group orbit of hyperbolic fixed points should be non-zero.

So for the study of equivariant systems, there should be a modified index which should have properties derived from the properties in the non-equivariant case adjusted to the G-action.

For fixed points, there have been several approaches in constructing a suitable degree. All constructions known to me use equivariant extensions of G-maps to some representation sphere and then assign an element of some G-homotopy group as index. There are some unsatisfactory points to this definition. Though homotopy groups serve some geometric intuition, they are hard to understand topologically. Furthermore, the geometric intuition also vanishes when one stabilizes, which is usually done and necessary for sufficient nontriviality of the index. Also, the problems become even greater when dealing with compact Lie groups instead of finite groups.

My work (to be done) focuses on three aspects. I want to define an equivariant periodic orbit index which has all the desired properties. This might be done in a similar way the orbit index is constructed from the fixed point index, but maybe one needs a conceptually different approach. This leads to the second aspect. I try to find a more topological viewpoint on equivariant index theory, namely the use of equivariant homology theories might be a very fruitful attempt. It generalizes the concepts developed so far, it inserts more geometric objects, such as G-vector bundles, in the area of interest and it is easier to deal with in a topological sense, so there might be theoretical results obtainable in general from a homological approach which are very hard to see in homotopy theory. The last aspect is of course the application of the equivariant orbit index to prove equivariant bifurcation results and to investigate related topics. Especially the case of an action of compact Lie groups of dimension greater than zero is poorly understood from an index theoretic viewpoint.

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