A Topological Tool for Computer Vision

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January 23, 2012

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1 Introduction

Let $\mathcal{M} \subset \mathbb{R}^n$ denote a compact, piecewise smooth, closed m-submanifold. Let $\mathcal{M}^{k+1}$ denote the set of ordered $(k+1)$-tuples of points in $\mathcal{M}$. We refer to a function $\varphi : \mathcal{M}^{k+1} \to \mathbb{R}$ a *measuring function.*

Definition 1.1. We say that any two points $P = (P_0, \ldots, P_k), Q = (Q_0, \ldots, Q_k) \in \mathcal{M}^{k+1}$ are $(\varphi \leq x)$-homotopic if either $P = Q$, or there is a continuous map $H : [0,1] \to \mathcal{M}^k$ with $H(0) = P$ and $H(1) = Q$ such that

$$\varphi\left(H(t)\right) \leq x, \quad \forall\, t \in [0,1].$$

The notion of $(\varphi \leq x)$-homotopy defines an equivalence relation on any subset $U \subset \mathcal{M}^{k+1}$. We denote the set of equivalence classes on U with respect to the $(\varphi \leq x)$-homotopy by $U_{\langle \varphi \leq x \rangle}$.

Now, let $\varphi^{-1}(-\infty, x] \subset \mathcal{M}^{k+1}$ denote the preimage of $(-\infty, x]$ under φ. Next we define a *size function* corresponding to a measuring function.

Let $|M|$ denote the cardinality of a set M, then we define a size function by:

$$\begin{aligned} &l_{(\varphi,\mathcal{M})} : \mathbb{R}^2 \to \mathbb{N} \cup \{0, \infty\}, \\ &l_{(\varphi,\mathcal{M})}(x,y) := \left|\varphi^{-1}(-\infty, x]_{\langle \varphi \leq y \rangle}\right|. \end{aligned}$$

Let us observe that:

1. $l_{(\varphi,\mathcal{M})}$ is non decreasing in x;
2. $l_{(\varphi,\mathcal{M})}$ is non increasing in y;
3. $l_{(\varphi,\mathcal{M})} = 0$ for all $x < \min\limits_{P \in \mathcal{M}^{k+1}} \varphi(P)$;
4. $l_{(\varphi,\mathcal{M})} = \infty$ if there is a non isolated point $P \in \mathcal{M}^{k+1}$ with:

$$y < \varphi(P) < x.$$

Example 1.2. Let $\mathcal{M} \subset \mathbb{R}^2$ be the contour in (Figure 1) and $\varphi : \mathcal{M}^1 \to \mathbb{R}$ the distance from the point $B \in \mathbb{R}^2$ shown in the figure.

Example 1.3. Consider the international sign alphabet in (Figure 2). In (Figure 3), part (f) the size graph for the contours of the letter "w" is shown. Here the measuring function $L(v)$ for $v \in \alpha([0,1])$, where α is the curve that represents the contour, is the length of the curve within a ball centered on v of radius $c > 0$.

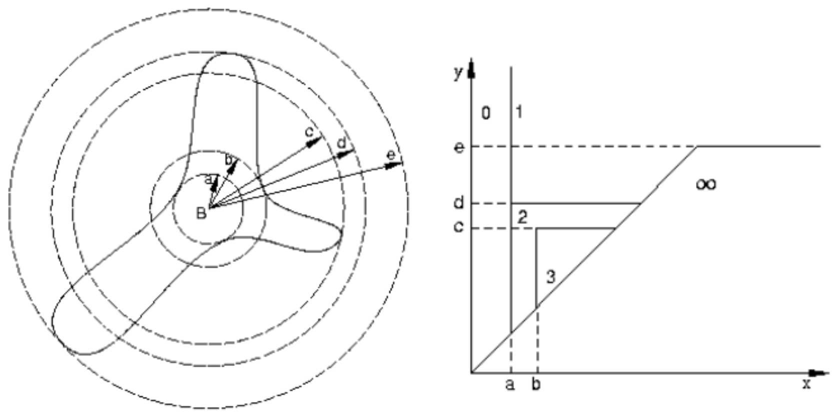

Figure 1

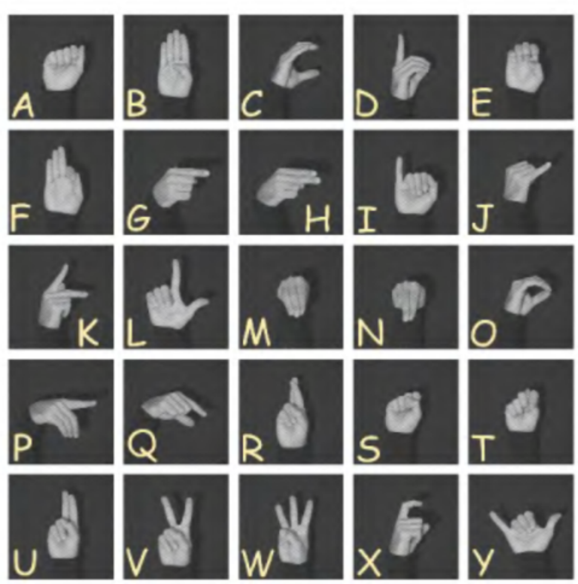

Figure 2

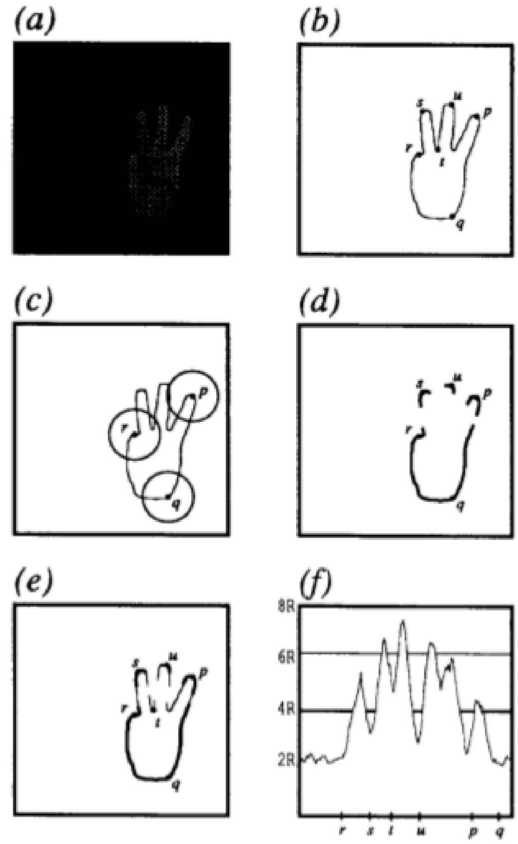

Figure 3

2 Results relevant for the Application

This simple theorem 2.1 has obvious implications for applications, but also for theoretical approach to size and submanifold distance notions.

Theorem 2.1. *Let φ, ψ denote two measuring functions on $\mathcal{M}^{k+1}$. Let $\|\cdot\|_{\mathcal{M}^{k+1}}$ denote the uniform distance on $\mathcal{M}^{k+1}$ and assume $\|\varphi - \psi\|_{\mathcal{M}^{k+1}} \leq \varepsilon$ for some $\varepsilon \geq 0$. Then for any $x, y \in \mathbb{R}$ we have:*

$$l_{(\varphi,\mathcal{M})}(x-\varepsilon, y+\varepsilon) \leq l_{(\psi,\mathcal{M})}(x, y) \leq l_{(\varphi,\mathcal{M})}(x+\varepsilon, y-\varepsilon)$$

Proof. Assume $\varphi^{-1}(-\infty, x]_{\langle\varphi\leq x-\varepsilon\rangle}$ is not empty, otherwise the inequality is trivial. Consider the map:

$$\begin{aligned}&\iota : \varphi^{-1}(-\infty, x-\varepsilon]_{\langle\varphi\leq y+\varepsilon\rangle} \to \psi^{-1}(-\infty, x]_{\langle\psi\leq y\rangle},\\ &\iota([P]_{\varphi,x-\varepsilon,y+\varepsilon}) := [P]_{\psi,x,y}\,.\end{aligned}$$

First of all, take $P \in \varphi^{-1}(-\infty, x-\varepsilon]$, then it's:

$$-\varepsilon \leq \psi\,(P) - \varphi\,(P) \leq \varepsilon \;\Leftrightarrow\; \psi\,(P) \leq \varepsilon + \varphi\,(P) \leq x$$

thus $P \in \psi^{-1}(-\infty, x]$.

Now choose another point $Q \in \varphi^{-1}(-\infty, x-\varepsilon]$ and assume $\iota[P] = \iota[Q]$, thus either $P = Q$ or there is a homotopy H that connects P with another point Q such that $\psi \circ H \leq y$. If $P \neq Q$, then we have:

$$\phi \circ H \leq \psi \circ H + \varepsilon \leq y + \varepsilon,$$

hence $[P] = [Q]$ in $\varphi^{-1}(-\infty, x-\varepsilon]_{\langle\varphi\leq y+\varepsilon\rangle}$. □

Now we assume φ is a Morse measuring function, thus $\mathscr{C}^2$ and with regular second derivative, and denote $V := -\nabla\varphi$.

Definition 2.2. Let $H : [0,1] \to \mathcal{M}^{k+1} \in \mathscr{C}^1(0,1)$ with $H(0)$,$H(1)$ critical points of V. We say H is a flow homotopy if $H(0) = H(1)$ or $\frac{dH}{dt}(t) = k_t V(H(t))$ with $k_t \in \mathbb{R} \setminus \{0\}$ for all $t \in (0,1)$. A homotopy obtained by a piecewise composition of flow homotopies will be called piecewise flow homotopy (p.f. homotopy).

From the definition it's clear that for any flow homotopy the function $\varphi \circ H$ has it's maxima only on the boundary. Thus, if we choose critical values in $\varphi^{-1}(-\infty, x]$ then it's clear that for any p.f. homotopy $\varphi \circ H \leq x$ will hold. This simplifies very much the computation of size functions, as we can restrict ourselves to critical points of the measuring function, which is what the next Theorem states. But before we can state the Theorem we have to add another definition.

Definition 2.3. Consider the graph $\mathscr{G}$ with verticies $\{C_1, \ldots, C_N\}$ the critical points of V and edges that connect any C_i, C_j if there is a flow homotopy that connects C_i with C_j. For every $x \in \mathbb{R}$ define $\mathscr{G}_y$ the subgraph of $\mathcal{G}$ obtained by deleting thouse verticies of $\mathcal{G}$ on which φ takes values greater then x.

Moreover, for any $(x, y) \in \mathbb{R}^2$ let $\mathscr{L}_\varphi(x, y)$ denote the number of connected components of $\mathscr{G}_y$ which contain at least one vertex of $\mathscr{G}_x$.

Theorem 2.4. *For any $x, y \in \mathbb{R}$ we have $l_{(\varphi,\mathcal{M})}(x, y) = \mathscr{L}_\varphi(x, y)$.*

Proof. See [Frosini 1996]. □

The immediate implication of the theorem is stated in the corollaries below, which can be found in [Frosini 1996]. From here the connection to *persistent homology* is immediate. Consult also [Weinberger] and compare to (Figure 3) part (f). Another good source is also [Kaczinski].

We denote $\mathcal{D} := \{(x, y) \in \mathbb{R}^2 \, : \, x \leq y\}$ and $\hat{l}_{(\varphi,\mathcal{M})} := l_{(\varphi,\mathcal{M})}|_\mathcal{D}$.

Corollary 2.5. *$\hat{l}_{(\varphi,\mathcal{M})}$ is a locally right-constant function, i.e. there is an $\varepsilon > 0$ for $(x, y) \in \mathring{\mathcal{D}}$ with $(x + \varepsilon, y) \in \mathcal{D}$ and $\hat{l}_{(\varphi,\mathcal{M})}(x, y) = \hat{l}_{(\varphi,\mathcal{M})}(x + \varepsilon, y)$.*

Corollary 2.6. *A necessary condition for $(x, y) \in \mathcal{D}$ to be a discontinuity point of $l_{(\varphi,\mathcal{M})}|_\mathcal{D}$ is that at least one of x, y is a critical value of φ.*

3 Theoretical Results

A natural way to deal with shapes and similarity of objects is to define a distance that would vanish for similar objects. One possible approach is as follows.

Definition 3.1. Let Σ_n be the set of $\mathscr{C}^\infty$ compact n-manifolds without boundary and embedded in $\mathbb{R}^m$. Define the equivalence relation $\sim$ by $\mathcal{M} \sim \mathcal{N}$ if there exists a similarity transformation $\theta : \mathbb{R}^m \to \mathbb{R}^m$, (i.e. a rotation, translation, scaling of reflection), such that $\theta(\mathcal{M}) = \mathcal{N}$.

Denote with $D(\mathcal{M}, \mathcal{N})$ the set of $\mathscr{C}^\infty$ diffeomorphisms from $\mathcal{M}$ to $\mathcal{N}$, and denote $\Sigma_n^\theta := \Sigma_n/_\sim$ the quotient of Σ_n up to equivalence $\sim$. Then we define the distance σ by:

$$\sigma\left([\mathcal{M}], [\mathcal{N}]\right) := \begin{cases} \infty & \text{if } D(\mathcal{M}, \mathcal{N}) = \emptyset \\ \inf_{\varphi \in D(\mathcal{M}, \mathcal{N})} \log\left(\sup_{P \in \mathcal{M}} \|\varphi(P)\| \sup_{Q \in \mathcal{N}} \|\varphi^{-1}(Q)\| \right) & \text{else.} \end{cases}$$

Although this distance reflects quite directly our expectations, it's obvoiusly not that simple to deal with. The more remarkable is the result that folows.

Theorem 3.2. *Let $\mathcal{M}, \mathcal{N} \in \Sigma_n^\theta$, $\delta(M) := \sup_{p,q \in \mathcal{M}} \|p - q\|$ the diamiter for any submanifold $\mathcal{M}$.*

If $f^L(\mathcal{N}, \xi, \mu) < f^L(\mathcal{M}, x, y)$ for $\xi, \mu, x, y > 0$ then:

$$\sigma([\mathcal{M}], [\mathcal{N}]) \geq \log\left(\min\left\{ \frac{\xi \cdot \delta(M)}{x \cdot \delta(N)}, \frac{y \cdot \delta(N)}{\mu \cdot \delta(M)} \right\} \right).$$

Here f^L is a size function defined for closed curves on a manifold with respect to a measuring function that measures the length of a curve.

Proof. [Frosini 1990], [Frosini 1999]. □

A simpler distance and an equivalent result for it can be found in [Frosini 1999].

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