Symmetric Topological Complexity and Embedding Problems for Projective Spaces

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[1] J. Gonzáles and P. Landweber, *Symmetric topological complexity of projective and lens spaces* Algebraic and Geometry Topology 9 (2009), 473-494

Main Result: $\mathrm{TC^S}$ and Embedding Dimension for P^r

Theorem (Gonzáles and Landweber, 2009)

$\mathrm{TC^S}$	:	Symmetric Topological Complexity
P^r	:	r-dimensional projective space
$E(r)$	:	Euclidean embedding dimension for P^r

$$\mathrm{TC^S}(P^r) = E(r) + 1, \quad r \in \{1, 2, 4, 8, 9, 13\} \quad r > 15$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \dots\}$$

(Proof Strategy)

- $\mathrm{TC^S}(P^r)$ Characterization: $\mathrm{TC^S}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$.
- $E(r)$ Characterization: Haefliger's metastable range: $2m \geq 3(n+1)$
  - Smooth embedding $M \subset \mathbb{R}^m$, $\dim(M) = n$.
  - $\mathbb{Z}_2$ equivariant map $M^* := M \times M - \Delta_M \to \mathbb{S}^{m-1}$.

Symmetric Topological Complexity TC^{S}: Definition

Definition (Schwarz genus)

Fibration $p : E \to B$
genus(p) is the smallest number of open sets U covering B s.t. p admits a continuous section on each U.

Definition (Topological Complexity)

- X Topological space
- $\mathrm{ev} : P(X) \to X \times X$ endpoints evaluation map
- $P(X)$ path space $X^{[0,1]}$ with compact open topology

$$\boxed{\mathsf{TC}(X) := \mathrm{genus}(ev)}$$

Definition (Symmetric Topological Complexity)

$\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$ $\quad P_1(X) := \{\text{paths } \gamma \in X^{[0,1]}, \gamma(0) \neq \gamma(1)\}$
$\mathrm{ev}_2 : P_2(X) \to B(X,2)$ $\quad P_2(X) := P_1(X)/\mathbb{Z}_2$
$B(X,2) := (X \times X - \Delta_X)/\mathbb{Z}_2$

$$\boxed{TC^S(X) := \mathrm{genus}(\mathrm{ev}_2) + 1}$$

Main Result Step 1: $\mathrm{TC}^{\mathrm{S}}(P^r)$ Characterization

Theorem (Gonzáles and Landweber, 2009)

$$\mathrm{TC}^{\mathrm{S}}(P^r) = E(r) + 1, \quad r \in \{1, 2, 4, 8, 9, 13\} \quad r > 15$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \dots\}$$

(Proof Strategy)

- $\mathrm{TC}^{\mathrm{S}}(P^r)$ Characterization: $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1.$
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(Remark: TC vs TC^{S} for Projective Spaces P^r)

Immersion problem and Topological Complexity (via Axial Maps)
Embedding problem and Symmetric Topological Complexity (via Levels)

Characterizing TC^{S} with Levels

Definition (Level of an involution)

The level of a $\mathbb{Z}_2$ action on a space X:

$$\text{level}(X, \mathbb{Z}_2) = \min\{\ell > 0 : \exists\, \mathbb{Z}_2 \text{ equivariant map } X \to \mathbb{S}^{\ell-1}\}$$

Theorem

For all values of r, $\mathsf{TC}^{\mathsf{S}}(P^r) = \text{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$

(Proof Ideas)

Three ingredients:

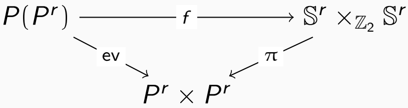
- For a $\mathbb{Z}_2$ action on X which admits $\mathbb{Z}_2$ equivariant map $X \to \mathbb{S}^{n-1}$, and for the canonical projection $p : X \to X/\mathbb{Z}_2$: (see Schwarz)
$$\text{genus}(p) = \text{level}(X, \mathbb{Z}_2)$$
- Characterizing genus(ev_i) for $\text{ev}_1 : P_1(X) \to X \times X - \Delta_X$.
- Characterizing genus(ρ) for $\rho : P^r \times P^r - \Delta_{P^r} \to B(P^r, 2)$ the canonical projection.

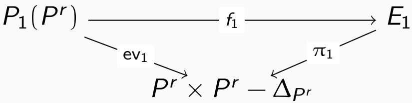
Characterizing genus(ev_i): Main Plan

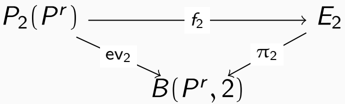
Proposition

For $i \in \{1, 2\}$ $\quad \mathrm{genus}(\mathrm{ev}_i) = \mathrm{genus}(\pi_i)$

(sketch definition of π_1 and π_2)







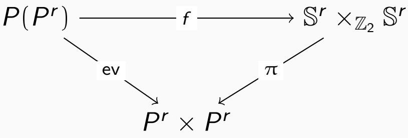
$P_1(P^r) := \{\text{paths } \gamma \in P^{r\,[0,1]}, \gamma(0) \neq \gamma(1)\}$
$P_2(P^r) := P_1(P^r)/\mathbb{Z}_2$
$B(P^r, 2) := (P^r \times P^r - \Delta_{P^r})/\mathbb{Z}_2$

Characterizing genus(ev_i): Constructing f

Proposition

For $i \in \{1, 2\}$ $\quad$ $\mathrm{genus}(\mathrm{ev}_i) = \mathrm{genus}(\pi_i)$

(Constructing $f : P(P^r) \to \mathbb{S}^r \times_{\mathbb{Z}_2} \mathbb{S}^r$)

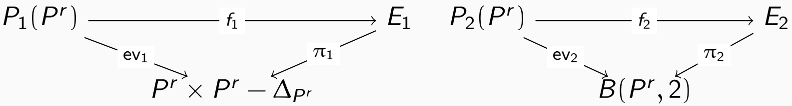


For a path $\gamma \in P(P^r)$, let $\hat{\gamma} : [0, 1] \to \mathbb{S}^r$ be any lifting through the canonical projection $\mathbb{S}^r \to P^r$, then $f(\gamma)$ is the class of $(\hat{\gamma}(0), \hat{\gamma}(1))$ in the Borel construction

$$\mathbb{S}^r \times_{\mathbb{Z}_2} \mathbb{S}^r := (\mathbb{S}^r \times \mathbb{S}^r)/(-x, y) \sim (x, -y)$$

Characterizing genus(ev_i): Constructing g_1

(Defining f_1 and f_2)



Proposition

For $i \in \{1, 2\}$ $\quad$ $\mathrm{genus}(\mathrm{ev}_i) = \mathrm{genus}(\pi_i)$

(Proof Idea: $\mathbb{Z}_2$ equivariant map $g_1 : E_1 \to P_1(P^r)$)

g_1 run backwards with respect to f_1. Explicit construction of g_1:
Model for E_1: $(\mathbb{S}^r \times \mathbb{S}^r - \tilde{\Delta})/(x, y) \sim (-x, -y)$
$\tilde{\Delta} = \{(x, y) \in \mathbb{S}^r \times \mathbb{S}^r | x \neq \pm y\}$
g_1 maps the class of a pair (x_1, x_2) into the curve $[0, 1] \to \mathbb{S}^r \to P^r$ with the first map $t \mapsto v(tx_1 + (1 - t)x_2)$ and v is the normalization map.

Characterizing TC^{S} with Levels: Idea of Proof

Theorem

For all values of r, $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$

(Proof Ideas)

Three ingredients:

- For a $\mathbb{Z}_2$ action on X which admits $\mathbb{Z}_2$ equivariant map $X \to \mathbb{S}^{n-1}$, and for the canonical projection $p : X \to X/\mathbb{Z}_2$: (see Schwarz)
$$\mathrm{genus}(p) = \mathrm{level}(X, \mathbb{Z}_2)$$
- Characterizing $\mathrm{genus}(\mathrm{ev}_i)$ for $\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$.
$$\mathrm{genus}(ev_i) = \mathrm{genus}(\pi_i)$$
- Characterizing $\mathrm{genus}(\rho)$ for $\rho : P^r \times P^r - \Delta_{P^r} \to B(P^r, 2)$ the canonical projection.
$$\mathrm{genus}(\rho) = \mathrm{genus}(\pi_2)$$

Characterizing TC^{S} with Levels: Idea of Proof - Final Step

Theorem

For all values of r, $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$

(Proof Ideas)

- $TC^S(X) := \mathrm{genus}(\mathrm{ev}_2) + 1$, $\mathrm{ev}_2 : P_2(X) \to B(X, 2)$
- Characterizing $\mathrm{genus}(\mathrm{ev}_i)$ for $\mathrm{ev}_1 : P_1(X) \to X \times X - \Delta_X$.

$$\mathrm{genus}(ev_i) = \mathrm{genus}(\pi_i)$$

- Characterizing $\mathrm{genus}(\rho)$ for $\rho : P^r \times P^r - \Delta_{P^r} \to B(P^r, 2)$ the canonical projection.

$$\mathrm{genus}(\rho) = \mathrm{genus}(\pi_2)$$

- For the canonical projection $p : X \to X/\mathbb{Z}_2$: (see Schwarz)

$$\mathrm{genus}(p) = \mathrm{level}(X, \mathbb{Z}_2)$$

$$\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$$

Main Result Step 2: $E(r)$ Characterization

Theorem (Gonzáles and Landweber, 2009)

$$\mathrm{TC}^{\mathrm{S}}(P^r) = E(r) + 1, \quad r \in \{1, 2, 4, 8, 9, 13\} \quad r > 15$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \dots\}$$

(Proof Strategy)

- $\mathrm{TC}^{\mathrm{S}}(P^r)$ Characterization: $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1.$
- $E(r)$ Characterization: Haefliger's metastable range: $2m \geq 3(n+1)$
  - Smooth embedding $M \subset \mathbb{R}^m$, $\dim(M) = n$.
  - $\mathbb{Z}_2$ equivariant map $M^* := M \times M - \Delta_M \to \mathbb{S}^{m-1}$.

Embeddings and Haefliger's Metastable Range

Theorem (Haefliger's Metastable Range)

M smooth n-dimensional manifold and $2m \geq 3(n+1)$, then there is a surjective map from the set of isotopy classes of smooth embeddings $M \subset \mathbb{R}^m$ onto the set of $\mathbb{Z}_2$-equivariant homotopy classes of maps $M^* \to \mathbb{S}^{m-1}$

(Our usage of the Haefliger Metastable Range)

The existence of a smooth embedding $M \subset \mathbb{R}^m$ is equivalent to the existence of a $\mathbb{Z}_2$-equivariant map $M^* \to \mathbb{S}^{m-1}$

(Explicit Surjective map $M = P^r$)

Embedding $g : P^r \to \mathbb{R}^d$ determines a $\mathbb{Z}_2$-equivariant map $\tilde{g} : P^r \times P^r - \Delta_{P^r} \to \mathbb{S}^{d-1}$:

$$\tilde{g}(a,b) = \frac{g(a) - g(b)}{|g(a) - g(b)|}$$

Proposition (Gonzáles and Landweber, 2009)

For $r \in \{8, 9, 13\}$ or $r > 15$, an axial map $P^r \times P^r \to P^s$ can exist only when $2s \geq 3(r+1)$.

Theorem (Symmetric Axial Maps and Embeddings)

The existence of a symmetric axial map $P^r \times P^r \to P^s$ implies the existence of a smooth embedding $P^r \subset \mathbb{R}^{s+1}$ provided $2s > 3r$.
Gonzáles and Landweber, 2009

The existence of a smooth embedding $P^r \subset \mathbb{R}^s$ implies the existence of a symmetric axial map $P^r \times P^r \to P^s$.
Haefliger, Hirsch, 1961, 1962

(Consequence)

$a_S(r)$ smallest integer k for which exist a symmetric axial map $P^r \times P^r \to P^k$

$$E(r) = a_S(r) + \delta, \qquad \delta = \delta(r) \in \{0, 1\}.$$

Main Result: Final Step

(Main Ideas)

- Haefliger's metastable range: $2m \geq 3(n+1)$
  - Smooth embedding $M \subset \mathbb{R}^m$ $\dim(M) = n$
  - $\mathbb{Z}_2$ equivariant map $M^* := M \times M - \Delta_M \to \mathbb{S}^{m-1}$.
- $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$.

Theorem (Gonzáles and Landweber, 2009)

$$\mathrm{TC}^{\mathrm{S}}(P^r) = E(r) + 1, \quad r \in \{1, 2, 4, 8, 9, 13\} \quad r > 15$$

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Cases Outside the Metastable Range

(Main Properties for analyzing some missing cases $r \leq 15$)

$$\mathrm{TC}(P^r) \leq \mathrm{TC}^{\mathrm{S}}(P^r) \leq E(P^r) + 1$$

$$\mathrm{TC}(P^r) \leq \mathrm{TC}^{\mathrm{S}}(P^r) \leq E_{\mathrm{TOP}}(P^r) + 1$$

$$l(r) \leq \mathrm{TC}^{\mathrm{S}}(P^r) \leq u(r)$$

(Estimations for $\mathrm{TC}^{\mathrm{S}}(P^r)$)

r	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**10**	**11**	**12**	**14**	**15**
u(r)	**3**	**5**	6	**9**	10	10	11	18	19	22	24	24
l(r)	**3**	**5**	5	**9**	9	9	9	17	17	19	23	23

Main Result: Final Step

(Main Ideas)

- Haefliger's metastable range: $2m \geq 3(n+1)$
  - Smooth embedding $M \subset \mathbb{R}^m$ $\dim(M) = n$
  - $\mathbb{Z}_2$ equivariant map $M^* := M \times M - \Delta_M \to \mathbb{S}^{m-1}$.
- $\mathrm{TC}^{\mathrm{S}}(P^r) = \mathrm{level}(P^r \times P^r - \Delta_{P^r}, \mathbb{Z}_2) + 1$.

Theorem (Gonzáles and Landweber, 2009)

$$\mathrm{TC}^{\mathrm{S}}(P^r) = E(r) + 1, \quad r \in \{1, 2, 4, 8, 9, 13\} \quad r > 15$$

$$\{\mathbf{1}, \mathbf{2}, 3, \mathbf{4}, 5, \mathbf{6}, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, \ldots\}$$

Complex Projective Spaces

Theorem

$\mathrm{TC}^{\mathrm{S}}(\mathbb{C}P^n) = 2n+1.$

(Sketch Ideas)

$$
\begin{array}{ccccc}
P(P^r) & \longleftarrow & P_1(\mathbb{C}P^n) & \longrightarrow & P_2(\mathbb{C}P^n) \\
\downarrow{\scriptstyle \mathrm{ev}} & & \downarrow{\scriptstyle \mathrm{ev}_1} & & \downarrow{\scriptstyle \mathrm{ev}_2} \\
\mathbb{C}P^n \times \mathbb{C}P^n & \longleftarrow & \mathbb{C}P^n \times \mathbb{C}P^n - \Delta_{\mathbb{C}P^n} & \longrightarrow & B(\mathbb{C}P^n, 2)
\end{array}
$$

- Diagram of pullback squares
- Common fiber: path connected space $\Omega\mathbb{C}P^n$
- (Schwarz) $\mathrm{TC}^{\mathrm{S}}(\mathbb{C}P^n) = \mathrm{genus}(ev_2) + 1 \leq \frac{\dim(Y)}{2} + 2$
  Y: CW-Complex having the homotopy type of $B(\mathbb{C}P^n, 2)$.
- (Farber,Grant, 2007) Observation: for a smooth closed m-dimensional manifold M, $B(M,2)$ has the homotopy type of a $(2m-1)$-dimensional CW-complex.

Lens Spaces

(Problems with $L^{2n+1}(m)$)

- No clear relation between the TC^{S} of $(L^{2n+1}(m))$ and its embedding dimension (or to the level of the switching involution $L^{2n+1}(m) \times L^{2n+1}(m) - \Delta_{L^{2n+1}(m)}$).
- For high torsion lens spaces $L^{2n+1}(m)$, their TC^{S} is equal to

$$4n + \epsilon, \qquad \epsilon \in \{2, 3\},$$

the level of the corresponding switching involution is at most

$$2n + 3$$

(See Rees, 1971).