On Groupoid C^*-Algebras, Persistent Homology and Time-Frequency Analysis

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Abstract

We study some topological aspects in time-frequency analysis in the context of dimensionality reduction using C^*-algebras and noncommutative topology. Our main objective is to propose and analyze new conceptual and algorithmic strategies for computing topological features of datasets arising in time-frequency analysis. The main result of our work is to illustrate how noncommutative C^*-algebras and the concept of Morita equivalence can be applied as a new type of analysis layer in signal processing. From a conceptual point of view, we use groupoid C^*-algebras constructed with time-frequency data in order to study a given signal. From a computational point of view, we consider persistent homology as an algorithmic tool for estimating topological properties in time-frequency analysis. The usage of C^*-algebras in our environment, together with the problem of designing computational algorithms, naturally leads to our proposal of using AF-algebras in the persistent homology setting. Finally, a computational toy example is presented, illustrating some elementary aspects of our framework. Due to the interdisciplinary nature of this work, we include a significant amount of introductory material on recent developments in groupoid theory and persistent homology.

Keywords: time-frequency analysis, groupoids C^*-algebras, Morita equivalence, persistent homology, AF-algebras, dimensionality reduction.

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1 Introduction

A crucial problem in modern developments of signal and data analysis is to construct precise and efficient methods for extracting, decomposing, and manipulating a signal. Time-frequency analysis is a fundamental strategy in signal processing, and at the core of its philosophy lies the concept of partitioning a signal f in segments $x_b = fg_b$, using $g_b(t) = g(t-b)$ for a window function g. Wavelet analysis, Gabor transforms and a large number of variations of these concepts have appeared over the last decades, illustrating the importance of the time-frequency philosophy. However, despite the success of these developments, modern problems in engineering fields are increasingly demanding more accurate and flexible tools for dealing with the ever increasing complexity of dynamical systems, signals, and datasets arising in many applications domains.

Over the last few years a new range of tools in pure and applied mathematics have emerged for the analysis of geometrical and topological structures. On the one hand, in application and engineering fields, there has been an important development of new strategies for the analysis of datasets $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ embedded in high-dimensional spaces. In this context, new algorithms have been proposed for manifold learning and dimensionality reduction, some of them using concepts from differential geometry and spectral decompositions. From a topological perspective, a dataset $X = \{x_i\}_{i=1}^m$ can now be analyzed with the powerful tools of persistent homology, which has emerged as an important subfield of computational topology. On the other hand, the vast universe of noncommutative geometry and noncommutative topology has investigated for several decades the relations between spaces X and C^*-algebras, and the large range of ideas in these developments has shown its potential in various applications fields.

Our main motivation in this project is to study a particular type of interaction of these topics, where our main concept is to study a signal f using the geometrical and topological properties of datasets $X_f = \{x_i\}$, arising from time-frequency representations. Our research direction differs substantially from modern developments in signal processing, as our main objective is to consider the interplay of topological and algebraic properties of spaces X_f in order to understand a signal f.

Our contributions in this paper are both conceptual and computational perspectives, and can be summarized as follows. From a conceptual point of view, we define the notion of a functional cloud $X_f = \{x_i\}_{i=1}^m$ (that we also denote by $M_f = \{x_i\}$ for non necessarily discrete cases) which can be considered as a geometrical/topological summary of the functional chunks x_i of a signal f. This concept encodes information of a signal f in a time-frequency framework, and we study a special version defined as a quotient space $M^G_{V_\psi f} = F_{V_\psi f}/G$ for $F_{V_\psi f}$ the graph of $V_\psi f|_{\mathrm{supp} V_\psi f}$, a time-frequency transform of f (wavelet, Gabor, etc), and G a groupoid (a powerful concept that generalizes the notion of equivalence relations, groups, group actions, etc). When considering a functional cloud $M^G_{V_\psi f}$ as a quotient space, we can study its structure in the framework of noncommutative topology. A main result of our work is to illustrate how noncommutative C^*-algebras, and the concept of Morita equivalence, can be applied as a new type of analysis layer in signal processing. The basic idea can be summarized as (Theorem 2.6):

$$\text{for a signal} \quad f = \textstyle\sum_{i=1}^k f_i \quad \text{we have} \quad C_0(M^G_{V_\psi f}) \overset{\mathrm{m}}{\sim} \mathcal{A} \rtimes_{lt,r} G,$$

with $\mathcal{A} = \left\{[h_{ij}] \in M_k(C_0(F_{V_\psi f})), h_{ij} \in C_0(F_{V_\psi f_i} \cap F_{V_\psi f_j})\right\}$ a noncommutative C^*-algebra.

In this situation, we consider the signal $f \in \mathcal{H}$, as an element of a Hilbert space $\mathcal{H}$, and its time-frequency representation is described abstractly as a voice transform $V_\psi f$. This concept generalizes many time-frequency operations (wavelets, Gabor analysis, etc) using $V_\psi f : \mathcal{G} \to \mathbb{C}$ for $V_\psi f(x) = \langle f, \pi(x)(\psi)\rangle$, with a group representation $\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, and $\psi \in \mathcal{H}$. The functional cloud $M^G_{V_\psi f} = F_{V_\psi f}/G$ for $F_{V_\psi f} = \mathrm{graph}(V_\psi f|_{\mathrm{supp} V_\psi f})$, is constructed with a groupoid G acting on $F_{V_\psi f}$, and we are interested in the analysis of the C^*-algebra $C_0(M^G_{V_\psi f})$, the space of continuous functions vanishing at infinity on $M^G_{V_\psi f}$. As it turns out, $C_0(M^G_{V_\psi f})$ is Morita equivalent (denoted by $\overset{m}{\sim}$) to $\mathcal{A} \rtimes_{lt,r} G$. The C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$ is a crossed product encoding the groupoid C^*-dynamical system represented by the action of the groupoid G on $C_0(F_{V_\psi f})$. A great advantage of the noncommutative C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$, is that it expose and reveals information on the time-frequency dynamics of the mixing process $f = \sum_{i=1}^k f_i$, contrary to the C^*-algebra $C_0(M^G_{V_\psi f})$ which completely ignores this information. Here, the crucial noncommutativity structure of $\mathcal{A} \rtimes_{lt,r} G$ reveals also how to understand the time-frequency interferences between the different signal components f_i, $i = 1, \ldots, k$. Our results are based on recent developments in operator algebras and groupoid theory. In particular, we use a recent generalization by J.H. Brown [5, 6] of the work of P. Green [29] and M. Rieffel [51], together with the Renault's equivalence theorem as explained by P.S. Muhly, J. Renault, and D. Williams in [45, 46]. Additionally, a description of noncommutative C^*-algebras related to open coverings of a manifold, as explained by A. Connes in [12, 13], plays a basic role in our setting.

A second main contribution of our work is inspired by the need to implement and apply, in computationally feasible algorithms, the concepts we have just developed for signal analysis via C^*-algebras. Here, our proposal (Section 3.2) is to use the framework of persistent homology, designed to analyze topological properties of finite datasets $X = \{x_i\}_{i=1}^m$. For this task, we use AF-algebras as an important family of C^*-algebras, particularly useful for studying finite structures, as required in applications of signal processing and data analysis. The core idea of our proposal is to construct an AF-algebra for each simplicial complex present in a filtration arising in the persistent homology algorithm. Here, we follow the large body of work prepared, in the setting of noncommutative geometry, on the analysis of AF-algebras, poset structures and Bratteli diagrams, as explained by G. Landi and his collaborators [3, 20, 19, 37]. The basic question is to investigate the feasibility of combining these tools with the framework of persistent homology for the analysis of geometrical and topological features of finite datasets. We remark that posets are truly noncommutative spaces and, therefore, the noncommutative features play again an important role in our setting.

The outline of this paper is as follows. In Sections 1.1, and 1.2 we present the main motivations of our work, where we discuss the importance of combining dimensionality reduction methods with signal transforms. In Section 2.1 we present the basic concepts of our framework, together with several examples motivating our definitions. Due to the relative lack of prevalence of groupoid theory in signal processing, we present an overview of these concepts in Section 2.2. In Section 2.3 we present our basic results illustrating the usage of C^*-algebra structures in signal analysis. In Section 3 we describe basic ideas on persistent homology, together with our proposal of integrating AF-algebras technology in this setting. Finally, in Section 4, we discuss a toy example illustrating a (very) limited set of features of our theoretical developments.

1.1 Motivations and Objectives

A basic motivation in our environment are time-frequency representations where a signal f is analyzed by considering a partition in chunks $x_b = fg_b$, using $g_b(t) = g(t-b)$ for a window function g, This is typical scenario in the short term Fourier transform (STFT), where a signal $f \in L^2(\mathbb{R})$ is analyzed using

$$V_g f(b,\omega) = \langle f, g_{b,\omega}\rangle = \int f(t)\overline{g_{b,\omega}(t)}\,dt \quad \text{where} \quad g_{b,\omega}(t) = g(t-b)e^{2\pi i\omega t}.$$

In the last decades, this fundamental procedure has been generalized to a large framework including wavelet theory and modern discrete methods of frame decompositions. However, despite the voluminous research activity in this area, many signals in modern applications fields remain difficult to analyze. Just to mention one example, an accurate separation or characterization of polyphonic acoustic signals in speech or music analysis still remains a very challenging task. Part of the problem lies in the fact that it is still difficult to cleanly characterize (e.g. with a few amount of wavelet coefficients) many realistic signals with modern frame decomposition methods. In our setting, we consider each signal (or family of signals) as en entity that can be analyzed with a combination of standard time-frequency transforms with geometrical and topological invariants.

The procedure we follow is to consider the dataset of chunks X_f, constructed with the time-frequency segmentation, as a main object of study. For instance, in the case of finite signals with have a finite set of the form

$$X_f = \{x_i\}_{i=1}^m \qquad \text{for} \qquad x_i = (f(t_{k(i-1)+j}))_{j=0}^{n-1} \in \mathbb{R}^n$$

for $k \in \mathbb{N}$ being a fixed *hop-size*. Here, the regular sampling grid $\{t_\ell\}_{\ell=0}^{km-k+n-1} \subset [0,1]$ is constructed with considering the Nyquist-Shannon theorem for f. Notice that X_f may be embedded in a very high-dimensional ambient space $\mathbb{R}^n$, even though the dimension of X_f itself may be small. For instance, in audio analysis, for 44kHz signals, $n = 1024$ is commonly used, and therefore, in the case of signals whose time-frequency representations are not sparse, the usage of dimensionality reduction methods could be of interest. With this particular scheme, the STFT of f can be interpreted as a transformation of the set X_f by taking the (windowed) Fourier transform of each x_i.

A second family of examples (similar in spirit to time-frequency analysis) arises in image processing. One strategy would be to consider a dataset $X_f = \{x_i\}_{i=1}^m$ constructed from a grayscale image $f : [0,1]^2 \to [0,1]$, along with a finite covering of small squares (each containing n pixels) $\{O_i \subset [0,1]^2\}_{i=1}^m$, centered at pixels positions $\{k_i\}_{i=1}^m \subset [0,1]^2$. As in the previous situation, when considering band-limited images, the domain $[0,1]^2$ can be sampled uniformly and the dataset can then be defined as

$$X_f = \{f(O_i) \in \mathbb{R}^n\}_{i=1}^m,$$

where n is the size of the squares O_i, and m denotes the number of pixels k_i. As before, our aim is to analyze the geometry of the image data X_f to gain useful information about the properties of the image f. For instance, we consider a grayscale image $f : [0,1]^2 \to [0,1]$ using a covering with small squares, or patches, $\{O_i \subset [0,1]^2\}_{i=1}^m$, each containing n pixels. In this toy example we assume that the corresponding point cloud data $X_f = \{f(O_i) \in \mathbb{R}^n\}_{i=1}^m$ lies in some manifold $\mathcal{M} \subset \mathbb{R}^n$. If an image f is composed of an homogeneous texture, the dataset X_f is a cluster whose elements have similar geometrical characteristics. In a simplified scenario, the idea would be to use a representative patch $\phi \in \mathbb{R}^n$ in order to generate all elements of X_f. The main task is to find a family of transformations (the *modulation maps*) $s(\alpha) : \mathbb{R}^n \to \mathbb{R}^n$, parametrized by a low dimensional space Ω, such that for any patch $y \in X_f$, there is some $\alpha \in \Omega$ with $y = s(\alpha)\phi$. We remark that several methods in image processing have recently been proposed with a loosely related philosophy (see e.g. the *patch-based texture analysis* as part of classical *texture synthesis* methods [36]).

1.2 Dimensionality Reduction and Signal Analysis

In dimensionality reduction [38], we study a *point cloud data* defined as a finite family of vectors $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ embedded in an n-dimensional Euclidean space. The fundamental assumption is that X lies in $\mathcal{M}$, a (low dimensional) *space* (manifold or topological space i.e. CW-complex, simplicial complex) embedded in $\mathbb{R}^n$. We have therefore, $X \subset \mathcal{M} \subset \mathbb{R}^n$ with $p := \dim(\mathcal{M}) \ll n$. An additional key concept is the consideration of a *ideal model* representing $\mathcal{M}$, and denoted by Ω, embedded in a low dimensional space $\mathbb{R}^d$ (with $d < n$), together with a homeomorphism (diffeomorphism) $\mathcal{A} : \mathbb{R}^d \supset \Omega \to \mathcal{M} \subset \mathbb{R}^n$. The space Ω represents an ideal representation of $\mathcal{M}$ that could be used for analysis procedures in a low-dimensional environment. For instance, in the case of $\mathcal{M}$ being the well-know Swiss roll dataset, the space Ω is a rectangle. However, in practice, we can only try to approximate Ω with a dimensionality reduction map $P : \mathbb{R}^n \supset \mathcal{M} \to \Omega' \subset \mathbb{R}^d$, where Ω' is an homeomorphic copy of Ω.

Now we discuss the interactions of dimensionality reduction tools with signal transformations. A basic characteristic of short term Fourier analysis is the high dimensionality of the Euclidean space where the time-frequency data is embedded. In this context, for many applications, a combination with dimensionality reduction methods could be useful for improving the quality of the data analysis. Our motivation examples in time-frequency analysis can be naturally related to the dimensionality reduction framework by considering X_f to be a subset of $\mathcal{M}$, a (low dimensional) space, embedded in the high dimensional Euclidean space $\mathbb{R}^n$. We have therefore, $X_f \subset \mathcal{M} \subset \mathbb{R}^n$ with $p := \dim(\mathcal{M}) \ll n$. We recall that there is a well-known framework for studying properties of sets X_f in the context of nonlinear time series and dynamical systems (see e.g. [34]). But in our situation, we are additionally considering a close interaction with signal processing transforms T, together with dimensionality reduction techniques P (Principal component analysis, Isomap, LTSA, etc). The construction of time-frequency data can be described as the application of a map $T : \mathcal{M} \supset X_f \to T(X_f) \subset \mathcal{M}_T$, where $\mathcal{M}_T := T(\mathcal{M})$, and $T(x_i)$ is the signal transformation of x_i (Fourier transform, wavelet, etc). The following diagram shows the basic situation:

$$\begin{array}{ccc}
\mathbb{R}^d \supset \Omega & \xrightarrow{\mathcal{A}} & \mathcal{M} \supset X_f \subset \mathbb{R}^n \\
 & & \Big\downarrow T \\
\mathbb{R}^d \supset \Omega' & \xleftarrow[P]{} & \mathcal{M}_T \supset T(X_f) \subset \mathbb{R}^n
\end{array}$$

2 Functional Clouds and C^*-Algebras

Our objective is to design tools for signal processing using properties of spaces constructed from a signal, in the same spirit as the construction of time-frequency data. As we will also shortly explain, a similar construction occur also in the setting of nonlinear time series and the Taken's theorem. We present in this section basic definitions, where a core idea is to study the segmentation of a signal f, as classically performed in wavelet or short term Fourier transforms. Two basic concepts are the notions of a functional cloud M_f of a signal f, and its related foliated partition F_f. The information encoded in these structures contains the interplay between local and global properties of f, and our plan is to use geometrical and topological tools for their analysis. An additional notion of a modulation map provides the interaction with the dimensionality reduction and manifold learning framework. Some basic properties of a functional cloud and a foliated partition are discussed in Section 2.3, where we study their topology by applying elementary notions of C^*-algebras and their K-theory. For instance, in Proposition 2.3 we consider the case of a signal decomposition $f = \sum f_i$, and a simplified scenario illustrating the topological interaction between the spaces M_f, $\{M_{f_i}\}_i$, and F_f.

2.1 Motivating and Defining Functional Clouds

Time-frequency transforms are fundamental tools in modern developments of harmonic analysis. An important task in this field is to construct adequate strategies for decomposing a function in order to study their time-frequency behavior. The basic procedure is to split a signal f in consecutive segments (sometimes denominated patches or chunks) that can be used to perform a global analysis of the function f. We now introduce an abstraction of these ideas by denoting (for lack of better names), a *functional cloud* and a *foliated partition* as our basic objects of study. Given a real function in a locally compact group, we construct a functional cloud as a quotient space of a foliated partition with an adequate equivalence relation. We will also generalize the equivalence relation to a more powerful concept of groupoid in order to use the rich theory readily available in this field, as well as preparing the terrain for new potential application problems.

Definition 2.1 (Functional cloud and foliated partition of a function)**.** Given a locally compact group $\mathcal{G}$, and a continuous function $\mathcal{F} : \mathcal{G} \to \mathbb{C}$, we define the *functional cloud* for $\mathcal{F}$ and a measurable compact set A with $0 \in A \subset \mathcal{G}$, as the set $M_{\mathcal{F},A} \subset L^1(A)$ with

$$M_{\mathcal{F},A} := \bigcup_{x \in \mathcal{G}} \{\mathcal{F}_x : A \to \mathbb{C}\},$$

for $\mathcal{F}_x(y) := \mathcal{F}(x + y)$, $\forall\, y \in A$. The related *foliated partition* is defined as the corresponding disjoint union:

$$F_{\mathcal{F},A} := \bigsqcup_{x \in \mathcal{G}} \{\mathcal{F}_x : A \to \mathbb{C}\}.$$

We will abuse the notation, and we use also $F_{\mathcal{F},A}$ for the disjoint union of the graphs of the functions $\mathcal{F}_x$:

$$F_{\mathcal{F},A} := \bigsqcup_{x \in \mathcal{G}} F_x \subset \mathcal{G} \times A \times \mathbb{C}, \quad F_x := \mathrm{graph}(\mathcal{F}_x) = \{(y, \mathcal{F}_x(y)), y \in A\} \subset A \times \mathbb{C}.$$

Remark 2.1 (Notational comments)**.** Note that in the set $M_{\mathcal{F},A}$, no repeated elements are taken into account. Namely, only one representative $\mathcal{F}_x$ is used for different elements $x_1 \neq x_2$ where $\mathcal{F}_{x_1}(y) = \mathcal{F}_{x_2}(y)$, $\forall\, y \in A$. In contrast, in the disjoint union $F_{\mathcal{F},A}$ we keep track of repeated elements $\mathcal{F}_x$ for different values $x \in \mathcal{G}$. An obvious interaction can be established using an adequate equivalence relation R and a quotient space $M_{\mathcal{F},A} = F_{\mathcal{F},A}/R$. As we will see later on, our main object of study is an important generalization of this construction using $M^G_{\mathcal{F},A} = F_{\mathcal{F},A}/G$ for a groupoid G acting on $F_{\mathcal{F},A}$. To avoid clutter the notation, we denote $M_{\mathcal{F},A}$ by $M_{\mathcal{F}}$ and $F_{\mathcal{F},A}$ by $F_{\mathcal{F}}$ when no confusion arises. We also remark that we use both notations $X_{\mathcal{F}}$ and $M_{\mathcal{F}}$ to denote a functional cloud of $\mathcal{F}$, but $X_{\mathcal{F}}$ is mostly preferred for the case of finite signals.

Remark 2.2 (Motivations)**.** The idea of functional cloud captures a basic time-frequency strategy by segmenting a function $f \in L^2(\mathbb{R})$ in chunks f_x, constructed with the translations $A_x = \{x + y, y \in A\}$ of a set A that can be defined as the support of a given window function. For instance, the standard wavelet procedure for computing the product $\langle f, \psi_{a,b}\rangle$, with a wavelet ψ, can be considered as a local analysis of f in a region defined by $\psi_{a,b}$. Remember that the region of influence of $\psi_{a,b}$ is defined by the scale a and the translation factor b. In the concept of a functional cloud of f, the set A can be related to the support of ψ, but it also plays a generalization role for the scale factor a. The objective is to consider M_f as a set which encodes the local behavior of f (using the set A as a measuring tool) and whose geometrical properties are of interest.

Notice that in this particular concept, we do not take into account different resolution scales, as a single set A is used for constructing M_f. Currently, our main focus is to study a single scale level, and we leave for future work the analysis of geometrical and topological interactions between clouds $M_{f,A}$ with different scales A.

The conceptual motivation of a functional cloud is also in close relation to the concepts of a phase space and attractors in dynamical systems, as seen in the setting of the Taken's theorem (Example 2.5). Here, we use the term functional cloud in order to stress the relation to the concept of a point cloud data as used in dimensionality reduction. In the standard philosophy of time series analysis, as seen in dynamical systems, the interactions with classical Fourier analysis are usually not that taken into consideration. Here, we want to consider situations where a combination of these techniques could be of interest. We now present a family of examples where the concept of a cloud plays a significant role.

Example 2.1 (Cloud of a discrete 1d signal)**.** A typical example of a finite functional cloud is the point cloud dataset constructed by drawing samples from a signal f. More precisely, we consider, for a bandlimited signal $f \in L^2(\mathbb{R})$, the set of signal patches $X_f = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$, $x_i = (f(t_{k(i-1)+j}))_{j=0}^{n-1} \in \mathbb{R}^n$. The construction of this cloud X_f involves the set of integers $A = \{0, \ldots, n-1\}$, and the identification $C(A, \mathbb{R}) = \mathbb{R}^n$, and we can see A as a subset indexing the values ℓ. Here, the regular sampling grid $\{t_\ell\}_{\ell=0}^{km-k+n-1}$ is constructed when considering the Nyquist-Shannon theorem for f.

Example 2.2 (Cloud of an Image)**.** A straightforward generalization of the previous example applies for the case of an image $f : [0,1]^2 \to [0,1]$. Here, the cloud M_f depends on a set of pixels represented by a representative patch A. In this particular example, the usefulness of a topological analysis of M_f lies, for instance, in the study of its connectivity (e.g. clustering aspects) for estimating qualitative as well as quantitative differences between different regions in the image f.

Example 2.3 (Cloud of a sinusoid)**.** Let $f : \mathbb{T} \to \mathbb{R}$ with $f(x) = \sin(x), x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and let $A = \{x \mod (2\pi), -\epsilon < x < \epsilon\}$. If $0 < \epsilon < 2\pi$ the functional cloud M_f is homeomorphic to a circle. This can be seen by considering the map $\phi : M_f \to \mathbb{T}$, $\phi(f_x) = x$. Notice that $f_x(z) - f_{x+\delta}(z) = \delta \cos(\xi)$, for $\xi \in]z + x, z + x + \delta[$, and we can ensure that ϕ is continuous, with the uniform norm in the space $C(A, \mathbb{R})$ using the estimation $\|f_x - f_{x+\delta}\|_\infty \leq |\delta|$. When considering the inverse function $\phi^{-1}(x) = f_x$, we can use the computation $f_{x+\delta}(z) - f_x(z) = f(z + x + \delta) - f(z + x)$ for $x - \epsilon < z < x + \epsilon$, and as f is continuous, we have $\lim_{\delta\to 0} f_{x+\delta}(z) = f_x(z)$, and as f_x is continuous we have ϕ^{-1} continuous. Finally, notice that ϕ is bijective only for the cases $0 < \epsilon < 2\pi$.

Example 2.4 (Cloud of a modulated path)**.** Consider an embedded manifold $\Omega \subset [-1, 1]^d$, and a path $\phi : [0, 1] \to \Omega$. We construct $f : [0, 1] \to \mathbb{R}$ as

$$f(x) = \sum_{i=1}^{d} \sin\Big(\int_0^x (\alpha_c^i + \gamma\phi_i(t))\, dt\Big),$$

such that the entries of the vector $\alpha_c = (\alpha_c^i)_{i=1}^d \in \mathbb{R}^d$ are center frequencies where a number of d frequency bands are located, and the value γ plays the role of a *bandwidth parameter*. The notation $\phi_i(x)$ refers to the i-th coordinate of $\phi(x)$. A basic question in this example is, given an open set A, what are the conditions for the path ϕ such that the functional cloud $M_f \subset C(A, \mathbb{R})$ "approximates" an homeomorphic copy of Ω. Here, the meaning of approximation can be considered, for instance, in the context of persistent homology. If the functional cloud of f is a path in $C(A, \mathbb{R})$ the question would be, given an adequate finite sampling of M_f, whether its persistent homology corresponds to the persistent homology of a finite sampling of Ω.

Another point of view for comparing the homology of Ω with a finite sampling of M_f can be expressed with conditions for approximating the homology of submanifolds with high confidence from random samples (see the work of Niyogi, Smale and Weinberger [47]). For this setting, the question would be to ask if a given finite sampling of M_f fulfills the conditions in [47] for reconstructing the homology of an homeomorphic copy of Ω in $C(A, \mathbb{R})$.

Example 2.5 (Time series and dynamical systems)**.** In dynamical systems and time series analysis, a similar segmentation procedure for a signal is implemented as in the functional cloud concept. In the framework of the celebrated Taken's theorem [52], we consider a dynamical system $\phi \in \mathrm{Diff}(M)$, defined as a diffeomorphism ϕ of a manifold M, a smooth function $h : M \to \mathbb{R}$, and a *delay coordinate map* $F(h, \phi) : M \to \mathbb{R}^n$, constructed by drawing consecutive samples from a time series:

$$F(h, \phi)(x) := (h(x), h(\phi(x)), h(\phi^2(x)), \ldots, h(\phi^{n-1}(x))), \quad x \in M.$$

The main result is that under suitable conditions, the map $F(h, \phi)$ is an embedding. Given a signal $f(k) = h(\phi^k(x)), k \in \mathbb{Z}$, the Taken's theorem provides a conceptual framework that justifies the estimation of geometrical and topological properties of M using the signal f. The key notion in this construction turns out to be a functional cloud $M_{f,A}$ for A an n-dimensional set.

Example 2.6 (Voice transform, wavelets and short term Fourier analysis)**.** In time-frequency analysis and voice transforms, a crucial component is the way a locally compact group $\mathcal{G}$ acts in a Hilbert space of functions $\mathcal{H}$. This action is an irreducible and unitary group representation, $\pi : \mathcal{G} \to \mathcal{U}(\mathcal{H})$, (for $\mathcal{U}(\mathcal{H})$ unitary operators in $\mathcal{H}$) that fulfills square integrable conditions. With this representation the *voice transform* is constructed as [10, Formula 17.3, p387]:

$$V_\psi : \mathcal{H} \to C(\mathcal{G}) \quad \text{with} \quad V_\psi f(x) = \langle f, \pi(x)(\psi)\rangle, \; f, \psi \in \mathcal{H}, x \in \mathcal{G}.$$

The short term Fourier transform (STFT) and wavelet transforms are typical examples where ψ corresponds to a window function for the former and to a wavelet for the latter. These transforms represent an interplay between $\mathcal{H}$ and $C(\mathcal{G})$, that allows to analyze the function f, by porting its information to a setting defined by $\mathcal{G}$. Another way to rephrase this procedure is that the transformation V_ψ "unfolds" data present in f, using $\mathcal{G}$ as an analysis environment. In the case of the STFT transform, the Weyl-Heisenberg group represents the time-frequency background to which information from f is translated. In the case of the wavelet transform, the affine group provides a time-scale representation of a function. In these situations, a fundamental objective is to understand the components of f, using $\mathcal{G}$ and V_ψ as observation tools. Discretization aspects of these transforms have been designed in the setting of coorbit theory and frames of a Hilbert space $\mathcal{H}$ [22, 23, 24]. Recall that a frame of $\mathcal{H}$ is a collection of vectors $F = \{f_i\}_{i\in I} \subset \mathcal{H}$, such that there exist two constants $0 < A \le B$, with $A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2$, for any $f \in \mathcal{H}$.

In the particular case of Gabor analysis, the voice transform is defined as

$$V_g f(b, w) = \int_{\mathbb{R}} f(t)g(t-b)e^{-2\pi i t w} dt,$$

$f \in L^2(\mathbb{R})$ and we can define two basic functional clouds in this setting. The cloud $X_f = \{x_b\}_b$ is defined with chunks $x_b = f g_b$ by splitting the function f, and using $g_b(t) = g(t-b)$ for a window function g. We can also define a cloud of the corresponding spectral view using

$$X_{V_g f} = \{V_g f(b, .)\}_b.$$

Due to the orthogonality property of the Fourier transform, the geometrical and topological properties of $X_f = \{x_b\}_b$, are the same as the ones of the set $X_{V_g f} = \{V_g f(b, .)\}_b$ (i.e. X_f and $X_{V_g f}$ are isometric spaces). But it is crucial to notice that highly nontrivial geometrical and topological changes can occur in $X_{V_g f}$ when applying time-frequency operations to the function f (e.g. filters and convolution operations). The interplay between the geometry of $X_{V_g f}$ and the time-frequency properties of f is a main topic in our research (a toy-example of this interaction will be discussed in Section 4, and can also be found in [31] and its corresponding simulation).

Studying M_f with F_f

In order to study the functional cloud M_f, the foliated partition F_f is used for estimating its geometrical and topological properties. The motivation for this strategy is based on standard procedures of noncommutative geometry. A prototypical situation is to study the geometry of a quotient space $X = Y/\sim$ using a C^*-algebra constructed with the equivalence relation $\sim$. The relation between a foliated partition and a function cloud can be interpreted as quotient space using an equivalence relation (and more generally a groupoid). These conceptual interactions will be the main topic we will investigate.

The basic idea of a foliated partition F_f is to study each segment f_x of the function f, by keeping track their relationship with respect to the parameter $x \in \mathcal{G}$. This is in contrast with the construction of a functional cloud M_f, where we study the interactions between the vectors f_x, irrespectively of their positions x. Basic standard procedures in time-frequency analysis are related to this concept. For instance, when studying the time-frequency representation of a signal f, an important objective is to keep track to the time evolution of different frequency components of f. For example, the topic of *partial tracking* is a classical signal processing task which keeps track of harmonic information in a signal. Speech analysis is a typical example, where vocal information is represented in the time-frequency plane by varying harmonical components. The time-frequency data in this context is obtained by considering the Fourier transform of each segment f_x in the foliated partition. We now describe an important particular scenario that justifies the terminology *foliated partition*.

Proposition 2.1 (Foliated partition as a foliated manifold)**.** *Let F_f be a foliated partition for a continuous function $f : \mathcal{G} \to \mathbb{R}$, where $\mathcal{G}$ is a finite dimensional Hilbert space, and A an open set with $0 \in A \subset \mathcal{G}$. The set F_f is a manifold of dimension $2\dim(\mathcal{G})$, and it has a foliation structure of dimension $\dim(\mathcal{G})$.*

Proof. This easily follows by considering the graph of the function $\phi : \mathcal{G} \times A \to \mathbb{R}$, $\phi(x, y) = f_x(y)$. As f is continuous and $f_x(y) = f(x + y)$, ϕ is a continuous function. The graph of ϕ can be identified with $\{(x, y, f_x(y), (x, y) \in \mathcal{G} \times A\}$, and therefore it is also identified with the foliated partition F_f. The map ϕ exhibits a chart from the open set $\mathcal{G} \times A$ to F_f, which actually relates these two sets homeomorphically, and we obtain a manifold structure for F_f. As $\mathcal{G} \times A$ is an open subset of $\mathcal{G} \times \mathcal{G}$, the dimension of F_f is therefore equal to $2\dim(A)$. The foliation structure of F_f is a straightforward consequence of its construction, where the leaves are given by F_x, $x \in \mathcal{G}$, and their dimension is $\dim(\mathcal{G})$. ■

Example 2.7 (Foliated partition for the sinusoidal example)**.** In our previous Example 2.3 of the function $f(x) = \sin(x), x \in \mathbb{T} = \mathbb{R}/\mathbb{Z}$ with $A = \{x \mod (2\pi), -\epsilon < x < \epsilon\}$, the foliated partition F_f can be identified with a surface ($\dim(F_f) = 2$) described by the graph of the function $\phi : \mathbb{T}\times]-\epsilon, \epsilon[\to \mathbb{R}$, $\phi(x, y) = \sin(x + y)$, and for the corresponding leaves we have $\dim(F_x) = 1$.

Remark 2.3 (Relating F_f and M_f)**.** There is an obvious relation between a foliated partition F_f and a functional cloud M_f. If we define an equivalence relation in F_f, as $R = \{(u, v) \in F_f \times F_f, d(p(u), p(v)) = 0\}$, for the projection map $p : F_f \to M_f$, $p((x, y, f_x(y)) = f_x$, and d is the metric induced by the uniform norm in $C(A, \mathbb{R})$, we have an identification (as sets) between F_f/R and M_f. This remark has important implications when using the foliated partition F_f for studying the geometry and topology of M_f. Indeed, the relations between a space X and a quotient X/R for an equivalence relation R (or more generally with a groupoid G) is an important source of examples in the noncommutative geometry world. In this field, there is a very important machinery for studying quotients X/R using C^*-algebras defined on the spaces X and R. This framework provides important tools for studying pathological quotients, sometimes called "bad quotients" (e.g. X/R non Hausdorff while X being Hausdorff). A prototypical example is the noncommutative torus, defined as the quotient of the 2-torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ with the Kronecker foliation constructed from the differential equation $dy = \theta dx$ [12]. But it is very important to remark that the tools from noncommutative geometry are also useful for studying objects in the commutative world.

Before describing the application of this strategy in our setting of foliated partitions and functional clouds, we describe an important related concept which makes more explicit the interaction of these problems with modern tools from manifold learning and dimensionality reduction.

Definition 2.2 (Modulation Maps [32, 30, 31, 33])**.** Let $\{\phi_k\}_{k=1}^d \subset \mathcal{H}$ be a set of vectors in an Euclidean space $\mathcal{H}$, and $\{s_k : \Omega \to \mathcal{C}_{\mathcal{H}}(\mathcal{H})\}_{k=1}^d$ a family of smooth maps from a space Ω to $\mathcal{C}_{\mathcal{H}}(\mathcal{H})$ (the continuous functions from $\mathcal{H}$ into $\mathcal{H}$). We say that $\mathcal{M} \subset \mathcal{H}$ is a $\{\phi_k\}_{k=1}^d$*-modulated space* if

$$\mathcal{M} = \left\{ \sum_{k=1}^{d} s_k(\alpha)\phi_k,\ \alpha \in \Omega \right\}.$$

In this case, the map $\mathcal{A} : \Omega \to \mathcal{M}$, $\alpha \mapsto \sum_{k=1}^d s_k(\alpha)\phi_k$, is denoted *modulation map*.

The concept of a *modulation map*, summarizes the well-known concept of modulation in signal processing, using a geometrical and topological language. The fundamental objective of a modulation map is to construct spaces $\mathcal{M}$ using generating functions $\{\phi_k\}$ and a parametrization space Ω. This concept is a related, but different component in the machinery of a functional cloud and foliated partitions. An explicit example of this concept is given by a *frequency modulation map*, which considers $\phi(t) = \sin(t)$ and a modulation with the coordinates of points in some manifold Ω [32, 30, 31]. Our previous example of the cloud of a sinusoid also fits in this setting.

Example 2.8 (Cloud of a sinusoid as a modulated space)**.** We use again our previous example for the cloud of a sinusoid to show a modulated space with $f(x) = \sin(x), x \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ and $B =]-\epsilon, \epsilon[\mod (\mathbb{T})$. Here, the map ϕ is a modulation map when $0 < \epsilon < 2\pi$, and for this case, the functional cloud M_f is homeomorphic to a circle.

2.2 Basics on Groupoids Crossed Products and C^*-Dynamical Systems

In noncommutative geometry, the fundamental interplay between locally compact Hausdorff topological spaces and (commutative) C^*-algebras as explained in the Gelfand-Naimark theory [25, 54] has been extended to an important framework using noncommutative C^*-algebras [37, 28, 12]. The multiple conceptual and application developments over the last decades is a evident indication of its breath and increasing importance in mathematics. A basic example of a noncommutative space, is the noncommutative torus [12, 13], which can be defined as a crossed product C^*-algebra $\mathcal{A}_\theta = C^1(S^1) \rtimes_{R_\theta} \mathbb{Z}$, for a rotation $R_\theta(x) = (x+\theta) \operatorname{mod} 1$, $x \in \mathbb{T}$. In this construction, the algebra $C^1(S^1)$ can be replaced with other algebras depending on the type of analysis and resolution required: for geometrical, topological and measure theoretical properties, the spaces that can be used are $C^\infty(S^1) \subset C^1(S^1) \subset L^1(S^1)$, respectively. The noncommutative torus is just one particular example in the world of noncommutative geometry, and it belongs to the general theory of spaces of leaves of foliations. But an even more general setting can be described with the powerful theory of groupoids. The important application of groupoids in noncommutative geometry is given by the concept of noncommutative quotients, and a particular example is the analysis of quotient spaces $X = Y/\sim$ of an equivalence relation $\sim$ in Y. We remark that there is an important family of C^*-algebras (the AF-algebras) particularly useful for studying finite structures, as required in applications of signal processing and data analysis. We now introduce some basic tools from groupoid theory we need for our setting.

Groupoids C^*-algebras

A groupoid G can be defined as a small category where each morphism has an inverse [12, 15]. More explicitly, we say that a groupoid over a set X is a set G together with two maps $r, s : G \to X$, called the *range and source maps*, and a composition law (or product) $\circ : G^{(2)} \to G$ denoted $\gamma \circ \eta = \gamma\eta$, where

$$G^{(2)} = \{(\gamma, \eta) \in G \times G, r(\gamma) = s(\eta)\},$$

and $r(\gamma\eta) = r(\gamma), s(\gamma\eta) = s(\eta)$, $\gamma(\eta\xi) = (\gamma\eta)\xi$. We additionally have an embedding $e : X \to G$ and an inversion map $i : G \to G$ with $e(r(\gamma)) = \gamma = \gamma e(s(\gamma))$, and $i(\gamma)\gamma = e(s(\gamma))$, $\gamma i(\gamma) = e(r(\gamma))$. We have an hierarchy of sets defined as $G^{(0)} = e(X) \simeq X$ (the *unit space*), $G^{(1)} = G$, and $G^{(2)}$ as previously defined. For $u \in G^{(0)}$ we define $G_u = s^{-1}(u)$ and $G^u = r^{-1}(u)$.

An alternative way to introduce a groupoid is to start with a subset $G^{(2)}$ of $G \times G$ as the set of composable pairs, an inverse operation $G \to G, \gamma \mapsto \gamma^{-1}$ for each $\gamma \in G$, and define the maps r and s with $r(\gamma) = \gamma\gamma^{-1}$, $s(\gamma) = \gamma^{-1}\gamma$. From the axioms, the maps r, s have a common image $G^{(0)}$ as the unit space, meaning that $\gamma s(\gamma) = r(\gamma)\gamma = \gamma$, for each $\gamma \in G$ [7, Section 2.1].

The *isotropy group* for a unit $u \in X$ is defined as

$$G_u^u = \{\gamma \in G, s(\gamma) = r(\gamma) = u\} = s^{-1}(u) \cap r^{-1}(u),$$

and in general, we define $G_v^u = r^{-1}(u) \cap s^{-1}(v)$. The *isotropy group bundle* is defined as $G' = \{\gamma \in G, s(\gamma) = r(\gamma)\}$. When the groupoid is seen as a category, the set of objects is $\operatorname{Ob}(G) = G^{(0)}$, and the morphisms are identified with G itself [35, Definition 2.1].

An *homomorphism of groupoids* G and Γ is a map $\phi : G \to \Gamma$, such that $(\gamma, \eta) \in G^{(2)}$, then $(\phi(\gamma), \phi(\eta)) \in \Gamma^{(2)}$, and $\phi(xy) = \phi(x)\phi(y)$. We have, in particular, $\phi(\gamma^{-1}) = (\phi(\gamma))^{-1}$, and $\phi(G^{(0)}) \subset \Gamma^{(0)}$ [7, p73].

We will consider groupoids where both $G^{(0)}$ and $G^{(1)}$ have topologies such that the maps $(\gamma, \eta) \mapsto \gamma\eta$ from $G^{(2)} \to G$, and $\gamma \mapsto \gamma^{-1}$ from G to G are continuous [26, Definition 1.8]. In the following, we denote by G a second countable locally compact Hausdorff groupoid.

Important examples of groupoids are equivalence relations, groups, and group actions. For instance, with an equivalence relation $R \subset X \times X$, we define a groupoid $G = G^{(1)} = R$, $G^{(0)} = X$, $r(x, y) = x, s(x, y) = y$. For a group Γ, we can define the groupoid $G = \Gamma$, $G^{(0)} = \{e\}$ (the unit of Γ), and the groupoid composition is the group product. For a group Γ acting on a set X, we can define $G = X \times \Gamma$, $G^{(0)} = X$, $r(x, g) := x, s(x, g) := xg$, for all $(x, g) \in X \times \Gamma$, and the product is defined as $(x, g)(xg, h) = (x, gh)$. Another important example of groupoid is a *group bundle* defined as a disjoint union of groups $\{\Gamma_i\}_{i \in U}$ indexed by a set U. The composition between two elements is defined by the corresponding group composition if the elements are in the same group. A groupoid is a group bundle if $d(x) = r(x)$ for all $x \in G$, and for this case, the groupoid G equals its isotropy group bundle G' (see [7, 2.3 p76]).

Definition 2.3 (Haar Systems for Groupoids [7, Section 2.4])**.** The concept of a Haar system generalizes, to groupoids, the notion of a Haar measure for locally compact groups. A Haar system is family $\{\lambda^u\}_{u \in G^{(0)}}$ of Radon Measures on G with $\text{supp}(\lambda^u) = G^u$, and $u \mapsto \int f(\gamma) d\lambda^u(\gamma)$ is a continuous function from $G^{(0)}$ to $\mathbb{C}$ for all $f \in C_c(G)$ (the space of complex-valued continuous functions with compact support). Additionally, we require

$$\int f(\eta) d\lambda^{r(\gamma)}(\eta) = \int f(\gamma\eta) d\lambda^{s(\gamma)}(\eta),$$

for all $f \in C_c(G)$, and all $\gamma \in G$.

Remark 2.4 (Hilbert bundles and direct integrals)**.** In groupoid theory and groupoid representations, the concept of Hilbert bundles has the same fundamental role as the concept of Hilbert space in group representations [7, Section 2.6]. A Hilbert bundle is constructed with a family of Hilbert spaces $\mathcal{H} = \{H(x)\}_{x \in X}$ indexed by X, which can be more precisely denoted as a disjoint family $X * \mathcal{H} := \{(x, \chi), \chi \in H(x)\}$ (see [44, Chapter 3]). In general, for X_1, X_2, two spaces with maps $\tau_i : X_i \to T, i = 1, 2$, one defines [5]

$$X_1 * X_2 := \{(x, y) \in X_1 \times X_2 : \tau_1(x) = \tau_2(y)\}.$$

If X is an analytic Borel space, then $X * \mathcal{H}$ is denominated a *analytic Borel Hilbert bundle* with the natural projection $\pi : X * \mathcal{H} \to X$, and the corresponding set of Borel sections is denoted as $B(X * \mathcal{H})$ [5, Definition 3.61 p109].

The concept of a *direct integral* of the spaces $\{H(x)\}_{x \in X}$ is defined for an analytic Borel Hilbert bundle $X * \mathcal{H}$, and μ a measure in X, as (see [5, Definition 3.80 p118], [7, p83])

$$\mathcal{L}^2(X * \mathcal{H}, \mu) = \{f \in B(X * \mathcal{H}), \int_X ||f(x)||^2_{H(x)} d\mu(x) < \infty\}.$$

The space $\mathcal{L}^2(X * \mathcal{H}, \mu)$, denoted also as $\int_X^{\oplus} H(x) d\mu(x)$, is a Hilbert space with the product $\langle f, g \rangle = \int \langle f(x), g(x) \rangle_{H(x)} d\mu(x)$. When X is a discrete space $\int_X^{\oplus} H(x) d\mu(x)$ is just $\bigoplus_{x\in X} H(x)$, and in general case, if the $H(x)$'s are the fibers of the vector bundle $\mathcal{H}$, the direct integral $\int_X^{\oplus} H(x)d\mu(x)$ is the space of sections that are square integrable with respect to μ [25, p223].

Given a Borel bundle $X * \mathcal{H}$ are constructed with a *Borel field of operators* defined with a family of bounded linear maps $T(x) : H(x) \to H(x)$, which can be used to define the operator $\int_X^{\oplus} T(x)d\mu(x)$, also denoted as $T \in B(L^2(X * \mathcal{H}, \mu))$, and denominated the *direct integral* of $T(x)$ (see [5, Definition 3.88 p120, Proposition 3.91]). Recall that in harmonic analysis, the direct integral plays a basic concept in the decomposition of representations of groups. For instance, in the case of locally compact Abelian groups, a unitary representation is equivalent to a direct integral of irreducible representations, and in more general situations, as locally compact groups, a similar mechanism (the Plancherel theorem) is implemented for the regular representation [25, Theorem 7.36, Section 7.5].

Remark 2.5 (Isomorphism groupoid for a Borel Hilbert bundle and groupoid representations)**.** Given a Borel Hilbert bundle, its fibred structure gives rise to an isomorphism groupoid that will be used to define unitary groupoid representations. The *isomorphism groupoid* for an analytic Hilbert bundle $X * \mathcal{H}$ is $\mathrm{Iso}(X * \mathcal{H}) = \{(x, V, y), V : H(y) \to H(x)\,\text{unitary}\}$, with composable pairs $\mathrm{Iso}(X * \mathcal{H})^{(2)} = \{((x, V, y),(w, U, z)) \in \mathrm{Iso}(X * \mathcal{H}) \times \mathrm{Iso}(X * \mathcal{H}), y = w\}$, and the composition is defined as $(x, V, y)(y, U, z) = (x, VU, z)$, $(x, V, y)^{-1} = (y, V^*, z)$ [5, Definition 3.67 p111], [7, p83].

Definition 2.4 (Groupoid representation)**.** A *groupoid representation* of a locally compact Hausdorff groupoid G is a triple $(\mu, G^{(0)} * \mathcal{H}, L)$ with μ a quasi invariant measure in $G^{(0)}$, $G^{(0)} * H$ is an analytic Borel Hilbert bundle, and $L : G \to \mathrm{Iso}(G^{(0)} * \mathcal{H})$ a Borel groupoid homomorphism with $L(\gamma) = (r(\gamma), L_\gamma, s(\gamma))$, for a unitary $L_\gamma : H(y) \to H(x)$ [5, Definition 3.76 p117], [7, p83].

Definition 2.5 (Groupoid C^*-algebra)**.** Given a Haar system $\{\lambda^u\}_{u\in G^{(0)}}$ of a locally compact Hausdorff groupoid G, we define, for $f, g \in C_c(G)$, the convolution as

$$(f * g)(\gamma) = \int f(\gamma\eta)g(\eta^{-1})\, d\lambda^{s(\gamma)}(\eta) = \int f(\eta)g(\eta^{-1}\gamma)\, d\lambda^{r(\gamma)}(\eta)$$

and the involution by $f^*(x) = \overline{f(x^{-1})}$. With these operations, $C_c(G)$ is a topological $*$-algebra (see [7, Section 3.1] and [49]). In order to define a C^*-algebra with $C_c(G)$, we can select several norms giving rise to the full and reduced C^*-algebras for the groupoid G. The basic step for constructing these norms, is to consider a *representation* of $C_c(G)$, defined as a $*$-homomorphism from $C_c(G)$ into $B(H)$, the bounded operators for some Hilbert space H. Every groupoid representation $(\mu, G^{(0)} * \mathcal{H}, L)$ can be related to a representation of $C_c(G)$ with $H = \int_{G^{(0)}}^{\oplus} H(x)d\mu(x)$, for $\mathcal{H} = \{H(x)\}_{x\in G^{(0)}}$ [7, p87].

The analogue in groupoid theory of the regular representation of a group is a representation of $C_c(G)$ given by an operator $\mathrm{Ind}\mu$ in $L^2(G)$ with $\mathrm{Ind}\mu(f)\xi(x) = (f * \xi)(x)$. The *reduced norm* is constructed as $\|f\|_{red} = \|\mathrm{Ind}\mu(f)\|$, making $C_c(G)$ into a C^*-algebra (see [7, p87] for details).

Groupoids actions and orbit spaces

The concept of a groupoid action G on X generalizes the concept of a group action by considering partially defined maps on pairs $(\gamma, x) \in G \times X$. This is a natural consequence of the partially defined multiplication in a groupoid [26, Section 1.2].

Definition 2.6 (Groupoid action)**.** A (left) action of a groupoid G in a set X is a surjection $r_X : X \to G^{(0)}$, together with a map

$$G * X = \{(\gamma, x) \in G \times X, s(\gamma) = r_X(x)\} \to X, \quad (\gamma, x) \mapsto \gamma x,$$

with the following three properties (see [44, Chapter 2] and [26, Definition 1.55]):

1- $r(\gamma x) = r(\gamma)$ for $(\gamma, x) \in G * X$.

2- If $(\gamma_1, x) \in G * X$ and $(\gamma_2, \gamma_1) \in G^{(2)}$, then $(\gamma_2\gamma_1, x), (\gamma_2, \gamma_1 x) \in G * X$ and

$$\gamma_2(\gamma_1 x) = (\gamma_2\gamma_1)x.$$

3- $r_X(x)x = x$ for all $x \in X$.

With these conditions, we say that X is a (left) G-space. We can define in a similar way right actions and right G-spaces by denoting with s_X the map from X to $G^{(0)}$, and using $X * G = \{(x, \gamma) \in X \times G : s_X(x) = r(\gamma)\}$ instead of $G * X$.

The action of a groupoid in a set defines an equivalence relation that can be used to construct the *orbit space*, which represents a main object to study.

Definition 2.7 (Orbit space for groupoid actions)**.** Given a left G-space X, we define the *orbit equivalence relation* on X defined by G with $x \sim y$ if and only if there exist a $\gamma \in G$, with $\gamma \cdot x = y$, and the corresponding quotient space is the *orbit space*, and denoted by X/G with elements $G \cdot x$ or $[x]$. The same notation is used for right G-spaces, but in situations where X is both a left G-space and right H-space, the orbit space with respect to the G-action is denoted $G\backslash X$ and the orbit space with respect to the H-action is denoted by X/H [26, Definition 1.67].

In the particular case where $X = G^{(0)}$, the equivalence relation can be defined as $u \sim v$ iff $G^u_v \neq \emptyset$. The orbits $[u]$ for $u \in G^{(0)}$ are the corresponding equivalence classes and the orbit space is denoted by $G^{(0)}/G$. The graph of the equivalence relation can be described as $R = \{(r(\gamma), s(\gamma)), \gamma \in G\}$. We say that the subset $A \subset G^{(0)}$ is *saturated* if it contains the orbits of its elements, and we say that the groupoid G is *transitive* or *connected* if it has a single orbit. Alternatively, we say that G is *transitive* or *connected* if there is a morphism between any pair of elements in $G^{(0)}$ [35, Example 2.2.2, p50] [42, p20] For each orbit $[u]$ of a groupoid G, the set $G|_{[u]}$ is a transitive groupoid denominated *transitive component* of G. An important property is that each groupoid is a disjoint union of its transitive components (see [7, p73] for details). In a similar topic we also mention that, seen as a category, each groupoid is equivalent (but not isomorphic) to a category of disjoint union of groups (see [16, Appendix A] for a very short but nice survey on this topic).

We can now state some basic results we need on the characterization of a C^*-algebra of a transitive groupoid:

Theorem 2.1 (Muhly-Renault-Williams: Transitive Groupoids and their C^*-algebras [45, 7])**.** *Let G be a transitive, locally compact, second countable and Hausdorff groupoid, then the (full) C^*-algebra of G is isomorphic to $C^*(H) \otimes K(L^2(\mu))$, where H is the isotropy group G_u^u at any unit $u \in G^{(0)}$, and μ a measure on $G^{(0)}$, $C^*(H)$ denotes the group C^*-algebra of H, and $K(L^2(\mu))$ denotes the compact operators on $L^2(\mu)$.*

Definition 2.8 (Free and proper groupoid actions)**.** The action of a groupoid G in a set X is *free* if the map $\Phi : G * X \to X \times X$, $(\gamma, x) \mapsto (\gamma x, x)$, is injective [5, Conventions 1.1]. This can also rephrased by saying that $\gamma x = x$ implies γ is a unit ($\gamma \in G^{(0)}$) [26, Definition 1.83]. The action is *proper* if the map Φ is proper (meaning that the inverse images of compact sets are compact). A main property for proper actions is that the orbit space X/G is locally compact and Hausdorff if G acts property on the locally compact space Hausdorff space X [26, Proposition 1.85].

Groupoid dynamical systems and groupoid crossed products

A natural consequence of the fibred properties of a groupoid is the usage of fibred C^*-algebras when generalizing the concept of dynamical systems to the groupoid language.

Remark 2.6 ($C_0(X)$-algebras and C^*-bundles)**.** A $C_0(X)$-*algebra* is a C^*-algebra with a nondegenerate homomorphism Φ_A from $C_0(X)$ (the space of continuous functions vanishing at infinity on X) into $Z(M(A))$, where $M(A)$ denotes the multiplier algebra of A, and $Z(A)$ denotes the center of A. Here, Φ_A is nondegenerate when $\Phi_A(C_0(X)) \cdot A = \text{span}\{\Phi_A(f)a, f \in C_0(X), a \in A\}$ is dense in A (see [56] for details).

Remember that the multiplier algebra $M(A)$ of A is the maximal C^*-algebra containing A as an essential ideal (see [4, Chapter 4]). For instance, if A is unital, $M(A) = A$. If $A = C_0(X)$, the continuous functions with compact support in a locally compact Hausdorff space, then $M(A) = C_b(X)$, the continuous functions bounded on X. If A is the space of compact operators on a separable Hilbert space H, $M(A) = B(H)$, the C^*-algebra of all bounded operators on H. Recall also that the *center* of an algebra A is the commutative algebra $Z(A) = \{x \in A, xa = ax \; \forall a \in A\}$. This concept plays a crucial role as, for instance, in the theory of Von Neumann algebras (algebras of bounded operators on a Hilbert space). Von Neumann algebras with a trivial center are called *factors*, and these are basic building blocks for general Von Neumann algebras via direct integral decompositions.

An *upper semicontinuous C^*-bundle* over X, a locally compact Hausdorff space, is a topological space $\mathscr{A}$ with a continuous open surjection $p_{\mathscr{A}} = p : \mathscr{A} \to X$ such that the fiber $A(x) = p^{-1}(x)$ is a C^*-algebra with the following conditions. First, the map $a \mapsto \|a\|$ is upper-continuous from $\mathscr{A}$ to $\mathbb{R}^+$ (i.e. for all $\epsilon > 0$, the set $\{a \in \mathscr{A}, \|a\| < \epsilon\}$ is open). The operations sum, multiplication, scalar multiplication, and involution in the algebra $\mathscr{A}$ are continuous. Finally, if $\{a_i\}$ is a net in $\mathscr{A}$ with $p(a_i) \to x$, and $\|a_i\| \to 0$, then $a_i \to 0_x$, with 0_x the zero element of $A(x)$.

Two fundamental properties of $C_0(X)$-algebras are the fact that there is a one to one correspondence between $C_0(X)$-algebras and upper-semicontinuous bundles C^*-bundles [5, Definition 3.12 p91], and that the primitive ideal space of a $C_0(X)$-algebra is fibred over X [5, p93].

The interaction between $C_0(X)$-algebras and upper-semicontinuous bundles C^*-bundles is given by the fact that the C^*-algebra $A = \Gamma_0(X, \mathscr{A})$ of continuous sections of $\mathscr{A}$ vanishing at infinity, is a $C_0(X)$-algebra, as we now explicitly rephrase in the following example.

Example 2.9 ($C_0(X)$-algebras)**.** A basic example of a $C_0(X)$-algebra is given by $A = C_0(X, D)$, where D is any C^*-algebra, X is a locally compact Hausdorff space, and $\Phi_A(f)(a)(x) = f(x)a(x)$, for $f \in C_0(X)$, $a \in A$. For this example, each fiber $A(x)$ is identified with D. [26, Example 3.16, p91]

As previously mentioned, a fundamental example of a $C_0(X)$-algebra is $A = \Gamma_0(X, \mathscr{A})$, for a $\mathscr{A}$ upper-semicontinuous C^*-bundle, with $\Phi_A(\phi)f(x) = \phi \cdot f(x) = \phi(x)f(x)$, for $\phi \in C_0(X)$, and $f \in A$. [26, Example 3.18, p91]

Definition 2.9 (($\mathscr{A}, G, \alpha$) Groupoid dynamical system)**.** If G is a groupoid with Haar system $\{\mu^u\}_{u\in G^{(0)}}$, and $\mathscr{A}$ is an upper-semicontinuous C^*-bundle over $G^{(0)}$. An action α of G on $A = \Gamma_0(X, \mathscr{A})$ is a family of $*$-isomorphisms $\{\alpha_\gamma\}_{\gamma\in G}$ with $\alpha_\gamma : A(s(\gamma)) \to A(r(\gamma))$, for all $\gamma \in G$, $\alpha_{\gamma\eta} = \alpha_\gamma\alpha_\eta$ for all $(\gamma, \eta) \in G^{(2)}$, and the map $G * \mathscr{A} \to \mathscr{A}$, $(\gamma, a) \mapsto \alpha_\gamma(a)$ is continuous. With these conditions, the triple $(\mathscr{A}, G, \alpha)$ is a *groupoid dynamical system* (see [5, Definition 2.2]).

Example 2.10 (($\mathscr{C}_X, G, \mathrm{lt}$) Groupoid dynamical system)**.** A basic example of a groupoid dynamical system is $(\mathscr{C}_X, G, \mathrm{lt})$, where G is a groupoid acting on a second countable locally compact Hausdorff space X, and the upper semi-continuous C^*-bundle $\mathscr{C}_X = G^{(0)} * \{C_0(r_X^{-1}(u))\}_{u\in G^{(0)}}$ is associated with the $C_0(G^{(0)})$-algebra $C_0(X)$. The action of G on X induces an action of G on $\mathscr{C}_X$ by left translation [6, Example 3.30, Proposition 3.31, p25],

$$\mathrm{lt}_\gamma(f) : C_0(r_X^{-1}(s(\gamma))) \to C_0(r_X^{-1}(r(\gamma))), \quad x \mapsto f(\gamma^{-1} \cdot x).$$

Remark 2.7 (Reduced crossed product of a groupoid dynamical system)**.** With a groupoid dynamical system $(\mathscr{A}, G, \alpha)$, we can construct a convolution algebra that can be completed to the reduced crossed product, which is one possible generalization of the concept of a crossed product. An important tool for this task is the *pullback bundle* $r^*\mathscr{A}$ of a bundle $\mathscr{A}$ over X, with bundle map $p_{\mathscr{A}} : \mathscr{A} \to X$. The pullback bundle is defined as

$$r^*\mathscr{A} := \{(\gamma, a), r(\gamma) = p_{\mathscr{A}}(a)\},$$

for $r : G \to X$. The corresponding bundle map for $r^*\mathscr{A}$ is $q : r^*\mathscr{A} \to G$, with $q(\gamma, a) = \gamma$ [26, Definition 3.33 p97].

The first step for constructing the groupoid crossed product is a property [5, Proposition 2.4] ensuring that, given a groupoid G with Haar system $\{\lambda^u\}_{u\in G^{(0)}}$, the set of continuous compactly supported sections of $r^*\mathscr{A}$, denoted by $\Gamma_c(G, r^*\mathscr{A})$, is a $*$-algebra with respect to the operations

$$(f * g)(\gamma) := \int_G f(\eta)\alpha_\eta(g(\eta^{-1}\gamma))d\lambda^{r(\lambda)}(\eta), \quad f^*(\gamma) := \alpha_\gamma(f(\gamma^{(-1)})^*).$$

The second step we consider here is to complete $\Gamma_c(G, r^*\mathscr{A})$ with the *reduced norm* $\|f\|_r = \sup\{\|\mathrm{Ind}\pi(f)\|, \pi \text{ is a } C_0(G^{(0)}) \text{ linear representation of } A\}$ (see [5, p4] for details). We define the completion of $\Gamma_c(G, r^*\mathscr{A})$ with the norm $\|\|_r$ as the *reduced crossed product* of the dynamical system $(\mathscr{A}, G, \alpha)$, and we denoted it with $\mathscr{A} \rtimes_{\alpha,r} G$. Notice that this procedure is the important strategy of using representations of an algebra in order to construct a meaningful norm that leads to a C^*-algebra construction denominated *enveloping C^*-algebra* (see [28, Definition 12.2, p523]).

Remark 2.8 (Morita equivalence)**.** Representation theory plays a crucial role in the interplay between topological spaces and algebraic structures. The Gelfand-Naimark theorem identifies a locally compact Hausdorff space X with a commutative C^*-algebra $A = C_0(X)$ (continuous functions vanishing at infinity) by considering an homeomorphism between X and the set of *characters* $\hat{A}$ identified with the set of unitary equivalence classes of irreducible $*$-representations [37, 54]. The set $\hat{A}$ is also known as the *structure space* and, for commutative C^*-algebras, it coincides with the *primitive spectrum* $\mathrm{Prim}(A)$, defined as the space of kernels of irreducible $*$-representations of A [37, Section 2.3]. The set of characters of a Banach algebra A is also known as the *Gelfand spectrum*, also denoted by $\mathrm{sp}(A)$ (see [28, Definition 1.3, p5]).

This conceptual interaction between representation theory, commutative C^*-algebras and topological spaces has its origins in the Morita theory, as described in the context of representation theory of rings [1, Chapter 6] [48]. Recall that modules are intimately related to representations of rings, and this fact motivates the concept of Morita equivalence relation between two rings R and S, defined as an equivalence of categories between ${}_RM$ and ${}_SM$, (the categories of modules over R and S, respectively).

These ideas can be extended to the context of C^*-algebras, but for this task, we require more subtle procedures, and a crucial role is played by the landmark ideas of M. Rieffel who introduced the concept of *strong* Morita equivalence. Given two C^*-algebras, A and B, the basic concept behind a Morita equivalence is the notion of a *$A - B$ equivalence bimodule M* (also known as *imprimitivity bimodule*), defined as a $A - B$ bimodule such that M is a full left Hilbert A-module and full right Hilbert B-module, and we have an associativity formula ${}_A\langle x, y\rangle z = x\langle y, z\rangle_B$, for $x, y, z \in M$ (see [35, Definition 2.4.3], [48, Chapter 3]). A *right Hilbert B-module* for a C^*-algebra B is a right B-module M with a B-valued inner product $\langle\cdot,\cdot\rangle : M \times M \to B$, with corresponding generalizations of the standard notion of inner product (see [35, Definition 2.4.1]). With these notions, we say that two C^*-algebras A and B, are *(strongly) Morita equivalent* ($A \overset{\mathrm{m}}{\sim} B$) if there exist an equivalence $A - B$ bimodule (see also [28, Definition 4.9, p162]). We follow the explanations of [48, Remark 3.15], and we will usually omit the word *strongly* for this Morita equivalence concept. Many important properties are conserved under this equivalence relation. In particular, a crucial fact is that the structure space $\hat{A}$ is homeomorphic to the structure space $\hat{B}$ when A and B are (strong) Morita equivalent (see [28, p167]).

Remark 2.9 (Open covers of manifolds)**.** A basic example in noncommutative geometry is given by the *open covers of manifolds* (see [12, Chap 2, Example 2α] and [35, Example 2.5.3, p81]). Here, a Morita equivalence relation is established between the (noncommutative) C^*-algebra $C^*(R)$ and the (commutative) C^*-algebra $C_0(\mathcal{M})$, for a locally compact manifold $\mathcal{M}$, where the equivalence relation R is defined in the set $V = \bigsqcup U_i$ for a finite covering $\bigcup U_i = \mathcal{M}$, with $z \overset{\mathrm{R}}{\sim} z'$ iff $p(z) = p(z')$, using the canonical projection $p : V \to \mathcal{M}$. This example will be fundamental in our framework, as this Morita equivalence will be used to exploit the configuration introduced by an open cover in order to analyze the particular type of manifolds (the foliated partitions F_f) we are interested in.

Proper groupoid dynamical systems

A fundamental result used in noncommutative geometry is a property proposed by Green [29], which constructs a Morita equivalence relation between the C^*-algebra $C_0(H\backslash X)$ on the quotient space $H\backslash X$ of a group H acting on X, and the corresponding crossed product $C_0(X) \rtimes_{\rm lt} H$.

Theorem 2.2 (Green 1977 [29, Corollary 15], [6, Theorem 4.1, p61])**.** *If H is a locally compact Hausdorff group acting freely and properly on a locally compact Hausdorff space X, then $C_0(X) \rtimes_{lt} H$ is Morita equivalent to $C_0(H\backslash X)$.*

An important generalization of this property has been prepared by Rieffel [51] which considers the action of a group G in a (noncommutative) C^*-algebra. We describe now a generalization of this machinery in the setting of groupoid actions as prepared by Brown in [5].

Definition 2.10 (Proper dynamical system [5, Definition 3.1])**.** Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system and $A = \Gamma_0(G^{(0)}, \mathscr{A})$ its associated $C_0(G^{(0)})$-algebra. We say that $(\mathscr{A}, G, \alpha)$ a *proper dynamical system* if there exist a dense $*$-subalgebra $A_0 \subset A$ with the following two conditions.

1- we construct functions ${}_E\langle a, b\rangle$ that will generate a dense subspace E of $\mathscr{A} \rtimes_{\alpha,r} G$ (see Theorem 2.3). For this step, we require that, for each $a, b \in A_0$, the function ${}_E\langle a, b\rangle : \gamma \to a(r(\gamma))\alpha_\gamma(b(s(\gamma))^*)$, $\gamma \in G$, is integrable (see also [6, Section 4.1.1, p62]). Notice that with this requirement we use the sections a, b (in $\Gamma_0(G^{(0)}, \mathscr{A})$) to construct sections ${}_E\langle a, b\rangle$ defined in the groupoid G and considered in $\Gamma_c(G, r^*\mathscr{A})$.

2- We set a requirement for constructing the fixed point algebra A^α (see Theorem 2.3) by defining

$$M(A_0)^\alpha = \{d \in M(A), A_0 d \subset A_0, \overline{\alpha_\gamma}(d(s(\gamma))) = d(r(\gamma))\}.$$

We define now $\langle a, b\rangle_D \in M(A_0)^\alpha$ such that for all $c \in A_0$, $(c \cdot \langle a, b\rangle_D)(u) = \int_G c(r(\gamma))\alpha_\gamma(a^* b(s(\gamma)))d\lambda^u(\gamma)$.

We can now state the main result in [5, Theorem 3.9], generalizing to the groupoid language the property of Rieffel [51, Section 2] which generalizes to (non necessarily commutative) C^*-algebras the result of Green [29, Corollary 15].

Theorem 2.3 (Morita equivalence in proper dynamical systems [5, Theorem 3.9])**.** *Let $(\mathscr{A}, G, \alpha)$ be a proper dynamical system with respect to A_0, and let $D_0 = \mathrm{span}\{\langle a, b\rangle_D, a, b \in A_0\}$ is a dense subalgebra of $A^\alpha = \overline{D_0}$, the fixed point algebra which is the completion of D_0 in $M(A)$. Let also $E_0 = \mathrm{span}\{{}_E\langle a, b\rangle, a, b \in A_0\}$ be a dense subalgebra of $E = \overline{E_0}$, the completion of E_0 in $\mathscr{A} \rtimes_{\alpha,r} G$.*

With these conditions, A_0 is a $E_0 - D_0$ pre-imprimitivity bimodule, which can be completed to a $E - A^\alpha$ imprimitivity bimodule. As a consequence, the generalized fixed point algebra A^α is Morita equivalent to a subalgebra E of the reduced crossed product $\mathscr{A} \rtimes_{\alpha,r} G$.

Saturated groupoid dynamical systems

We now present a basic tool we need, as developed by J.H. Brown [5, 6], generalizing to the setting of groupoid theory, the results of Rieffel [51]. The core concept is the notion of saturated groupoid dynamical systems (Definition 2.11), whose requirements can be ensured when considering principal and proper groupoids (Definition 2.12). The main Theorem 2.4 considers the case of a general groupoid dynamical system $(\mathscr{A}, G, \alpha)$, but our main current interest is the particular case where $\mathscr{A} = C_0(G^{(0)})$, as described in the Theorem 2.5.

Definition 2.11 (Saturated Groupoid Dynamical System [5, Definition 5.1])**.** A dynamical system $(\mathscr{A}, G, \alpha)$, is saturated if ${}_{E_0}A_{0D_0}$ completes to a $\mathscr{A} \rtimes_{\alpha,r} G - A^{\alpha}$ imprimitivity bimodule.

Definition 2.12 (Principal groupoid and proper groupoid)**.** Given a groupoid G with its unit space $G^{(0)} = X$, if the natural action of G in X, $\gamma s(\gamma) = r(\gamma)$, is free (see Definition 2.8), we say that G is *principal*, and we say that G is *proper* if this action is proper (see Definition 2.8 and [5, Conventions 1.1]).

Theorem 2.4 (Principal and proper groupoids, saturated actions, and Morita Equivalence [5, Theorem 5.2])**.** *Let $(\mathscr{A}, G, \alpha)$ be a groupoid dynamical system and $A = \Gamma_0(G^{(0)}, \mathscr{A})$ the associated $C_0(G^{(0)})$-algebra. Then, if G is principal and proper, the action of G on A is saturated with respect to the dense subalgebra $C_c(G^{(0)}) \cdot A$. Therefore, A^{α} is Morita equivalent to $\mathscr{A} \rtimes_{\alpha,r} G$.*

Theorem 2.5 (Case $\mathscr{A} = C_0(G^{(0)})$ [5, Theorem 5.9])**.** *If the groupoid G is principal and proper, then, the dynamical system $(C_0(G^{(0)}), G, lt)$ is saturated with respect to the dense subalgebra $C_c(G^{(0)})$. As a consequence, we have the following Morita equivalence:*

$$C_0(G \backslash G^{(0)}) \overset{\mathrm{m}}{\sim} C_r^*(G) \quad \mathit{for} \quad C_r^*(G) := C_0(G^{(0)}) \rtimes_{lt,r} G.$$

Renault's equivalence for groupoid crossed products

A fundamental additional property we need in our framework is the concept of Morita equivalent dynamical systems which is helpful to ensure when two groupoid crossed products are Morita equivalent. Two dynamical systems $(\mathscr{A}, G, \alpha)$, $(\mathscr{B}, G, \beta)$ are *Morita equivalent* if there is a $\mathscr{A} - \mathscr{B}$ imprimitivity bimodule $\mathscr{H}$ over $G^{(0)}$ and a G action on $\mathscr{H}$ with adequate compatibility conditions (see [46, Definition 9.1, p54]).

Remark 2.10 (Renault's equivalence of groupoid crossed products)**.** An important consequence of the Renault's equivalence for groupoid crossed products (see [46, Theorem 5.5, p27], [50]) is the fact that a Morita equivalence between dynamical systems $(\mathscr{A}, G, \alpha)$, and $(\mathscr{B}, G, \beta)$ implies that the corresponding crossed products are Morita equivalent:

$$\mathscr{A} \rtimes_{\alpha,r} G \overset{\mathrm{m}}{\sim} \mathscr{B} \rtimes_{\beta,r} G.$$

2.3 Functional Clouds and Some Basic Properties

The concept of a functional cloud M_f can be described as the set of possible *local* states of a signal $f : \mathcal{G} \to \mathbb{R}$, where the *local* properties are measured with respect to a set $A \subset \mathcal{G}$ as a basic analysis unit. We now want to study the topology of M_f in order to analyze the different types of local components present in f. An important scenario is to consider the case where f is a combination of different signals $f = \sum f_i$. Here, the objective is to use tools from C^*-algebras and their K-theory in order to study how the topology of M_f is assembled from the pieces $\{M_{f_i}\}_i$. A very simplified scenario for this situation is presented in Proposition 2.3. The strategy we use for the analysis of the spaces M_f and F_f is to study the C^*-algebras of spaces $M_f^G = F_f/G$, using groupoids G with $G^{(0)} = F_f$. We use the Theorem 2.1 to describe simple geometrical relationships between F_f and M_f.

Notice that the space $M_f^G = F_f/G$ considers a generalization of the equivalence relation used to define M_f. A main advantage of this strategy is that we can directly apply the large body of work already available in groupoid theory and operator algebras. Additionally, this method prepares the terrain for addressing more complex problems crucial in concrete applications of signal processing.

The groupoid C^*-algebra of G contains information related to M_f, and the space $M_f^G = F_f/G$ is in relation to $\mathrm{Prim}(C^*(G))$, the primitive spectrum of the algebra $\mathcal{A}$, used as a basic tool in the Gelfand-Naimark theory [37, 25]. The following property includes these ideas, and it is inspired by the basic strategy presented in [35, Example 2.2.2].

Proposition 2.2. *Let M_f be a functional cloud, F_f the related foliated partition, and G a groupoid with $G^{(0)} = F_f$. If G is as a finite disjoint union $G = \bigsqcup_{i=1}^k G_i$, for G_i transitive groupoids, and $M_f^G := F_f/G$ is locally compact and Hausdorff, then by denoting with H_i the isotropry group at any unit $u \in G_i^{(0)}$, we have for the K-theory of M_f^G:*

$$K^0(M_f^G) \simeq K_0(C_0(F_f) \rtimes_{lt,r} G) \quad \textit{with} \quad K_0(C^*(G)) \simeq \bigoplus_{i=1}^{k} K_0(C^*(H_i)).$$

Proof. This is a direct application of the characterization of the C^*-algebra of a transitive groupoid in Theorem 2.1. For each transitive groupoid G_i we have the isomorphism $C^*(G_i) \simeq C^*(H_i) \otimes K(L^2(\mu_i))$, for a measure μ_i on H_i, and $C^*(G_i)$, $C^*(H_i)$, the C^*-algebras of G_i and H_i respectively. Therefore, given $G = \bigsqcup_{i=1}^k G_i$, we have

$$C^*(G) \simeq \bigoplus_{i=1}^{k} C^*(G_i)) \simeq \bigoplus_{i=1}^{k} C^*(H_i) \otimes K(L^2(\mu_i)).$$

The K-theory can now be computed using the stability of the functor K_0, that is $K_0(C^*(H_i) \otimes K(L^2(\mu_i))) \simeq K_0(C^*(H_i))$ (see [4, Corollary 6.2.11 p118]). With the relation between the topological and algebraic K-theory we can conclude that $K^0(M_f^G) \simeq K_0(C_0(M_f^G))$ (see [28, Corollary 3.21, p101] and the corresponding generalization to locally compact spaces in [28, p103]). Now, as

$$C_0(M_f^G) \overset{\mathrm{m}}{\sim} C_r^*(G) \quad \text{for} \quad C_r^*(G) := C_0(F_f) \rtimes_{\mathrm{lt},r} G$$

(due to the Theorem 2.5), and using the result from Exel [28, Theorem 4.30, p165], [21, Theorem 5.3] (ensuring that Morita equivalent C^*-algebras have isomorphic K-theory groups) we have

$$K^0(M_f^G) \simeq K_0(C_0(M_f^G)) \simeq K_0(C_0(F_f) \rtimes_{\mathrm{lt},r} G).$$

■

Remark 2.11 (noncommutative algebra $M_k(A)$)**.** In the following we use the standard notation $M_k(A)$ for the noncommutative algebra of $k \times k$ matrices with entries in an algebra A. Recall also that $M_k(A) = M_k(\mathbb{C}) \otimes A$.

Remark 2.12 (The discrete setting for groupoid algebras)**.** The Proposition 2.2 is partially inspired by the description of a groupoid algebra in a discrete setting (see [35, Example 2.2.2]). By denoting with $\mathbb{C}G$ the $*$-algebra with multiplication and $*$ operation as declared in Definition 2.5 (and ignoring for now its C^* properties), we can describe its structure using a canonical decomposition $G = \bigsqcup_i G_i$ for G_i transitive groupoids as:

$$\mathbb{C}G \simeq \bigoplus_i \mathbb{C}H_i \otimes M_{n_i}(\mathbb{C}),$$

where H_i is the isotropy group of a unit in $G_i^{(0)}$ (whose isomorphism class is independent of the chosen unit), and $M_{n_i}(\mathbb{C})$ is the noncommutative algebra of $n_i \times n_i$ matrices with complex entries. Each transitive groupoids G_i is assumed to be finite, and its cardinality is denoted by n_i.

Remark 2.13 (Relations to Persistent Homology)**.** In the previous constructions, we considered the groupoid as a generalized equivalence relation, but with these constructions in groupoid theory we can include more complex situations as required by applications. An important additional aspect to consider is the generalization of the groupoid construction used in the previous Example 2.12. If we consider the groupoid $G_\epsilon = \{(u,v) \in F_f \times F_f,\ d(p(u),p(v)) < \epsilon\}$, we can study the family $\{M_{f,A}^{G_\epsilon}\}_{\epsilon>0}$ as a filtration in the context of persistent homology (see Section 3). As we will see in the following Section, the framework of persistent homology can also be adapted to handle C^*-algebra structures.

Function decompositions, clouds and their K-theory

We now present a basic property where we study how the topology of the functional cloud of $f = \sum_{i=1}^n f_i$ interacts with the topology of the functional clouds of f_i. The long term objective is to design signal analysis and separation algorithms using topological or geometrical invariants of the functional clouds of f_i.

The next Proposition 2.3 represents just a first glance on how to study the topological interactions between M_f and $\{M_{f_i}\}_i$. Here, the mechanism is based on the simplified assumption that a group G is acting in F_f, and the study of the quotient F_f/G represents an approximation for M_f. There are two different, but interrelated aspects occurring. On the one hand, we have the group G acting on F_f, and on the other hand we have the decomposition of F_f with an open cover originated from the consideration of the function decomposition $f = \sum f_i$.

Proposition 2.3. *Let $f : \mathcal{G} \to \mathbb{R}$ be a continuous function, M_f a functional cloud and F_f its related foliated partition. Let $f = \sum_{i=1}^{k} f_i$ with $\operatorname{supp}(f_i) \subset U_i$, for $\{U_i\}_{i=1}^{k}$ an open cover of $\mathcal{G}$, and define*

$$V_i := \bigsqcup_{x \in U_i} F_x, \quad F_x = \{(y, f_x(y)), y \in A\},$$

for $f_x : A \to \mathbb{R}$ with $f_x(y) = f(x+y), x \in \mathcal{G}, y \in A$. If a locally compact group G is acting on F_f then for the K-theory of $M_f^G := F_f/G$ we have:

$$K^0(M_f^G) \simeq K_0(\mathcal{A} \otimes C^*(G)), \quad \mathcal{A} := \Big\{[h_{ij}] \in M_k(C_0(F_f)),\ h_{ij} \in C_0(V_i \cap V_j)\Big\}.$$

Proof. The proof is a simple application of two basic facts concerning the C^*-algebra of a quotient space given by a group action and the mechanism for studying quotients of *open covers of a manifold* (see Remark 2.9). First, recall that a group action $\alpha : G \times F_f \to F_f$ induces an action in the algebra $C_0(F_f)$, and the resulting dynamical system can be encoded in a cross product denoted as $C_0(F_f) \rtimes_\alpha G$, and, seen as a vector space, it can be written as

$$C_0(F_f) \rtimes_\alpha G = C_0(F_f) \otimes C^*(G).$$

The corresponding product defined for this algebra is given by $(a \otimes g)(b \otimes h) = ag(b) \otimes gh$ (see [35, Example 2.2.7]). As a consequence of Theorem 2.2 (see also [35, Theorem 2.5.1, p78]), we have $C_0(M_f^G) \overset{\text{m}}{\sim} C_0(F_f) \rtimes_\alpha G$. On the other hand, the open cover property (see Remark 2.9) ensures that the C^*-algebra $C_0(F_f)$ is Morita equivalent to $\mathcal{A}$ when considering the open cover $\{V_i\}_{i=1}^{k}$ of F_f. By combining these two properties, together with the result from Exel in [28, Theorem 4.30, p165], [21, Theorem 5.3] ensuring that Morita equivalent C^*-algebras have isomorphic K-theory groups, we have

$$K^0(M_f^G) \simeq K_0(C_0(M_{f,A}^G)) \simeq K_0(C_0(F_f) \otimes C^*(G)) \simeq K_0(\mathcal{A} \otimes C^*(G)).$$

∎

The simplified scenario of this proposition is just a first step where more general situations should consider not just a group G acting on F_f but a groupoid (see the Remark 2.13) for capturing more accurately the interactions between the components f_i. This is particularly important in applications, as illustrated in Example 2.4, where M_f is actually just an intermediate structure, and the principal goal is to understand the geometry and topology of the underlying parameter set Ω (see the Example 2.4).

Example 2.11 (Image Segmentation)**.** A typical application which illustrates the objectives in Proposition 2.3 is to consider an grayscale image $f : [0,1]^2 \to [0,1]$, where different areas $U_i \subset [0,1]^2$ correspond to different regions in the image. The main task is to understand how the topology of M_f is assembled from the different regions $f_i = f(U_i)$ and their clouds M_{f_i}. This requires not only to study the regions themselves, but also their contours or edges represented by $f(U_i \cap U_j), i \neq j$. The combination and interaction between these topologies is encoded in the algebra $\mathcal{A}$ together with the partition of F_f with the group G (in a more general setting, crucial for applications, a groupoid structure G defined in F_f should be considered, as explained in the previous paragraph).

Voice transforms and groupoids crossed products

We now consider a more general situation using the abstract machinery of time-frequency analysis represented by the theory of voice transforms. A crucial advantage of this setting is the fact that multiple time-frequency transforms (e.g. wavelets, Gabor analysis, etc) are represented in a single abstract environment. Additionally, we consider signals as vectors in an abstract Hilbert space, allowing a clean, general, and powerful environment for expressing our problems and the solutions strategies.

Given a mixture of signals $f = \sum_{i=1}^{k} f_i \in H$, for a Hilbert space H, the following property is an initial step in understanding the interaction between the functional clouds $M^G_{V_\psi f}$ and the components f_i of the signal f. One possible analogy for this scenario is to consider each signal f_i as a measurement originated from a particular physical event i, and f is the mixture of signals encoding the interactions of all k events.

Theorem 2.6. *Let $f = \sum_{i=1}^{k} f_i \in H$, for a Hilbert space H and $V_\psi : H \to C(\mathcal{G})$, a voice transform for a locally compact group $\mathcal{G}$. Let G be a principal and proper groupoid with unit space $G^{(0)} = F_{V_\psi f} := \mathrm{graph}(V_\psi f|_{suppV_\psi f})$, and consider an open cover $\{U_i\}_{i=1}^{k}$ of $\mathcal{G}$. If $\mathrm{supp}(V_\psi f_i) \subset U_i$, for $i = 1, \dots, k$, then the following Morita equivalence holds for $M^G_{V_\psi f} := F_{V_\psi f}/G$,*

$$C_0(M^G_{V_\psi f}) \overset{\mathrm{m}}{\sim} \mathcal{A} \rtimes_{lt,r} G, \quad \mathcal{A} := \Big\{ [h_{ij}] \in M_k(C_0(F_{V_\psi f})),\ h_{ij} \in C_0(F_{V_\psi f_i} \cap F_{V_\psi f_j}) \Big\}.$$

Proof. This follows directly from a combination of properties on Morita equivalence for groupoid C^*-algebras as discussed in the previous sections. First, notice that for a proper and principal groupoid G with $G^0 = F_{V_\psi f}$ we have a Morita equivalence

$$C_0(F_{V_\psi f}/G) \overset{\mathrm{m}}{\sim} C^*_r(G) \quad \text{for} \quad C^*_r(G) := C_0(F_{V_\psi f}) \rtimes_{\mathrm{lt},r} G,$$

using the Theorem 2.5. We have also a Morita equivalence $C_0(F_{V_\psi f}) \overset{\mathrm{m}}{\sim} \mathcal{A}$ for an open cover of the manifold $F_{V_\psi f}$, as explained in Remark 2.9. Notice that for any $f \in H$, $F_{V_\psi f}$ is indeed a manifold as defined for a voice transform (see Example 2.6 and Proposition 2.1). These two relations can be combined with the Renault's equivalence property, as discussed in Remark 2.10, in the following computation:

$$\begin{aligned} C_0(M^G_{V_\psi f}) &= C_0(F_{V_\psi f}/G) \\ &\overset{\mathrm{m}}{\sim} C_0(F_{V_\psi f}) \rtimes_{\mathrm{lt},r} G && \text{(Morita equivalence: Theorem 2.5)} \\ &\overset{\mathrm{m}}{\sim} \mathcal{A} \rtimes_{\mathrm{lt}} G, && \text{(Renault's equivalence: Remark 2.10)} \end{aligned}$$

$$\text{using} \quad C_0(F_{V_\psi f}) \overset{\mathrm{m}}{\sim} \mathcal{A} \qquad \text{(Open cover property: Remark 2.9).}$$

■

Remark 2.14 (Interpreting the Theorem 2.6)**.** There are several ways in which the Theorem 2.6 can be interpreted. Broadly speaking, one can see this property as an initial step for understanding how one can analyze and encode, with the noncommutative C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$, the interaction of different phenomena, measured with the signals f_i, and occurring at different time-frequency regions $F_{V_\psi f_i}$. In special cases, the signals f_i can be seen as measurements from different dynamical systems whose phase spaces contain the sets $M_{V_\psi f_i}$, as explained in the setting of the Taken's theorem (see the Example 2.5). With the rule $f = \sum_{i=1}^k f_i$, the spaces $M_{V_\psi f_i}$ are combined in a single structure $M_{V_\psi f}$ whose construction depends on two main properties encoded in the C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$. As we will now explain, these properties are clearly visible in the noncommutative C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$, but they are completely ignored in the commutative C^*-algebra $C_0(M^G_{V_\psi f})$. This feature illustrates the usefulness of the Morita equivalence property as a new analysis layer (constructed on top of the time-frequency machinery) for studying hidden features of a signal.

In order to explain the previous remark, we consider the particular case of Gabor analysis $V_g f(b, w) = \int_{\mathbb{R}} f(t)g(t-b)e^{-2\pi itw}dt$, $f \in L^2(\mathbb{R})$, and the cloud $M_f = \{x_b\}_b$ with chunks $x_b = fg_b$ using $g_b(t) = g(t-b)$ for a window function g. We consider also the cloud $M_{V_g f} = \{V_g f(b, .)\}_b$ (isometric to M_f) in the corresponding Fourier view (see the Example 2.6).

Now, the two main properties encoded in the C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$, can be described as the time-domain pattern for the function f represented with an adequate groupoid G, and the time-frequency relations between the signals f_i encoded with the C^*-algebra $\mathcal{A}$. A time-domain pattern encoded in a groupoid G takes into account the way different vectors $x_b = fg_b$ are repeated at different time positions $b \in \mathbb{R}$. Recall that the cloud M_f is the quotient $M_f = F_f/R$, for an adequate equivalence relation $R = \{(u, v) \in F_f \times F_f,\, p(u), p(v) = 0\}$ and the projection map $p : F_f \to M_f$, $p((x, y, f_x(y)) = f_x$. (see Remark 2.3). We use the important generalization $M^G_{V_\psi f} = F_{V_\psi f}/G$ for a groupoid G with $G^{(0)} = F_{V_\psi f}$, and the groupoid G can be used to encode the similarities and repetitions between the vectors x_b as the time parameter b changes, in order to construct a meaningful quotient space $M^G_{V_\psi f}$. The second important property encoded in the C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$ is the relationship between the time-frequency regions corresponding to each signal f_i, captured by the algebra $\mathcal{A} = \big\{[h_{ij}] \in M_k(C_0(F_{V_\psi f})),\, h_{ij} \in C_0(F_{V_\psi f_i} \cap F_{V_\psi f_j})\big\}$. The influence of each signal f_i in the whole system is encoded in a C^*-algebra $C_0(F_{V_\psi f_i})$ stored in a diagonal entry of $\mathcal{A}$. The time-frequency interference between different phenomena measured with f_i and f_j is encoded in a C^*-algebra $C_0(F_{V_\psi f_i} \cap F_{V_\psi f_j})$ located in the off-diagonal entries of the noncommutative C^*-algebra $\mathcal{A}$.

Remember that the time-frequency analysis machinery "unfolds" the information of a function f in order to understand its internal properties. In this context, in contrast to the usage of $C_0(M^G_{V_\psi f})$, the consideration of the (larger) Morita equivalent C^*-algebra $\mathcal{A} \rtimes_{lt,r} G$, can be used as a further analysis level for understanding the features of the function f and its internal structure.

Remark 2.15 (Towards an application example of the Theorem 2.6)**.** In the Example 2.4, we have defined a *cloud of a modulated path* M_f constructed with an embedded manifold $\Omega \subset [-1, 1]^d$, a continuous path $\phi : [0, 1] \to \Omega$, and the construction of a real function $f(x) = \sum_{i=1}^d \sin\Big(\int_0^x (\alpha_c^i + \gamma\phi_i(t))\, dt\Big)$, for a fixed center frequency vector $\alpha_c = (\alpha_c^i)_{i=1}^d \in \mathbb{R}^d$, bandwidth parameter γ, and the i-th coordinate of $\phi(x)$ denoted as $\phi_i(x)$. Given the function f, the main problem is to use a functional cloud $M_{f,A}$ in order to estimate geometrical and topological features of Ω. Notice that in this example, $M_f = M_{f,A} \subset C(A, \mathbb{R})$ is a curve, but under adequate conditions, we can construct a groupoid G with $G^{(0)} = F_{V_\psi f} = \mathrm{graph}(V_\psi f|_{\mathrm{supp} V_\psi f})$, such that the quotient space $M^G_{V_\psi f} = F_{V_\psi f}/G$ can be used to compute topological features of Ω.

More precisely, one strategy is to consider a *topological approximation scheme* as explained in [37, Section 3.1, p23], where a covering $\mathcal{U} = \{U_i\}$ of Ω is used to construct an equivalence relation R, defined as

$$x \overset{\mathrm{R}}{\sim} y \quad \text{if and only if} \quad x \in U_i \Leftrightarrow y \in U_i, \quad \forall\, U_i \in \mathcal{U}.$$

As explained in [37, Section 3.2, p27], topological features of the space Ω can computed with the approximation $P_{\mathcal{U}}(\Omega) = \Omega/R$, and in the limit, using finer coverings, the whole space Ω can be approximated [37, Section 3.3, p30]. Now, broadly speaking, by designing a groupoid G (with $G^{(0)} = F_{V_\psi f} = \mathrm{graph}(V_\psi f|_{\mathrm{supp} V_\psi f})$) mirroring the properties of a covering $\mathcal{U} = \{U_i\}$ of Ω, we can construct an approximation space $M^G_{V_\psi f}$ with similar topological features as $P_{\mathcal{U}}(\Omega) = \Omega/R$. With this scenario, we can now conjecture the feasibility of using $M^G_{V_\psi f} = F_{V_\psi f}/G$ for estimating topological features of Ω together with adequate groupoid G with $G^{(0)} = F_{V_\psi f}$, and adequate conditions on the density of $\mathrm{Image}(\phi) \subset \Omega$. We also notice that a main motivation for studying the topological features of the cloud $M^G_{V_\psi f}$ is to understand the parameter space Ω seen in the dimensionality reduction context. The Theorem 2.6 delivers a strategy for this task using the C^*-algebra $\mathcal{A} \rtimes_{lt} C(G)$ in order to study the properties of $M^G_{V_\psi f}$ with respect to the components f_i and Ω_i.

Now, given a family of manifolds $\{\Omega_i\}_{i=1}^k$, $\Omega_i \subset [-1,1]^d$, and paths $\phi_i : [0,1] \to \Omega$ with corresponding functions f_i (defined as in the previous paragraph), the sum $f = \sum_{i=1}^k f_i$ leads to the study of a space M_f (resp. F_f) resulting from a particular type of combination of the spaces M_{f_i} (resp. F_{f_i}). The analysis of M_f can be performed with the Theorem 2.6, which provides an explicitly understanding of the assembling process of the spaces $F_{V_\psi f_i}$ into a single structure $M^G_{V_\psi f}$, and whose topological properties can be studied with $\mathcal{A} \rtimes_{lt,r} G$. We will see very preliminary initial steps for such an setting in the toy example presented in section 4.

3 Persistence Homology and AF-Algebras

We need to study now the problem of implementing, in a practical and computationally feasible environment, the concepts we have developed for signal analysis with groupoid crossed products. For this task, we need to apply basic ideas of C^*-algebras for the analysis of finite structures. As we will see, we consider for this problem, the theory of AF-algebras, which has a rich and well developed theoretical framework. Another basic component in our strategy is persistent homology, which will be another crucial theoretical and algorithmic tool, with a readily available efficient computational setting.

First, we recall basic concepts on persistent homology as a important new development in computational topology for extracting qualitative information from a point cloud data $X = \{x_i\}$. As we have already discussed, our interest lies mostly on datasets arising from signal processing problems. The concepts of functional clouds, foliated partitions, and modulation maps have formalized this signal processing setting. As as we have seen, an important strategy for studying these geometrical objects is the setting of C^*-algebras, K-theory, and the corresponding philosophy of noncommutative geometry and topology. In order to implement these ideas in a computational framework we need to adapt the tools of persistent homology using the setting of C^*-algebras. We use the theory of posets and AF-algebras for the implementation, in the context of persistent homology, a framework for point cloud data analysis with C^*-algebras and K-theory.

Simplicial and persistent homology

We first recall elementary concepts on simplicial homology as a basic homology theory used for constructing algebraic data from topological spaces (see [27] for similar material).

Remark 3.1 (Simplicial complexes)**.** A basic component in this context is a (finite) *abstract simplicial complex* which is a nonempty family of subsets K of a vertex set $V = \{v_i\}_{i=1}^m$ such that $V \subseteq K$ (here we simplify the notation and we identify the vertex v with the set $\{v\}$) and if $\alpha \in K, \beta \subseteq \alpha$, then $\beta \in K$. The elements of K are denominated *faces*, and their *dimension* is defined as their cardinality minus one. Faces of dimension zero and one are called vertices and edges respectively. A *simplicial map* between simplicial complexes is a function respecting their structural content by mapping faces in one structure to faces in the other. These concepts represent combinatorial structures capturing the topological properties of many geometrical constructions. Given an abstract simplicial complex K, an explicit topological space is defined by considering the associated *geometric realization* or *polyhedron*, denoted by $|K|$. These are constructed by thinking of faces as higher dimensional versions of triangles or tetrahedrons in a large dimensional Euclidean spaces and gluing them according to the combinatorial information in K.

Remark 3.2 (Homology groups)**.** A basic analysis tool of a simplicial complex K, is the construction of algebraic structures for computing topological invariants, which are properties of $|K|$ that do not change under homeomorphisms and even continuous deformations. From an algorithmic point of view, we compute topological invariants of K by translating its combinatorial structure in the language of linear algebra. For this task, a basic scenario is to consider the following three steps. First, we construct the *groups of*

k-*chains* C_k, defined as the formal linear combinations of k-dimensional faces of K with coefficients in a commutative ring R (with e.g. $R = \mathbb{Z}$, or $R = \mathbb{Z}_p$). We then consider linear maps between the group of k-chains by constructing the *boundary operators* ∂_k, defined as the linear transformation which maps a face $\sigma = [p_0, \cdots p_n] \in C_n$ into C_{n-1} by $\partial_n \sigma = \sum_{k=0}^{n} (-1)^k [p_0, \cdots, p_{k-1}, p_{k+1}, \cdots p_n]$. As a third step, we construct the *homology groups* defined as the quotient $H_k := \ker(\partial_k)/\mathrm{im}(\partial_{k+1})$. Finally, the concept of *number of* k-*dimensional holes* are defined using the rank of the homology groups, $\beta_k = \mathrm{rank}(H_k)$ (Betti numbers). For instance, in a sphere we have zero 1-dimensional holes, and one 2-dimensional hole. In the case of a torus, there are two 1-dimensional holes, and one 2-dimensional hole.

3.1 Basics on Persistent Homology

In many application problems a main objective is to analyze experimental datasets $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ and understand their content by computing qualitative information. Topological invariants are important characteristics of geometrical objects, and their properties would be fundamental tools for understanding experimental datasets. The major problem when computing topological invariants of datasets is their finite characteristics and the corresponding inherent instability when computing homological information. Indeed, minor variations (e.g. noise and error in measurements) on how topological structures are constructed from X, could produce major changes on the resulting homological information. Persistent homology [8, 18, 17] is an important computational and theoretical strategy developed over the last decade for computing topological invariants of finite structures. We now describe its motivations, main principles, and theoretical background.

Motivations

A major problem when using tools from simplicial homology for studying a dataset $X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$ is the fact that we do not have a simplicial complex structure at hand. If we assume that X is sampled from a manifold (e.g. $X \subset \mathcal{M}$, with $\mathcal{M}$ being a submanifold of $\mathbb{R}^n$), a main objective would be to compute homological information of $\mathcal{M}$ using only the dataset X. We remark that more generalized settings, where $\mathcal{M}$ is not necessarily a manifold, are fundamental cases for many applications and experimental scenarios. But we can discuss, for illustration purposes, the simplified situation of $\mathcal{M}$ being a manifold. We also notice that the crucial problem of finding density conditions for X to be a meaningful sampling set of a manifold $\mathcal{M}$ has been recently addressed in [47], and we discuss these issues later in this report.

Attempting to construct a simplicial complex structure from X can be a very difficult problem. A simple strategy would be to consider the homology of the spaces

$$\mathbb{X}_\epsilon = \cup_{i=1}^m B(x_i, \epsilon),$$

where a ball $B(x_i, \epsilon)$ of radius ϵ is centered around each point of X. A naive approach would be to try to find an optimal ϵ_o such that the homology of $\mathbb{X}_{\epsilon_o}$ corresponds to the homology of $\mathcal{M}$. But this approach is highly unstable, as different homological values might be obtained when considering small perturbations of ϵ_o.

In proposal in *persistent homology* is to consider topological information for all $\epsilon > 0$ simultaneously, and not just a single value ϵ_o. The key concept is that a general homological overview for all values $\epsilon > 0$ is a useful tool when studying the topology of finite datasets. From a computational point of view, estimating homological data for all continuous values $\epsilon > 0$ might sound unreasonable, but there are two crucial remarks for implementing these ideas in an efficient computational framework. On the one hand, despite the fact that we are considering a continuous parameter $\epsilon > 0$, it can be verified that for a given dataset X, there is actually only a finite number of non-homeomorphic simplicial complexes

$$K_1 \subset K_2 \subset \cdots \subset K_r$$

(which is the concept of a *filtration* to be explicitly defined later on) that can be constructed from $\{\mathbb{X}_\epsilon, \epsilon > 0\}$. On the other hand, another crucial property is that the persistent homology framework includes efficient computational procedures for calculating homological information of the whole family $K_1 \subset K_2 \subset \cdots \subset K_r$, [57].

We also remark that, given a parameter ϵ with corresponding set $\mathbb{X}_\epsilon$, there are various topological structures useful for studying homological information of X. In particular, an efficient computational construction is given by the *Vietoris-Rips complexes* $R_\epsilon(X)$, defined with X as the vertex set, and setting the vertices $\sigma = \{x_0, \ldots, x_k\}$ to span a k-simplex of $R_\epsilon(X)$ if $d(x_i, x_j) \leq \epsilon$ for all $x_i, x_j \in \sigma$. For a given ϵ_k the Vietoris-Rips complex $R_{\epsilon_k}(X)$ provides an element of the filtration $K_1 \subset K_2 \subset \cdots \subset K_r$, with $K_k = R_{\epsilon_k}(X)$. In conclusion, there is only a finite set of positive values $\{\epsilon_i\}_{i=1}^r$, that describe homological characteristics of X, each of which generate a Vietoris Rips complex $\{K_i\}_{i=1}^m$ representing the topological features of the family $\{\mathbb{X}_\epsilon, \epsilon > 0\}$. Therefore, the topological analysis of a point cloud data X boils down to the analysis of a filtration $K_1 \subset K_2 \subset \cdots \subset K_r$, which is the main object of study in persistent homology. We now describe the main conceptual ingredients in this framework.

Conceptual setting

The input in the persistent homology framework is a *filtration* of a simplicial complex K, defined as a nested sequence of subcomplexes $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_r = K$. Given a simplicial complex K, we recall that the boundary operators ∂_k connect the chain groups C_k, and define a *chain complex*, denoted by C_*, and depicted with the diagram:

$$\ldots C_{k+1} \xrightarrow{\partial_{k+1}} C_k \xrightarrow{\partial_k} C_{k-1} \to \ldots.$$

Recall that given a chain complex C_* one defines the k-cycle groups and the k-boundary groups as $Z_k = \ker \partial_k$, and $B_k = \mathrm{im}\partial_{k+1}$ respectively. As we have nested subgroups $B_k \subseteq Z_k \subseteq C_k$, the k-homology group $H_k = Z_k/B_k$ is well defined.

There are several basic definitions required for the setting of persistent homology. A *persistent complex* is defined as a family of chain complexes $\{C_*^i\}_{i\geq 0}$ over a commutative ring R, together with maps

$$f^i : C_*^i \to C_*^{i+1} \quad \text{related as} \quad C_*^0 \xrightarrow{f_0} C_*^1 \xrightarrow{f_1} C_*^2 \xrightarrow{f_2} \ldots,$$

or more explicitly, described with the following diagram

$$
\begin{tikzcd}
\vdots \arrow[d, "\partial_3"] & \vdots \arrow[d, "\partial_3"] & \vdots \arrow[d, "\partial_3"] & \\
C_2^0 \arrow[r, "f^0"] \arrow[d, "\partial_2"] & C_2^1 \arrow[r, "f^1"] \arrow[d, "\partial_2"] & C_2^2 \arrow[r, "f^2"] \arrow[d, "\partial_2"] & \dots \\
C_1^0 \arrow[r, "f^0"] \arrow[d, "\partial_1"] & C_1^1 \arrow[r, "f^1"] \arrow[d, "\partial_1"] & C_1^2 \arrow[r, "f^2"] \arrow[d, "\partial_1"] & \dots \\
C_0^0 \arrow[r, "f^0"] \arrow[d, "\partial_0"] & C_0^1 \arrow[r, "f^1"] \arrow[d, "\partial_0"] & C_0^2 \arrow[r, "f^2"] \arrow[d, "\partial_0"] & \dots \\
0 \arrow[r, "f^0"] & 0 \arrow[r, "f^1"] & 0 \arrow[r, "f^2"] & \dots
\end{tikzcd}
$$

We remark that, due to the applications we have in mind, we will assume that chain complexes are trivial in negative dimensions. Given a filtration of a simplicial complex K, a basic example of a persistent complex is given by considering the functions f^i as the inclusion maps between each simplicial complex in the nested sequence $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_r = K$. Another fundamental concept is a *persistent module*, defined as a family of R-modules M^i and homomorphisms $\phi^i : M^i \to M^{i+1}$. We say that the persistent module is of *finite type* if each M^i is finitely generated, and the maps ϕ^i are isomorphisms for $i \geq k$ and some integer k. The basic example of a persistent module is given by the homology of the simplicial complexes of a filtration. We now define the *p-persistent homology group* of K_i as the group

$$H_k^{i,p} = Z_k^i / (B_k^{i+p} \cap Z_k^i),$$

where Z_k^i and B_k^i stand respectively for the k-cycles and k-boundaries groups in C^i. This group can also be described in terms of the inclusions $K^i \subset K^{i+p}$, their induced homomorphisms $f_k^{i,p} : H_k^i \to H_k^{i+p}$, and the corresponding relation

$$\mathrm{im}(f_k^{i,p}) \cong H_k^{i,p}.$$

These persistent homology groups contain homology classes that are stable in the interval i to $i+p$: they are born before the "time" index i and are still alive at $i+p$. Persistent homology classes alive for large values of p, are stable topological features of X, while classes alive only for small values of p are unstable or noise-like topological components. We will see, in the following paragraphs, alternative views for explaining generalized versions of persistent objects as functors between special categories.

The output of the persistent homology algorithm are representations of the evolution, with respect to the parameter $\epsilon > 0$, of the topological features of X. These representations are depicted with *persistent diagrams* indicating, for each homology level k, the amount and stability of the different k-dimensional holes of the point cloud X. We now present a more precise explanation of the concepts related to persistent diagrams and some of its properties.

The main task we now describe is the analysis of persistent homology groups by capturing their properties in a single algebraic entity represented by a finitely generated module. Recall that a main objective of persistent homology is to construct a summary of the evolution (with respect to ϵ) of the topological features of X using the sets $\{\mathbb{X}_\epsilon, \epsilon > 0\}$. This property is analyzed when constructing, with the homology groups of the complexes K_i, a module over the polynomial ring $R = \mathbb{F}[t]$ with a field $\mathbb{F}$. The general setting for this procedure is to consider a persistent module $M = \{M^i, \phi_i\}_{i\geq 0}$ and construct the graded module $\alpha(M) = \bigoplus_{i\geq 0} M^i$ over the graded polynomial ring $\mathbb{F}[t]$, defined with the action of t given by the shift $t\cdot(m^0, m^1, \dots) = (0, \phi^0(m^0), \phi^1(m^1), \dots)$. The crucial property of this construction is that α is a functor that defines an equivalence of categories between the category of persistent modules of finite type over $\mathbb{F}$, and the category of finitely generated non-negatively graded modules over $\mathbb{F}[t]$. In the case of a filtration of complexes K_0 to K_r, this characterization of persistent modules provides the finitely generated $\mathbb{F}[t]$ module:

$$\alpha(M) = H_p(K_0) \oplus H_p(K_1) \oplus \cdots \oplus H_p(K_r).$$

These modules are now used in a crucial step that defines and characterizes the output of persistent homology. The main tool is the well-know structure theorem characterizing finitely generated modules over principle ideal domains (this is why we need $\mathbb{F}$ to be a field). This property considers a finitely generated non-negatively graded module $\mathfrak{M}$, and ensures that there are integers $\{i_1, \dots, i_m\}$, $\{j_1, \dots, j_n\}$, $\{l_1, \dots, l_n\}$, and an isomorphism:

$$\mathfrak{M} \cong \bigoplus_{s=1}^{m} \mathbb{F}[t](i_s) \oplus \bigoplus_{r=1}^{n} (\mathbb{F}[t]/(t^{l_r}))(j_r).$$

This decomposition is unique up to permutation of factors, and the notation $\mathbb{F}[t](i_s)$ denotes an i_s shift upward in grading. The relation with persistent homology is given by the fact that when a persistent homology class τ is born at K_i and dies at K_j it generates a torsion module of the form $\mathbb{F}[t]\tau/t^{j-i}(\tau)$. When a class τ is born at K_i but does not die, it generates a free module of the form $\mathbb{F}[t]\tau$.

We can now explain the concept of persistent diagrams using an additional characterization of $\mathbb{F}[t]$-modules. We first define a P-interval as an ordered pair (i, j) where $0 \leq i < j$ for $i, j \in \mathbb{Z} \cup \{\infty\}$. We now construct the function Q mapping a P-interval as $Q(i, j) = (\mathbb{F}[t]/t^{j-i})(i)$, $Q(i, \infty) = \mathbb{F}[t](i)$, and for a set of P-intervals $S = \{(i_1, j_1), \dots, (i_n, j_n)\}$, we have the $\mathbb{F}[t]$-module

$$Q(S) = \bigoplus_{\ell=1}^{n} Q(i_\ell, j_\ell).$$

This map Q turns out to be a bijection between the sets of finite families of P-intervals and the set of finitely generated graded modules over $\mathbb{F}[t]$.

Now, we can recap all these results by noticing that the concept of persistent diagrams can be described as the corresponding set of P-intervals associated to the finitely generated graded module over $\mathbb{F}[t]$, constructed with the functor α from a given filtration $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_r = K$. There are several graphical representations for persistent diagrams, and two well known examples are the so called barcodes, and triangular regions of index-persistent planes.

Remark 3.3 (Stability of persistent diagrams)**.** A crucial property in persistent homology is the concept of *stability of persistent diagrams*. We recall that for a topological space X, and a map $h : X \to \mathbb{R}$, we say that h is *tame* if the homology properties of $\{X_\epsilon, \epsilon > 0\}$, for $X_\epsilon = h^{-1}(]-\infty, \epsilon[)$, can be completely described with a finite family of sets $X_{a_0} \subset X_{a_1}, \subset \cdots \subset X_{a_r}$, where the positive values $\{a_i\}_{i=0}^r$ are *homology critical points.* If we denote the *persistent diagram* for X and $h : X \to \mathbb{R}$, as $\mathrm{dgm}_n(h)$, we have a summary of the *stable and unstable* holes generated by the filtration

$$X_{a_0} \subset X_{a_1}, \subset \cdots \subset X_{a_r}$$

(see [17]). With these concepts, the *stability of persistent diagrams* is a property indicating that small changes in the persistent diagram $\mathrm{dgm}_n(h)$ can be controlled with small changes in the tame function $h : X \to \mathbb{R}$ (see [11] for details on the stability properties of persistent diagrams).

An important theoretical and engineering problem to investigate is the sensibility of the persistent homology features of X_f when applying signal transformations to f. This is in relation to the question of finding useful signal invariants using the persistent diagram of X_f. For instance, in the case of audio analysis, a crucial task is to understand the effects in the persistent diagram of X_f when applying audio transformations to f as, for instance, delay filters or convolution transforms (e.g room simulations). This task requires both theoretical analysis and numerical experiments. For a conceptual analysis, a possible strategy is to consider these recent theorems explaining the stability of persistent diagrams.

Generalizations with functorial properties

In order to design useful generalizations of persistent homology, it is important to understand its setting in a deeper conceptual level. A recent formulation, providing the core features of persistent homology, has been presented in [8], and describes this concept as a functor between well chosen categories. Indeed, a crucial aspect of persistent homology is the association from an index set to a sequence of homology groups constructed from a filtration $\emptyset = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_r = K$. An important generalization of this construction considers a general partially ordered set P as an index set which we associate to a family of objects in a given category $\mathbf{C}$. Notice that we can consider the partially ordered set P as a category $\mathbf{P}$, whose objects are P, and a morphism from x to y is defined whenever $x \leq y$. With this setting, a P-persistent object in $\mathbf{C}$ is defined as a functor $\Phi : \mathbf{P} \to \mathbf{C}$, described also as a family of objects $\{c_x\}_{x \in P}$ in $\mathbf{C}$, and morphisms $\phi_{xy} : c_x \to c_y$, when $x \leq y$.

These concepts are of fundamental importance for extending the main ideas of persistent homology in more general situations. Notice that in standard persistent homology we use the partial ordered sets $P = \mathbb{N}$ or $P = \mathbb{R}$, but important extensions have been recently developed in the context of multidimensional persistence. Here, we consider multidimensional situations where the partial ordered sets are, for instance, $P = \mathbb{N}^k$ or $P = \mathbb{R}^k$, $k > 1$. These developments are motivated by multiple practical considerations, such as the analysis of point cloud using both density estimations and the Vietoris Rips Complex construction.

3.2 AF-Algebras in the Persistent Homology Framework

The objective now is to use some basic ideas on C^*-algebras discussed in Section 2 in combination with the framework of persistent homology. The main task is to use the basic input of persistent homology, a filtration $K_1 \subset K_2 \subset \cdots \subset K_r$, and construct an associated sequence of C^*-algebras. Given a simplicial complex, there are several strategies for constructing an associated C^*-algebra. We follow the method, presented in [37, 53], which consists of building a poset structure, together with its associated Bratelli diagram and AF-algebra. We remark that other alternatives are available, for instance, the concept of noncommutative simplicial complex has been introduced in [14].

There are two basic steps for implementing this program. First, we remark that there is a close interaction between the concept of simplicial complex and a poset [55]. Given a poset P, a simplicial complex $K(P)$ (the *order complex*), is constructed by considering the set of vertices as the elements of P, and its faces as the totally ordered subsets (chains) of P. Inversely, given a simplicial complex K, we can build a poset $P(K)$ (the *face poset*) by considering the nonempty faces ordered by inclusion (see [55] for additional details). The second step is to construct a Bratelli diagram from a poset, as discussed in [37], which represents an AF-algebra containing all information from a topological space encoded in an algebraic structure.

The framework of AF-algebras and posets can be considered as a finitary version of basic ideas in noncommutative geometry [37]. Recall that $\mathcal{A}$ is an approximately finite (AF) dimensional algebra if there exist an increasing sequence

$$\mathcal{A}_0 \overset{I_0}{\hookrightarrow} \mathcal{A}_1 \overset{I_1}{\hookrightarrow} \ldots \overset{I_{n-1}}{\hookrightarrow} \mathcal{A}_n \overset{I_n}{\hookrightarrow} \ldots$$

of finite dimensional C^*-subalgebras of $\mathcal{A}$, with I_k injective $*$-morphisms and $\mathcal{A} = \overline{\bigcup_n \mathcal{A}_n}$. Any finite dimensional C^*-algebras is of the form $\oplus_i M_{n_i}$, where M_{n_i} is the full $n_i \times n_i$ matrix algebra. The complete structure of an AF-algebra includes the matrix algebras $\mathcal{A}_k$ and the injective morphisms I_k, and can be encoded in a representation denominated Bratelli diagram (see [37]). We can now describe the interaction between simplicial complexes, posets, and their Bratelli diagrams in the framework of persistent homology. The following diagram is a summary of the three basic components:

$$\begin{array}{lccccccc}
\text{Simplicial Complexes:} & K_0 & \longrightarrow & K_1 & \longrightarrow & \ldots & \longrightarrow & K_n \\
 & \downarrow & & \downarrow & & & & \downarrow \\
\text{Face Posets:} & P(K_0) & \longrightarrow & P(K_1) & \longrightarrow & \ldots & \longrightarrow & P(K_n) \\
 & \downarrow & & \downarrow & & & & \downarrow \\
\text{AF-Algebras:} & \mathcal{A}^0 & \longrightarrow & \mathcal{A}^1 & \longrightarrow & \ldots & \longrightarrow & \mathcal{A}^n
\end{array}$$

Each horizontal arrow is an injective inclusion, and the vertical arrows represent the two main constructions: first we build posets from simplicial complexes, and then AF-algebras are computed from posets (using Bratelli diagrams as a main tool). Each AF-algebra $\mathcal{A}^k$, has its own decomposition with finite dimensional matrix algebras $\mathcal{A}_i^k$, and injective $*$-morphisms I_i^k:

$$\mathcal{A}_0^k \overset{I_0^k}{\hookrightarrow} \mathcal{A}_1^k \overset{I_1^k}{\hookrightarrow} \ldots \overset{I_{n-1}^k}{\hookrightarrow} \mathcal{A}_n^k \overset{I_n^k}{\hookrightarrow} \ldots$$

4 A Computational Toy Example

We present now an illustrative example of a filtering procedure and its interaction with topological measurements of a dataset X_f [31]. This toy example only shows a very limited amount of aspects of our conceptual developments, but the goal is to provide an initial sketch illustrating basic features of our setting. We consider the function $f = (1-\alpha)g + \alpha h$, $\alpha \in [0,1]$ to be a sum of two functions g and h, where the datasets X_g and X_h are sampled from spaces homeomorphic to a sphere $\mathbb{S}^2$ and a torus $\mathbb{T}^2$ respectively.

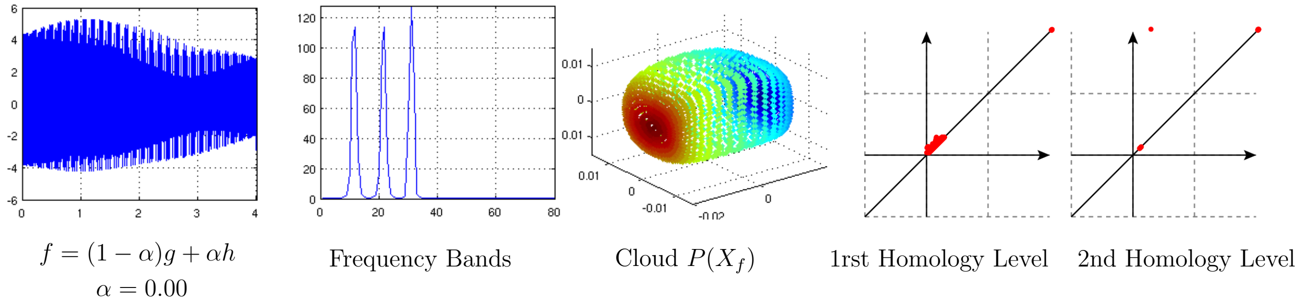


Figure 1: $f = g$, and $X_f \subset \mathcal{M}$ with $\mathcal{M}$ homeomorphic to $\mathbb{S}^2$

We construct X_g and X_h, as described in our examples on modulation maps in Section 1.2. For instance, in Fig. 1, each element x of the point cloud data X_f is a signal whose main frequency content is located in three frequency bands depicted in the second plot in the Fig. 1. We additionally design each element $x \in X_g$ and $y \in X_h$ such that that their frequency content do not overlap. For example, in Fig. 2, the second diagram shows the six different frequency bands for the signal $x + y$: the first three bands corresponding to a typical element $x \in X_g$, and the other bands correspond to elements $y \in X_h$.

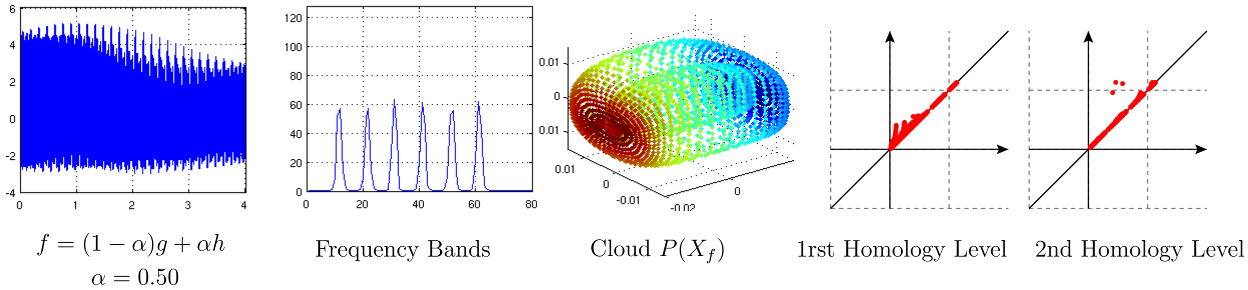
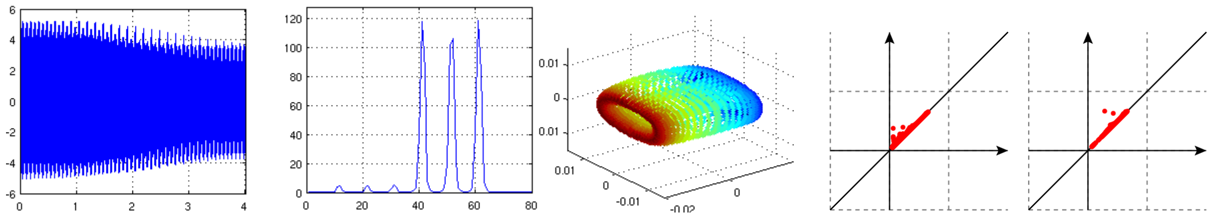


Figure 2: $f = (g + h)/2$, and X_f as an intermediate structure

For these examples, the variations of the parameter α corresponds to a filtering process, where we selectively remove (or add) the component g (or h) from the signal f. The topological effects can be seen by studying the persistent homology diagrams of X_f. For each Figure 1, 2, and 3, we have diagrams representing the first and second homology level. With this information we have an estimation for the number of one and two dimensional holes in X_f.



$f = (1-\alpha)g + \alpha h$
$\alpha = 1.00$

Frequency Bands

Cloud $P(X_f)$

1rst Homology Level

2nd Homology Level

Figure 3: $f = h$, and $X_f \subset \mathcal{M}$ with $\mathcal{M}$ homeomorphic to $\mathbb{T}^2$

In the case of Fig. 1, the persistent diagram for X_f shows a clear stable two dimensional hole, and only noise like one dimensional holes. As previously mentioned, this corresponds to a spherical structure for X_f. For the Fig. 3, we have two, closely related, one dimensional holes, and additionally two 2-dimensional holes, which (approximately) corresponds to a torus structure. The persistent homology diagrams for the intermediate structure $X_{(g+h)/2}$ is depicted in Fig. 2, where several two dimensional holes are present.

Conclusions and Future Work

Our main property, described in Theorem 2.6, explains basic conceptual interactions between a functional cloud $M^G_{V_\psi f} = F_{V_\psi f}/G$ for an element f in a Hilbert space H, and its components $f = \sum_{i=1}^k f_i$. In this property, we use a groupoid G with $G^{(0)} = F_{V_\psi f} := \mathrm{graph}(V_\psi f|_{\mathrm{supp} V_\psi f})$, and $V_\psi f$ the voice transform of f (e.g. wavelet, Gabor analysis, etc). These results are a first step in our strategy for using noncommutative C^*-algebras in time-frequency analysis. Among the many questions to analyze, an important issue is the consideration of other algebras, besides $C_0(F_{V_\psi f_i})$, for capturing different type of features. Recall that the spaces $C^\infty(F_{V_\psi f_i}) \subset C_0(F_{V_\psi f_i}) \subset L^1(F_{V_\psi f_i})$ can be used to encode geometrical, topological, and measure theoretical properties, respectively. The general framework prepared in the Theorem 2.4 could be a way to address these possibilities. We remark that new results have been recently achieved in the setting of AF-algebras and *spectral triples*, which is a fundamental tool for accessing geometrical data using C^*-algebras (see [12] for the concept of spectral triples, and [9] for its interaction with AF-algebras).

We also remark that related developments have been recently achieved in the integration of time-frequency analysis and noncommutative geometry as explained in [39, 41, 40]. These novel research directions are complementary to the ones we follow, but the same tools from noncommutative geometry and noncommutative topology are considered. We also remark that there is another important trend of research developments studying the interactions between operator algebras and wavelet theory with a particular focus on multiresolution analysis (see e.g. [2]).

We also notice that new developments in pattern classification are investigating new type of invariants based on algebraic criteria (see e.g. [43]). Our framework is designed to consider these directions, and the basic tool is to exploit the flexibility of C^*-algebras for representing interactions between geometrical/topological and algebraic structures.

We finally remark that the fundamental domain of time-frequency transforms in harmonic analysis, and the new developments in persistent homology and dimensionality reduction, have shown powerful perspectives in their own domains. However, an adequate integration of these tools is necessary in order to resolve modern application and theoretical problems in signal processing and data analysis. We argue that concepts based on noncommutative C^*-algebras can play a role in this interaction.

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