# New Perspectives in Signal Processing Combining Dimensional Reduction and Persistent Homology

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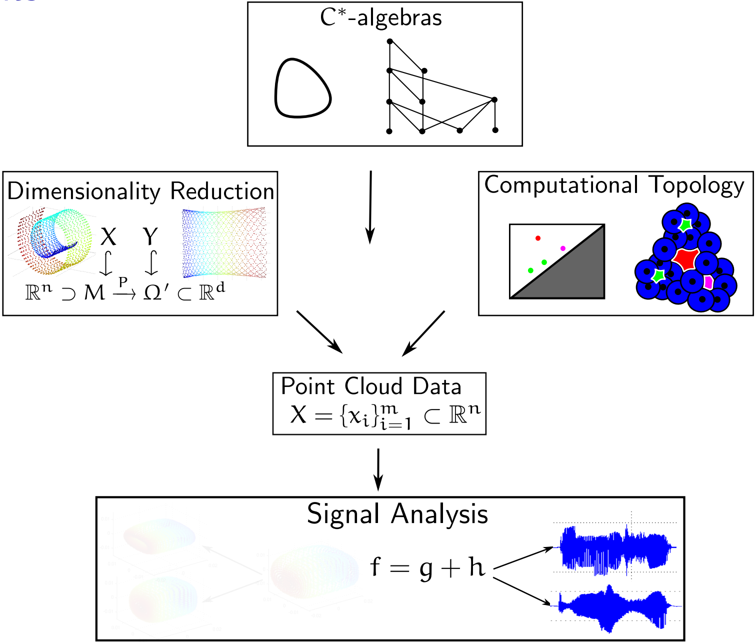


C*-algebras
Dimensionality Reduction
$X \quad Y$
$\mathbb{R}^n \supset M \xrightarrow{P} \Omega' \subset \mathbb{R}^d$
Computational Topology
Point Cloud Data
$X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$
Signal Analysis
$f = g + h$

# Dimensionality Reduction and Manifold Learning

Point Cloud Data $X = \{x_i\}_{i=1}^m \subset \mathcal{M} \subset \mathbb{R}^n$

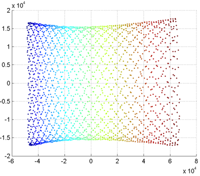
Hypothesis:

- $\mathcal{M}$ manifold, topological space (simplicial complex)
- $\dim(\mathcal{M}) = p \ll n$
- $\mathbb{R}^d \supset \Omega \xrightarrow{\mathcal{A}} \mathcal{M} \subset \mathbb{R}^n$ homeomorphism, $d < n$.

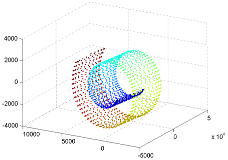
Objectives:

- Construct $Y = \{y_i\}_{i=1}^m \subset \Omega' \subset \mathbb{R}^d$, $d < n$
- $\mathbb{R}^n \supset \mathcal{M} \xrightarrow{P} \Omega' \subset \mathbb{R}^d$ homeomorphism (diffeomorphism)

$\mathrm{Y} \subset \Omega' \subset \mathbb{R}^2$




$\mathrm{X} \subset \mathcal{M} \subset \mathbb{R}^{1024}$



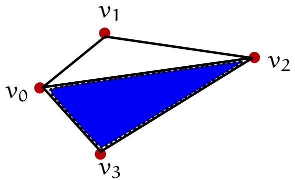

$$\mathbb{R}^d \supset \Omega \xrightarrow{\mathcal{A}} \mathcal{M} \subset \mathbb{R}^n$$

$P$

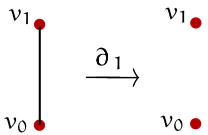
$$\mathbb{R}^d \supset \Omega'$$

# Simplicial Homology

Simplicial Complex $K = \{v_0, \ldots, v_3, [v_0, v_1], [v_1, v_2], \ldots, [v_0, v_2], [v_0, v_2, v_3]\}$




Group of k-Chains $C_k \cong \mathbb{R}^{d_k}$
$C_1 \cong \mathbb{R}^5, \quad C_0 \cong \mathbb{R}^4$




$\partial([v_0, v_1]) = v_1 - v_0$

Boundary Operator $\quad \partial_n : \mathbb{R}^k \to \mathbb{R}^\ell$
$\partial_n(\sigma) = \sum_{i=0}^{n} (-1)^i [v_0, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n]$

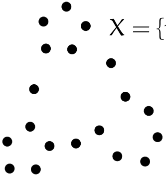
$\mathbb{R}^1 \xrightarrow{\partial_2} \mathbb{R}^5 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{0} 0$

k-Homology Group $H_k := \ker(\partial_k)/\mathrm{im}(\partial_{k+1}) \cong \mathbb{R}^d$

Betti Number $\beta_1 = \dim(H_1) = 1$

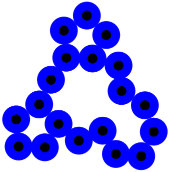
# Persistent Homology

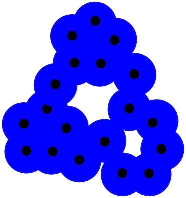
Point Cloud Data



$X = \{x_i\}_{i=1}^m$

$B(x_i, \epsilon)$ $\quad \epsilon < \epsilon' \quad$ $B(x_i, \epsilon')$



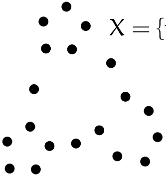


Vietoris-Rips Complex

$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_\ell = K$

# Persistent Homology

Point Cloud Data



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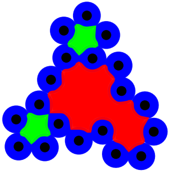
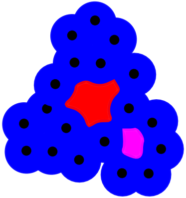
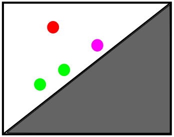




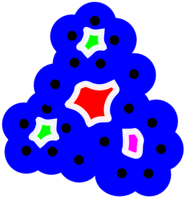
Vietoris-Rips Complex

$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_\ell = K$

persistent diagram (Homology 1rst level)



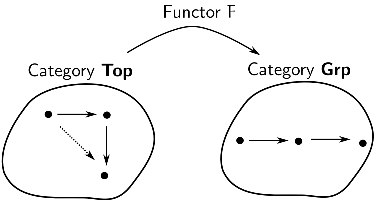
- stable 2-D hole
- unstable 2-D holes



# Simplicial and Persistent Homology (Revisited)

Categories and Functors

**Top**: Category of Topological Spaces
**Grp**: Category of Groups




**Proposition (Homology Functors)**

*The Homology Constructions are Functors* $H_k : Top \to Grp$

**Definition ($P$-persistent objects in $\mathcal{C}$)**

*$P$: partially ordered set.* **P**: *corresponding category for $P$.*
*A $P$-persistent object in $\mathcal{C}$ is a functor:*

$$\Phi : \mathbf{P} \to \mathcal{C}$$

*(family of objects $\{c_x\}_{x \in P} \subset \mathcal{C}$, and morphisms $\phi_{xy} : c_x \to c_y$, $x \le y$).*

# Persistence with AF-algebras

**Definition (C*-algebras)**

$(\mathcal{A}, +, \times)$ *Banach Space and an involution* $* : \mathcal{A} \to \mathcal{A}$ *with* $\|xx^*\| = \|x\|^2$
**Examples**: $M_n(\mathbb{C})$ **matrices,** $C(X)$ *continuous functions on a compact space,* $L(\mathcal{H})$ **bounded linear operators,** $L^1(\mathbb{R})$ *integrable functions, etc.*

**Definition (Finite Dimensional $C^*$-algebras and AF-algebras)**

*Finite dimensional algebra:* $\bigoplus_k M_{n_k}(\mathbb{C})$

*AF-algebra* $\quad \mathcal{A} : \ \mathcal{A}_0 \xrightarrow{I_0} \mathcal{A}_1 \xrightarrow{I_1} \dots$

**Definition (Persistence with AF-algebras)**

*Given a filtration* $K_0 \subset K_1 \subset \cdots \subset K_\ell$ *for a PCD* $X = \{x_i\}_{i=1}^m$*,*
*we construct a sequence of AF-algebras*

$$\mathcal{A}^0 \xrightarrow{f_0} \mathcal{A}^1 \xrightarrow{f_1} \dots \xrightarrow{f_{\ell-1}} \mathcal{A}^\ell$$

# The Gelfand-Naimark Functor

**Objective**

*Consider Persistence with AF-algebras for the analysis of a PCD*

$$X = \{x_i\}_{i=1}^{m} \subset \mathbb{R}^n$$

**Theorem (Gelfand-Naimark)**

`Haus`: *category of locally compact Hausdorff spaces*

`CsAlg` *category of commutative $C^*$-algebras.*

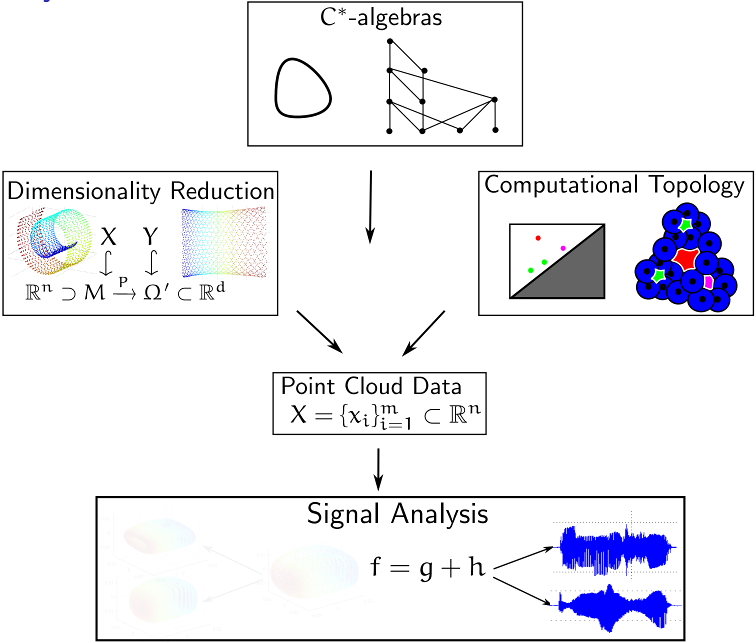
$$X \text{ l.c. H. topological space} \quad \leftrightarrow \quad C_0(X)\ C^*\text{-algebra}$$

**Equivalence of Categories**

$$\texttt{Haus} \quad \leftrightarrow \quad \texttt{CsAlg}$$

*but we can also do geometry (Serre-Swan Theorem) and much more.*

# PCD Analysis: Overview



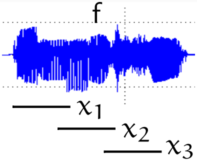
# Interaction with Signal Transforms: Motivations

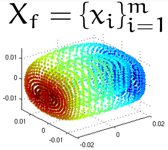
## (1-D Signal Analysis Example: Audio)

$$\mathcal{G}_g f(b,\omega) := \langle f, g_{b,\omega} \rangle = \int_{\mathbb{R}} f(t)\overline{g_{b,\omega}(t)}\, dt,$$

$$g_{b,\omega}(t) := g(t-b)e^{2\pi i \omega t}$$

$$X_f = \{x_i\}_{i=1}^m := \left\{ (f(t_j)g(t_j - k_i))_{j=1}^n \in \mathbb{R}^n \right\}_{i=1}^m$$



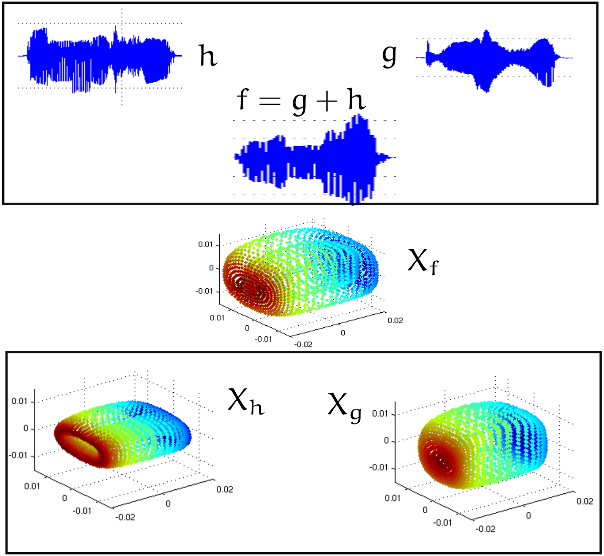



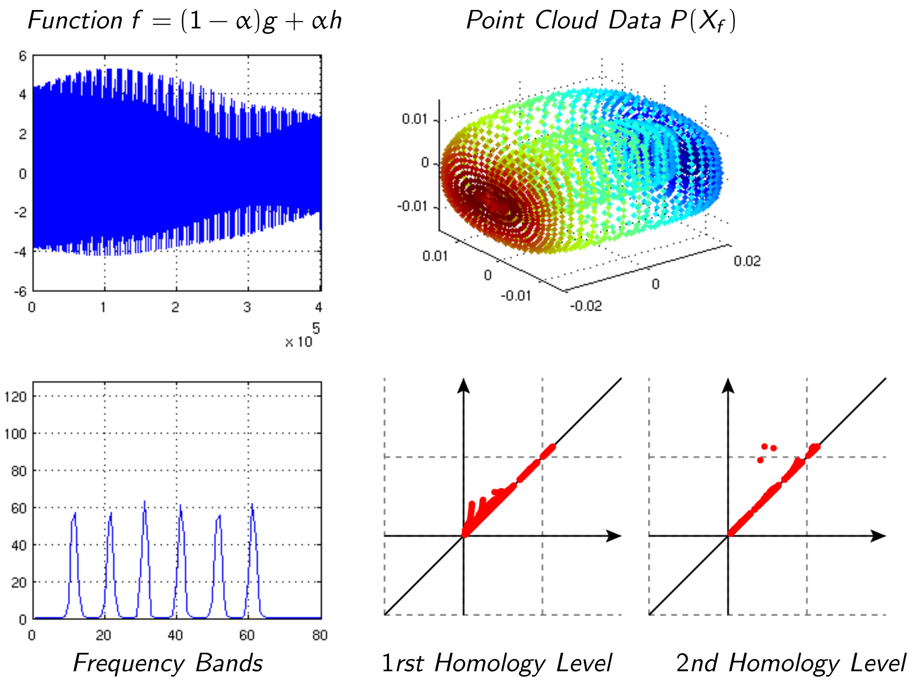

## (2-D Signal Analysis Example: Images)

$$F : [0,1]^2 \to [0,1] \quad (\textit{Grayscale image})$$

$$X_F := \{F(O_i) \in \mathbb{R}^n\}_{i=1}^m, \quad (\textit{Dataset of Image Patches})$$

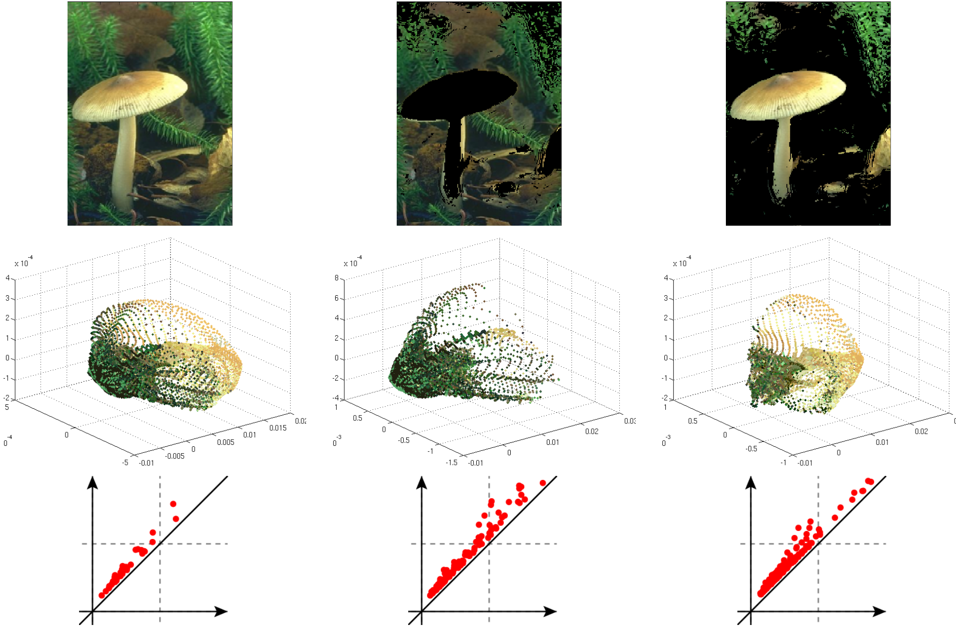
# Filtering and Homology Effects



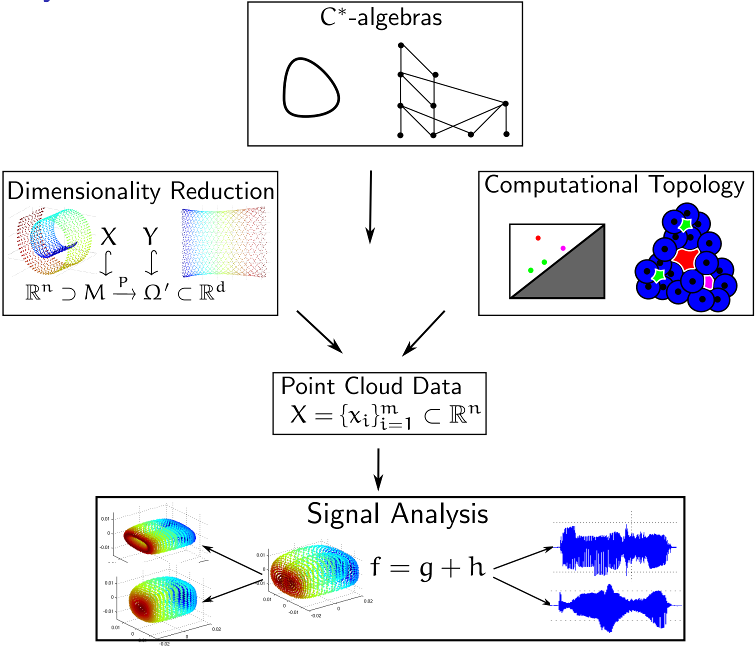


Function $f = (1-\alpha)g + \alpha h$
Point Cloud Data $P(X_f)$
6
4
2
0
-2
-4
-6
0
1
2
3
4
× 10^5
0.01
0
-0.01
0.01
0
-0.01
-0.02
0
0.02
120
100
80
60
40
20
0
0
20
40
60
80
Frequency Bands
1rst Homology Level
2nd Homology Level

# Image Processing Example



# PCD Analysis: Conclusion



# SIMULATIONS

# Appendix: Homology revisited

**Proposition (Homology Functors)**

*$\mathcal{C}$: Abelian Category*
*$C(\mathcal{C})$: category of complexes of $\mathcal{C}$*

$$(X, d_X): \qquad \cdots \to X_{k+1} \xrightarrow{d_{k+1}} X_k \xrightarrow{d_k} X_{k-1} \to \dots$$

*For $X \in C(\mathcal{C})$*
$Z_k(X) := Ker(d_k) \quad B_k(X) := Im(d_{k+1}) \quad H_k(X) := Z_k(X)/B_k(X)$

*Homology Functor* $\quad H_k : C(\mathcal{C}) \to \mathcal{C}$

**Definition ($P$-persistent objects in $\mathcal{C}$)**

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