



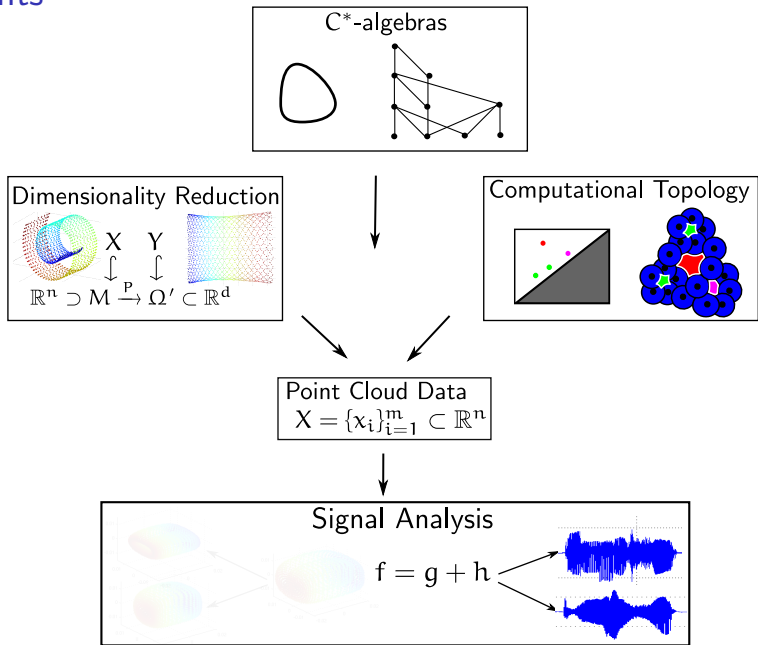
New Perspectives in Signal Processing Combining Dimensional Reduction and Persistent Homology

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Contents



Dimensionality Reduction and Manifold Learning

Point Cloud Data $X = \{x_i\}_{i=1}^m \subset \mathcal{M} \subset \mathbb{R}^n$

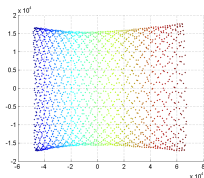
Hypothesis:

- ▶ \mathcal{M} manifold, topological space (simplicial complex)
- ▶ $\dim(\mathcal{M}) = p \ll n$
- ▶ $\mathbb{R}^d \supset \Omega \xrightarrow{\mathcal{A}} \mathcal{M} \subset \mathbb{R}^n$ homeomorphism, $d < n$.

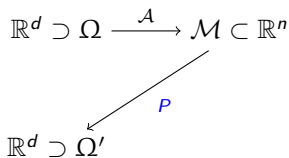
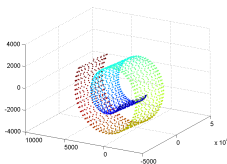
Objectives:

- ▶ Construct $Y = \{y_i\}_{i=1}^m \subset \Omega' \subset \mathbb{R}^d$, $d < n$
- ▶ $\mathbb{R}^n \supset \mathcal{M} \xrightarrow{P} \Omega' \subset \mathbb{R}^d$ homeomorphism (diffeomorphism)

$Y \subset \Omega' \subset \mathbb{R}^2$

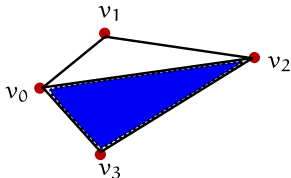


$X \subset \mathcal{M} \subset \mathbb{R}^{1024}$

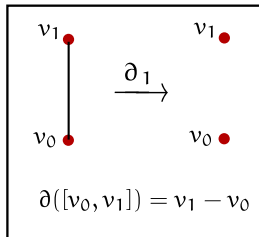


Simplicial Homology

Simplicial Complex $K = \{v_0, \dots, v_3, [v_0, v_1], [v_1, v_2], \dots, [v_0, v_2], [v_0, v_2, v_3]\}$



Group of k -Chains $C_k \cong \mathbb{R}^{d_k}$
 $C_1 \cong \mathbb{R}^5$, $C_0 \cong \mathbb{R}^4$



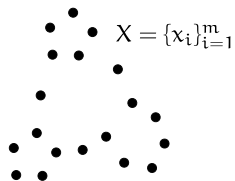
Boundary Operator $\partial_n : \mathbb{R}^k \rightarrow \mathbb{R}^\ell$ $\mathbb{R}^1 \xrightarrow{\partial_2} \mathbb{R}^5 \xrightarrow{\partial_1} \mathbb{R}^4 \xrightarrow{0} 0$
 $\partial_n(\sigma) = \sum_{i=0}^n (-1)^i [v_0, \dots, v_{i-1}, v_{i+1}, \dots, v_n]$

k -Homology Group $H_k := \ker(\partial_k) / \text{im}(\partial_{k+1}) \cong \mathbb{R}^d$

Betti Number $\beta_1 = \dim(H_1) = 1$

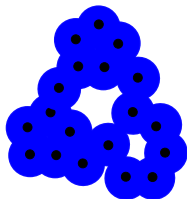
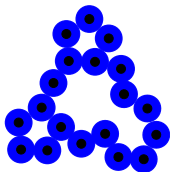
Persistent Homology

Point Cloud Data



Vietoris-Rips Complex

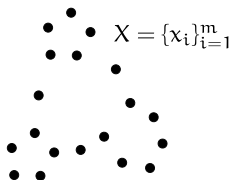
$B(x_i, \epsilon)$ $\epsilon < \epsilon'$ $B(x_i, \epsilon')$



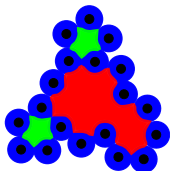
$\emptyset = K_0 \subset K_1 \subset \cdots \subset K_\ell = K$

Persistent Homology

Point Cloud Data

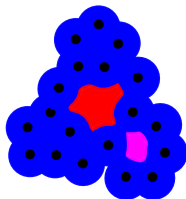


$B(x_i, \epsilon)$



$\epsilon < \epsilon'$

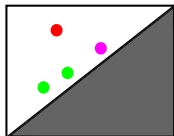
$B(x_i, \epsilon')$



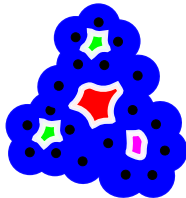
Vietoris-Rips Complex

$\emptyset = K_0 \subset K_1 \subset \dots \subset K_\ell = K$

persistent diagram (Homology 1st level)



- stable 2-D hole
- ● unstable 2-D holes

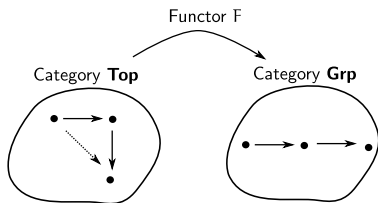


Simplicial and Persistent Homology (Revisited)

Categories and Functors

Top: Category of Topological Spaces

Grp: Category of Groups



Proposition (Homology Functors)

The Homology Constructions are Functors $H_k : \text{Top} \rightarrow \text{Grp}$

Definition (P -persistent objects in \mathcal{C})

P : partially ordered set. \mathbf{P} : corresponding category for P .

A P -persistent object in \mathcal{C} is a functor:

$$\Phi : \mathbf{P} \rightarrow \mathcal{C}$$

(family of objects $\{c_x\}_{x \in P} \subset \mathcal{C}$, and morphisms $\phi_{xy} : c_x \rightarrow c_y$, $x \leq y$).

Persistence with AF-algebras

Definition (C^* -algebras)

$(\mathcal{A}, +, \times)$ Banach Space and an involution $*$: $\mathcal{A} \rightarrow \mathcal{A}$ with $\|xx^*\| = \|x\|^2$

Examples: $M_n(\mathbb{C})$ matrices, $C(X)$ continuous functions on a compact space, $L(\mathcal{H})$ bounded linear operators, $L^1(\mathbb{R})$ integrable functions, etc.

Definition (Finite Dimensional C^* -algebras and AF-algebras)

Finite dimensional algebra: $\bigoplus_k M_{n_k}(\mathbb{C})$

AF-algebra $\mathcal{A} : \mathcal{A}_0 \xrightarrow{I_0} \mathcal{A}_1 \xrightarrow{I_1} \dots$

Definition (Persistence with AF-algebras)

Given a filtration $K_0 \subset K_1 \subset \dots \subset K_\ell$ for a PCD $X = \{x_i\}_{i=1}^m$, we construct a sequence of AF-algebras

$$\mathcal{A}^0 \xrightarrow{f_0} \mathcal{A}^1 \xrightarrow{f_1} \dots \xrightarrow{f_{\ell-1}} \mathcal{A}^\ell$$

The Gelfand-Naimark Functor

Objective

Consider *Persistence with AF-algebras* for the analysis of a PCD

$$X = \{x_i\}_{i=1}^m \subset \mathbb{R}^n$$

Theorem (Gelfand-Naimark)

Haus: category of locally compact Hausdorff spaces

CsAlg category of commutative C^* -algebras.

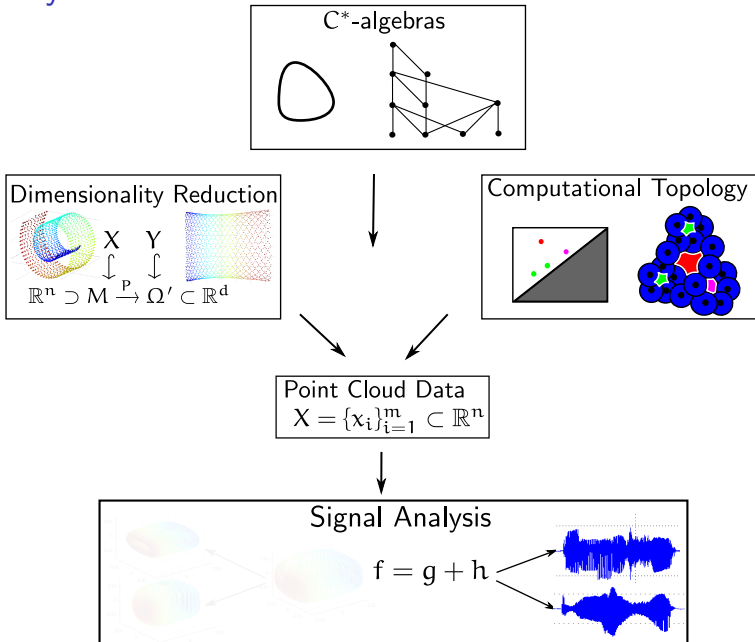
$$X \text{ l.c. H. topological space} \leftrightarrow C_0(X) \text{ } C^*\text{-algebra}$$

Equivalence of Categories

$$\text{Haus} \leftrightarrow \text{CsAlg}$$

but we can also do geometry (Serre-Swan Theorem) and much more.

PCD Analysis: Overview



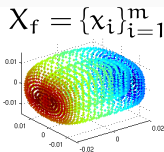
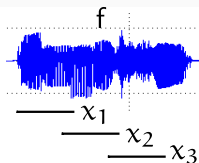
Interaction with Signal Transforms: Motivations

(1-D Signal Analysis Example: Audio)

$$\mathcal{G}_g f(b, \omega) := \langle f, g_{b, \omega} \rangle = \int_{\mathbb{R}} f(t) \overline{g_{b, \omega}(t)} dt,$$

$$g_{b, \omega}(t) := g(t - b) e^{2\pi i \omega t}$$

$$X_f = \{\chi_i\}_{i=1}^m := \{(f(t_j)g(t_j - k_i))_{j=1}^n \in \mathbb{R}^n\}_{i=1}^m$$

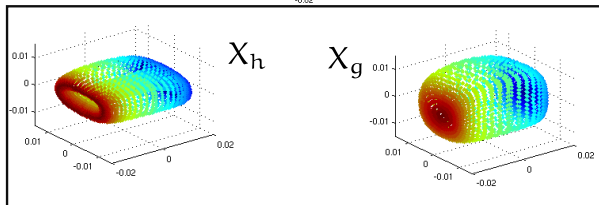
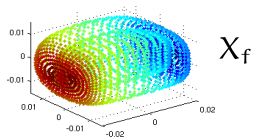
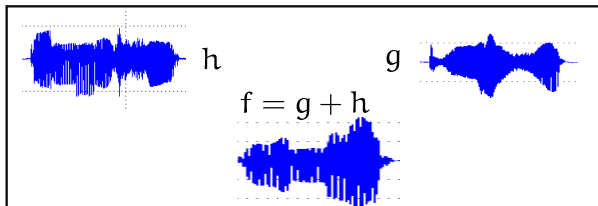


(2-D Signal Analysis Example: Images)

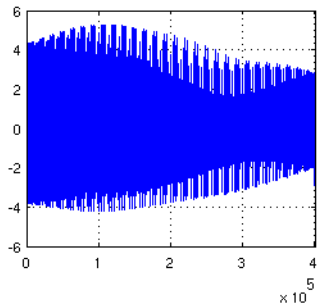
$$F : [0, 1]^2 \rightarrow [0, 1] \quad (\textit{Grayscale image})$$

$$X_F := \{F(O_i) \in \mathbb{R}^n\}_{i=1}^m, \quad (\textit{Dataset of Image Patches})$$

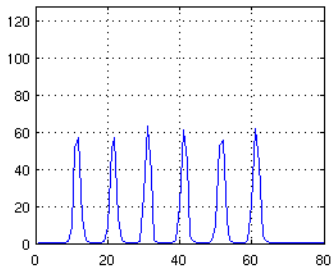
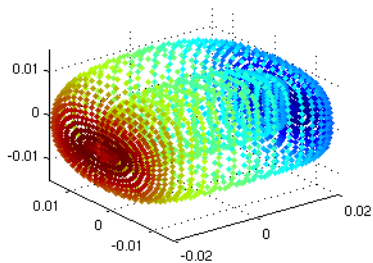
Filtering and Homology Effects



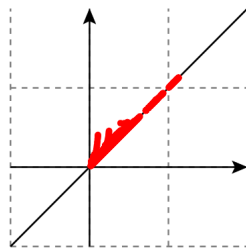
Function $f = (1 - \alpha)g + \alpha h$



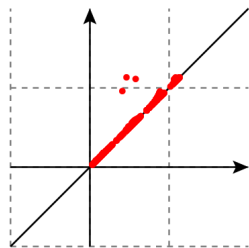
Point Cloud Data $P(X_f)$



Frequency Bands

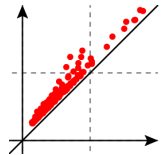
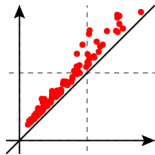
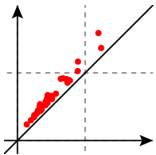
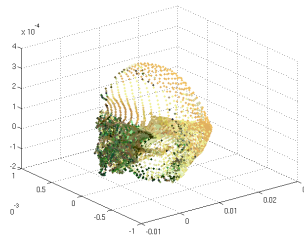
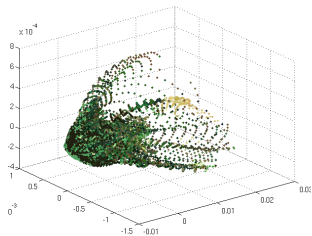
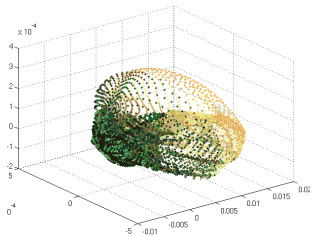


1st Homology Level

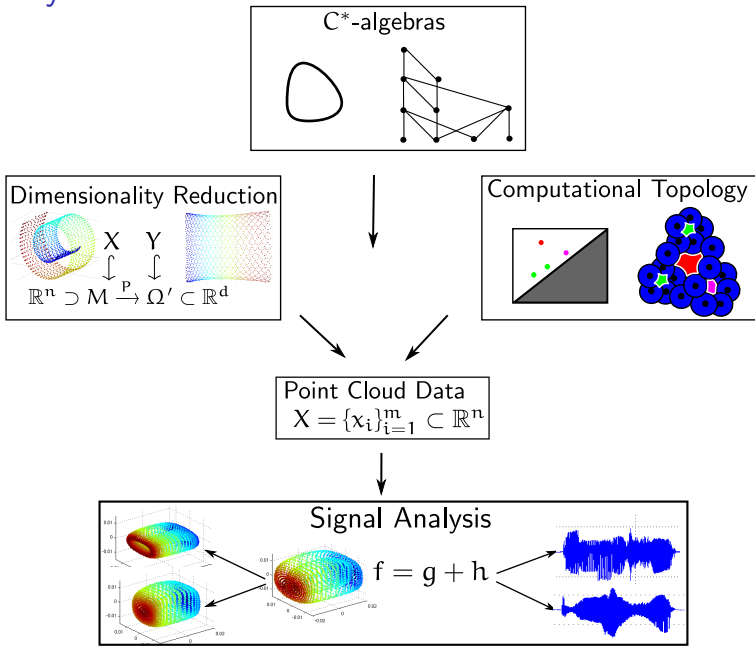


2nd Homology Level

Image Processing Example



PCD Analysis: Conclusion



SIMULATIONS

Appendix: Homology revisited

Proposition (Homology Functors)

\mathcal{C} : Abelian Category

$\mathcal{C}(\mathcal{C})$: category of complexes of \mathcal{C}

$$(X, d_X) : \quad \cdots \rightarrow X_{k+1} \xrightarrow{d_{k+1}} X_k \xrightarrow{d_k} X_{k-1} \rightarrow \cdots$$

For $X \in \mathcal{C}(\mathcal{C})$

$$Z_k(X) := \text{Ker}(d_k) \quad B_k(X) := \text{Im}(d_{k+1}) \quad H_k(X) := Z_k(X)/B_k(X)$$

$$\text{Homology Functor } H_k : \mathcal{C}(\mathcal{C}) \rightarrow \mathcal{C}$$

Definition (P -persistent objects in \mathcal{C})

P : partially ordered set. \mathbf{P} : corresponding category for P .

A P -persistent object in \mathcal{C} is a functor:

$$\Phi : \mathbf{P} \rightarrow \mathcal{C}$$

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