INFINITE RAMSEY THEORY

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Contents

1. Ramsey's theorem

1.1. The arrow-notation.

Definition 1.1. a) For a set X and a cardinal ν let

$$[X]^{\nu} = \{Y \subseteq X : |Y| = \nu\},$$

$$[X]^{\leq\nu} = \{Y \subseteq X : |Y| \leq \nu\},$$

and

$$[X]^{<\nu} = \{Y \subseteq X : |Y| < \nu\}.$$

b) For a natural number n and a cardinal μ, an *n-coloring on X with with μ colors* is a function $c : [X]^n \to \mu$. If μ is not specified (and not clear from the context), we usually mean $\mu = 2$. Also, if n is not specified (and not clear from the context), we mean $n = 2$. In particular, a *coloring on X* typically is a function $c : [X]^2 \to 2$ and an n-coloring on X typically is a function $c : [X]^n \to 2$.

c) If c is an n-coloring on X with μ colors, then $H \subseteq X$ is *c-homogeneous* or just *homogeneous* if c is constant on $[H]^n$. H is *homogeneous of color $i \in \mu$* if c is constant on H with value i.

d) Let κ and λ be cardinals. We write $\lambda \to (\kappa)^n_\mu$ if for every n-coloring on λ with μ colors there is a homogeneous set of size κ.

Lemma 1.2. *Let $\kappa' \leq \kappa$, $\lambda' \geq \lambda$, and $\mu' \leq \mu$. Then $\lambda \to (\kappa)^n_\mu$ implies $\lambda' \to (\kappa')^n_{\mu'}$*

Proof. Easy. □

Lemma 1.3. *Let κ, λ, n, μ be as before and let $n' \leq n$. Then $\lambda \to (\kappa)^n_\mu$ implies $\lambda \to (\kappa)^{n'}_\mu$*

Proof. Let c' be an n'-coloring on λ with μ colors. We define an n-coloring on λ as follows. Let $\{x_1, \dots, x_n\} \in [\lambda]^n$. We may assume that $x_1 < \dots < x_n$. Put $c(x_1, \dots, x_n) = c'(x_1, \dots, x_{n'})$.

Now, if $H \subseteq \lambda$ is c-homogeneous, then H is c'-homogeneous as well. This shows the lemma. □

1.2. The finite and infinite versions of Ramsey's theorem.

Theorem 1.4 (Ramsey's theorem, infinite version)**.** *For all $n, m \in \omega$ with $n, m > 0$, $\aleph_0 \to (\aleph_0)^n_m$.*

Proof. We first observe that $\aleph_0 \to (\aleph_0)^n_m$ follows from $\aleph_0 \to (\aleph_0)^n_2$ by induction on the number m of colors.

Assume $\aleph_0 \to (\aleph_0)^n_m$ and $\aleph_0 \to (\aleph_0)^n_2$. Let c be an n-coloring on ω with $m+1$ colors. Define an n-coloring c' on ω with 2 colors as follows:

Let

$$c'(x_1, \ldots, x_n) = \begin{cases} 0, & \text{if } c(x_1, \ldots, x_n) < m \\ 1, & \text{if } c(x_1, \ldots, x_n) = m. \end{cases}$$

Now, by $\aleph_0 \to (\aleph_0)^n_2$ there is an infinite c'-homogeneous set $H \subseteq \omega$. If c' has the value 1 on $[H]^n$, then H is c-homogeneous of color m. If the value of c' is 0 on all n-element subsets of H, then c assumes only m different colors on $[H]^n$ and the existence of an infinite c-homogeneous subset of H follows from $\aleph_0 \to (\aleph_0)^n_m$. This shows $\aleph_0 \to (\aleph_0)^n_{m+1}$.

Hence we may restrict our attention to the case $m = 2$. We show $\aleph_0 \to (\aleph_0)^n_2$ by induction on n. Note that for $n = 1$ the statement is just the familiar Pigeon Hole Principle. Now assume $\aleph_0 \to (\aleph_0)^n_m$. We will show $\aleph_0 \to (\aleph_0)^{n+1}_m$.

Let c be an $(n+1)$-coloring on ω with two colors. We define a strictly increasing sequence $(a_k)_{k\in\omega}$ of natural numbers, a decreasing sequence $(A_k)_{k\in\omega}$ of infinite subsets of ω and a sequence $(i_k)_{k\in\omega}$ of elements of 2.

Let $a_0 = 0$ and $A_0 = \omega$. Suppose we have already defined a_k and A_k. Let $c_{k+1} : [A_k \setminus (a_k + 1)]^n \to 2$ be defined by

$$c_{k+1}(x_1, \ldots, x_n) = c(a_k, x_1, \ldots, x_n).$$

Using the induction hypothesis $\aleph_0 \to (\aleph_0)^n_2$, there is an infinite c_{k+1}-homogeneous set $A_{k+1} \subseteq A_k \setminus (a_k + 1)$ of some color i_k. Let a_{k+1} be the least element of A_{k+1}. The finishes the recursive construction of the three sequences.

Let $A = \{a_k : k \in \omega\}$. Given $k_0, \ldots, k_n \in \omega$ with $k_0 < \cdots < k_n$ we have

$$c(a_{k_0}, \ldots, a_{k_n}) = c_{k_0+1}(a_{k_1}, \ldots, a_{k_n}) = i_{k_0}.$$

In other words, the color of the $(n+1)$-element set $\{a_{k_0}, \ldots, a_{k_n}\}$ only depends on its smallest element. Let $C \subseteq \omega$ be an infinite set such that for all $k \in C$, i_k is the same $i \in 2$. Let $H = \{a_k : k \in C\}$. Now H is infinite and c-homogeneous of color i. $\square$

Exercise 1.5. Show that every sequence $(x_n)_{n\in\omega}$ of real numbers has an infinite subsequence that is decreasing or increasing.

Theorem 1.6 (Ramsey's theorem, finite version)**.** *For all $n, m, k \in \omega$ with $m > 0$ there is $\ell \in \omega$ such that $\ell \to (k)^n_m$.*

It is convenient to introduce *Ramsey numbers* for the proof of this theorem.

Definition 1.7. Let $n > 0$. For each $k \in \omega$ let $R(k;n)$ denote the least cardinal ℓ such that $\ell \to (k)^n_2$. Note that Theorem 1.4 guarantees the existence of such a cardinal ℓ.

Proof of Theorem 1.6. An inductive argument similar to the one used in the proof of the infinite version of Ramsey's theorem shows that we may restrict our attention to the case $m = 2$. Also, it is easily seen that $2k \to (k)^1_2$.

Now assume that for some $n > 0$ and all $k \in \omega$, $R(k;n)$ is finite. Fix $k \in \omega$. We show that there is $\ell \in \omega$ with $\ell \to (k)^{n+1}_2$.

We start by defining a sequence $(\ell_j)_{j \in \omega}$ of natural numbers. Let $\ell_0 = 1$. Suppose ℓ_j has been chosen for some $j \in \omega$. Let

$$\ell_{j+1} = R(\ell_j; n) + 1.$$

This finishes the definition of the ℓ_j. Now let $\ell = \ell_{2k}$.

Claim 1.8. $\ell \to (k)^{n+1}_2$

To show the claim, let $c : [\ell]^{n+1} \to 2$. We define an increasing sequence $(a_j)_{j<2k}$ of elements of ℓ, a decreasing sequence $(A_j)_{j<2k}$ of subsets of ℓ and a sequence $(i_j)_{j<2k}$ of colors in 2.

Let $A_0 = \ell$. Suppose for some $j < 2k$ we have chosen $A_j \subseteq \ell$ of size ℓ_{2k-j}. Let a_j be the least element of A_j. Define

$$c_j : [A_j \setminus \{a_j\}]^n \to 2$$

by $c_j(x_1, \dots, x_n) = c(a_j, x_1, \dots, x_n)$. The set $A_j \setminus \{a_j\}$ is of size

$$\ell_{2k-j} - 1 = R(\ell_{2k-j-1}; n).$$

It follows that there is a c_j-homogeneous set $A_{j+1} \subseteq A_j \setminus \{a_j\}$ of some color $i_j \in 2$ of size $\ell_{2k-j-1} = \ell_{2k-(j+1)}$. This concludes the definition of the three sequences.

Now, given $j_0 < \dots < j_n < 2k$, we have

$$c(a_{j_0}, \dots, a_{j_n}) = c_{j_0}(a_{j_1}, \dots, a_{j_n}) = i_{j_0}.$$

In particular, the color of $\{a_{j_0}, \dots, a_{j_1}\}$ only depends on j_0. Choose a set $C \subseteq 2k$ of size k such that for some fixed $i \in 2$ and all $j \in C$ we have $i_j = i$. Let $H = \{a_j : j \in C\}$. H is of size k and c-homogeneous. $\square$

Exercise 1.9. Show that for $k \in \omega$ we have $9^k \to (k)^2_2$.

Hint: Go through the proof of Theorem 1.6 and do some explicit computations.

1.3. **Limitations.**

Theorem 1.10. *a)* $2^{\aleph_0} \not\to (3)^2_{\aleph_0}$
b) $2^{\aleph_0} \not\to (\aleph_1)^2_2$ *(Sierpiński)*

Proof. a) Consider the set 2^ω of all sequences of zeroes and ones of length ω. Given $\{x, y\} \in [2^\omega]^2$ let $c(x, y)$ be the least $n \in \omega$ such that $x(n) \neq y(n)$. It is easily checked that there is now c-homogeneous set of size three.

b) Let $<$ denote the usual linear order in $\mathbb{R}$. Choose a well-ordering $\prec$ of $\mathbb{R}$. Define $c : [\mathbb{R}]^2 \to 2$ by

$$c(x, y) = \begin{cases} 0, & \text{if } < \text{ and } \prec \text{ agree on } \{x, y\} \\ 1, & \text{otherwise.} \end{cases}$$

Now every homogeneous set of color 0 is an increasing well-ordered subset of $\mathbb{R}$ and every homogeneous set of color 1 is a reversely well-ordered subset of $\mathbb{R}$. But since not uncountable ordinal order-embeds into $\mathbb{R}$, the c-homogeneous sets cannot be uncountable. $\square$

Exercise 1.11. Show that for $n \in \omega$ with $n > 1$, $2^n \not\to (3)^2_n$.

Exercise 1.12. Sierpiński's example relies on the fact that ω_1, the first uncountable ordinal, does not order-embed into $\mathbb{R}$. Give a proof of that fact.

Hint: $\mathbb{R}$ has a countable dense subset.

1.4. **Compactness: The infinite Ramsey theorem implies the finite.**

Definition 1.13. A tree is a partial order $(T, <)$ such that for all $t \in T$ the set $\{s \in T : s < t\}$ is well-ordered by $<$. If $(T, <)$ is a tree and $t \in T$, then the *height* $\mathrm{ht}(t)$ is the ordertype of $\{s \in T : s < t\}$. Given an ordinal α, the α-th *level* of T is the set

$$\mathrm{Lev}_\alpha(T) = \{t \in T : \mathrm{ht}(t) = \alpha\}.$$

The height $\mathrm{ht}(T)$ of T is the least α with $\mathrm{Lev}_\alpha(T) = \emptyset$. For $t \in T$ let

$$\mathrm{succ}_T(t) = \{s \in T : t < s \wedge \mathrm{ht}(s) = \mathrm{ht}(t) + 1\}$$

denote the set of *immediate successors* of t. T is *finitely branching* if $\mathrm{succ}_T(t)$ is finite for all $t \in T$. T is *rooted* if it has a unique minimal element, i.e., if $\mathrm{Lev}_0(T)$ is a singleton. A *branch* of T is a maximal linearly ordered subset of T.

Theorem 1.14 (König's lemma)**.** *Every infinite, finitely branching, rooted tree has an infinite branch.*

Proof. Let T be an infinite, finitely branching, rooted tree. We choose a strictly increasing infinite chain $(t_n)_{n\in\omega}$ in T. By Zorn's lemma, this chain extends to an infinite branch of T.

Let t_0 be the *root* of T, i.e., the unique minimal element. Suppose for some $n \in \omega$, t_n has been defined such that

$$T(t_n) = \{s \in T : s < t_n \vee s = t_n \vee s > t_n\}$$

is infinite. Since $\mathrm{succ}_T(t_n)$ is finite and $T(t_n) = \bigcup\{T(s) : s \in \mathrm{succ}_T(t_n)\}$, there is $t_{n+1} \in \mathrm{succ}_T(t_n)$ such that $T(t_{n+1})$ is infinite. This finishes the construction of the sequence $(t_n)_{n\in\omega}$. $\square$

Exercise 1.15. $2^{<\omega}$ is the tree of all finite sequences of zeroes and ones. The order on $2^{<\omega}$ is proper set-theoretic inclusion. Let $S \subseteq 2^{<\omega}$ be such that for every $x \in 2^\omega$ there is $s \in S$ such that s is an initial segment of x, i.e., $s \subseteq x$. Show that there is a finite set $S_0 \subseteq S$ such that for all $x \in 2^\omega$ there is $s \in S_0$ with $s \subseteq x$.

Hint: Suppose this fails for some $S \subseteq 2^{<\omega}$ and consider the collection of those $t \in 2^{<\omega}$ that don't have an initial segment in S. Use König's lemma.

For topologists: This exercise essentially asks you to use König's lemma to show the compactness of the Hausorff space 2^ω.

We now give a proof of the finite version of Ramsey's theorem from the infinite.

Alternative proof of Theorem 1.6. Suppose for some $k, m, n \in \omega$ with $m, n > 0$ there is no $\ell \in \omega$ such that $\ell \to (k)^n_m$. Then for each $\ell \in \omega$, the set

$$C_\ell = \{c : c \text{ is an } n\text{-coloring on } \ell \text{ with } m \text{ colors without} \\ \text{a } c\text{-homogeneous set of size } k\}.$$

is finite and nonempty.

Let $T = \bigcup_{\ell \in \omega} C_\ell$. T, ordered by proper set-theoretic inclusion $\subsetneq$, is an infinite tree and for all $\ell \in \omega$ there is $\ell' \in \omega$ such that $\mathrm{Lev}_\ell(T) = C_{\ell'}$. It follows that T is finitely branching. The lowest level $\mathrm{Lev}_0(T)$ has exactly one element, the empty coloring.

Hence König's lemma applies to T and therefore T has an infinite branch $\{c_\ell : \ell \in \omega\}$. Now $c = \bigcup_{\ell \in \omega} c_\ell$ is an n-coloring on ω with m colors without a homogeneous set even of size k. But this constradicts $\aleph_0 \to (\aleph_0)^n_m$. □

1.5. **Uncountable versions of Ramsey's theorem.** In Lemma 1.10 we observed that the natural generalization of the infinite version of Ramsey's theorem to the uncountable fails. However, there is an uncountable version of Ramsey's theorem if you choose the cardinal on the left hand side of the arrow relation sufficiently large.

Definition 1.16. For a cardinal κ let $\beth_0(\kappa) = \kappa$. If $\beth_\alpha(\kappa)$ has been defined for some ordinal α, let $\beth_{\alpha+1}(\kappa) = 2^{\beth_\alpha(\kappa)}$. If α is a limit ordinal and $\beth_\beta(\kappa)$ has been defined for all $\beta < \alpha$, let

$$\beth_\alpha(\kappa) = \sup\{\beth_\beta(\kappa) : \beta < \alpha\}.$$

By $\beth_\alpha$ we denote $\beth_\alpha(\aleph_0)$.

Theorem 1.17 (Erdős-Rado)**.** *For all $n \in \omega$ and every infinite cardinal κ,*

$$(\beth_n(\kappa))^+ \to (\kappa^+)^{n+1}_\kappa.$$

In particular, $(2^\kappa)^+ \to (\kappa^+)^2_\kappa$ and therefore $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$.

Proof. First, we consider the case $n = 0$. We have to show

$$\kappa^+ \to (\kappa^+)^1_\kappa.$$

But this is just an instance of the pigeon hole principle: If κ^+ is partitioned into κ classes, one of the classes has to be of size κ^+.

Now let $n > 0$ and assume

$$(\beth_{n-1}(\kappa))^+ \to (\kappa^+)^n_\kappa.$$

Let $c : [\lambda]^{n+1} \to \kappa$, where $\lambda = (\beth_n(\kappa))^+$. For each $a \in \lambda$ let the coloring $c_a : [\lambda]^n \to \kappa$ be defined by

$$c_a(x_0, \ldots, x_{n-1}) = c(x_0, \ldots, x_{n-1}, a).$$

Claim 1.18. There is a set $A \subseteq \lambda$ of size $\beth_n(\kappa)$ such that for every set $B \subseteq A$ of size $\beth_{n-1}(\kappa)$ and every $b \in \lambda \setminus B$ there is $a \in A \setminus B$ such that c_a and c_b agree on $[B]^n$.

To show the claim, we construct an increasing chain $(A_\alpha)_{\alpha<(\beth_{n-1}(\kappa))^+}$ of subsets of λ of size $\beth_n(\kappa)$. We start with an arbitrary set $A_0 \subseteq \lambda$ of size $\beth_n(\kappa)$. Suppose we have chosen A_α for some $\alpha < (\beth_{n-1}(\kappa))^+$. Observe that there are

$$\beth_n(\kappa)^{\beth_{n-1}(\kappa)} = 2^{\beth_{n-1}(\kappa)} = \beth_n(\kappa)$$

subsets of A_α of size $\beth_{n-1}(\kappa)$. Given a set $B \subseteq A_\alpha$ of size $\beth_{n-1}(\kappa)$, there are

$$(\beth_{n-1}(\kappa))^\kappa \leq \beth_n(\kappa)$$

functions from $[B]^n$ to κ.

Choose $A_{\alpha+1} \subseteq \lambda$ such that $A_\alpha \subseteq A_{\alpha+1}$, $|A_{\alpha+1}| = \beth_n(\kappa)$ and such that for every $B \subseteq A_\alpha$ of size $\beth_{n-1}(\kappa)$ and every $b \in \lambda \setminus B$ there is $a \in A_{\alpha+1}$ such that c_a and c_b agree on $[B]^n$. This is possible since there are not too many $B \subseteq A_\alpha$ of size $\beth_{n-1}(\kappa)$ and functions $c_b \restriction [B]^n$.

If $\alpha < (\beth_{n-1}(\kappa))^+$ is a limit ordinal, let

$$A_\alpha = \bigcup\{A_\beta : \beta < \alpha\}.$$

This finishes the construction of the sequence $(A_\alpha)_{\alpha<(\beth_{n-1}(\kappa))^+}$.

Let $A = \bigcup\{A_\alpha : \alpha < (\beth_{n-1}(\kappa))^+\}$. Now, whenever $B \subseteq A$ is of size $\beth_{n-1}(\kappa)$, then there is $\alpha < (\beth_{n-1}(\kappa))^+$ such that $B \subseteq A_\alpha$. If $b \in \lambda \setminus B$, then by the choice of $A_{\alpha+1}$, there is $a \in A_{\alpha+1} \subseteq A$ such that c_a and c_b agree on $[B]^n$. This shows that A works for the claim.

Continuing the proof of the Erdős-Rado theorem, let $A \subseteq \lambda$ be as in the claim. Choose $a \in \lambda \setminus A$. Recursively, we construct a sequence $(x_\alpha)_{\alpha<(\beth_{n-1}(\kappa))^+}$ of pairwise distinct elements of A such that for all

$\alpha < (\beth_{n-1}(\kappa))^+$, c_{x_α} agrees with c_a on $[\{x_\beta : \beta < \alpha\}]^n$. This is possible by the choice of A.

Now let $X = \{x_\beta : \beta < (\beth_{n-1}(\kappa))^+\}$. Define $d : [X]^n \to \kappa$ by letting

$$d(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) = c_a(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) = c(a, x_{\alpha_1}, \dots, x_{\alpha_n}).$$

By the choice of the x_α, for all $\alpha_0 < \dots < \alpha_n < (\beth_{n-1}(\kappa))^+$,

$$\begin{aligned}(1)\quad c(x_{\alpha_0}, \dots, x_{\alpha_n}) &= c_{x_{\alpha_n}}(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) \\ &= c_a(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}) = d(x_{\alpha_0}, \dots, x_{\alpha_{n-1}}).\end{aligned}$$

By $(\beth_{n-1}(\kappa))^+ \to (\kappa^+)^n_\kappa$, there is a d-homogeneous set $H \subseteq X$ of size κ^+. H is in fact c-homogeneous by equation (1). $\square$

Exercise 1.19. Let κ be an infinite cardinal a consider the set 2^κ of all sequences of zeroes and ones of length κ. Let $<_{\text{lex}}$ denote the *lexicographic order* on 2^κ, i.e., let $x <_{\text{lex}} y$ if $x \neq y$ and for the smallest $\alpha < \kappa$ with $x(\alpha) \neq y(\alpha)$ we have $x(\alpha) < y(\alpha)$.

Show that 2^κ does not contain strictly increasing or decreasing sequences of length κ^+.

Instructions: Assume 2^κ contains a strictly increasing sequence of length κ^+. Let $\gamma \leq \kappa$ be the minimal ordinal such that there is a strictly increasing sequence $(x_\alpha)_{\alpha<\kappa^+}$ in 2^γ (with respect to the lexicographic ordering on 2^γ). For each $\alpha < \kappa^+$ let $\xi_\alpha < \gamma$ be the unique ordinal with $x_\alpha \upharpoonright \xi_\alpha = x_{\alpha+1} \upharpoonright \xi_\alpha$ and $x_\alpha(\xi_\alpha) = 0$ and $x_{\alpha+1}(\xi_\alpha) = 1$.

How many possibilities are there for ξ_α? Can γ really be minimal? Arrive at a contradiction.

Exercise 1.20. Show that for every infinite cardinal κ, $2^\kappa \not\to (\kappa^+)^2_2$.

Hint: Generalize Theorem 1.10 b) using Exercise 1.19.

Definition 1.21. Let n and m be natural numbers > 0. Let λ and $\kappa_0, \dots, \kappa_{m-1}$ be cardinals. Then

$$\lambda \to (\kappa_0, \dots, \kappa_{m-1})^n$$

iff for every coloring $c : [\lambda]^n \to m$ there is $i \in m$ and a c-homogeneous set of color i of size κ_i.

Theorem 1.22 (Dushnik-Miller)**.** *For every infinite cardinal κ,*

$$\kappa \to (\kappa, \omega)^2.$$

Proof. We only give the proof in the case that κ is regular. The proof of the singular case is slightly more involved and can be found in [2].

Let $c : [\kappa]^2 \to 2$ be a coloring. Let $A = c^{-1}(0)$ and $B = c^{-1}(1)$. For every $x \in \kappa$ let

$$B_x = \{y \in \kappa : x < y \wedge \{x, y\} \in B\}.$$

First assume that every set $X \subseteq \kappa$ has an element x such that

$$|B_x \cap X| = \kappa.$$

In this case we can construct an c-homogeneous set H of color 1 as follows:

Let $X_0 = \kappa$ and $x_0 \in X_0$ be such that $|B_{x_0} \cap X_0| = \kappa$. For each $n \in \omega$ let $X_{n+1} = B_{x_n} \cap X_n$ and choose $x_{n+1} \in X_{n+1}$ such that

$$|B_{x_{n+1}} \cap X_{n+1}| = \kappa.$$

Let $H = \{x_n : n \in \omega\}$. By the choice of the x_n, H is c-homogeneous of color 1.

Now assume that there is a set $X \subseteq \kappa$ of size κ such that for all $x \in X$ we have $|B_x \cap X| < \kappa$. If κ is regular, we can recursively construct a sequence $(x_\alpha)_{\alpha<\kappa}$ such that for all $\alpha < \beta < \kappa$ we have $\{x_\alpha, x_\beta\} \in A$.

Namely, let $x_0 \in X$ be arbitrary. Suppose for some $\alpha < \kappa$ we have already chosen x_β for all $\beta < \alpha$. Since κ is regular,

$$\left| \bigcup \{B_{x_\beta} : \beta < \alpha\} \cap X \right| < \kappa.$$

Let $x_\alpha \in X \setminus \bigcup_{\beta<\alpha} B_{x_\beta}$ be such that for all $\beta < \alpha$, $x_\alpha > x_\beta$. This finishes the construction of the sequence $(x_\alpha)_{\alpha<\kappa}$ and shows the existence of a c-homogeneous set of color 0 of size κ. $\square$

1.6. Weakly compact cardinals.

Definition 1.23. A cardinal κ is *weakly compact* if it is uncountable and satisfies $\kappa \to (\kappa)^2_2$.

Lemma 1.24. *Every weakly compact cardinal is inacessible, i.e., κ is regular and for all $\lambda < \kappa$ we have $2^\lambda < \kappa$.*

Proof. Let κ be weakly compact. We first show that κ is regular. Suppose not. Then for some $\delta < \kappa$ there is a partition $(A_\alpha)_{\alpha<\delta}$ of κ into sets of size $< \kappa$. Define $c : [\kappa]^2 \to 2$ by letting $c(x, y) = 1$ is x and y are in the same A_α and $c(x, y) = 0$ otherwise. If $H \subseteq \kappa$ is c-homogeneous, then either it intersect each A_α in at most one point or it is contained in a single A_α. In both cases it follows that $|H| < \kappa$, a contradiction.

In order to show that $2^\lambda < \kappa$ for all $\lambda < \kappa$ assume that this is not the case and let $\lambda < \kappa$ be such that $2^\lambda \geq \kappa$. Since $\lambda^+ \leq \kappa$ we have $2^\lambda \to (\lambda^+)^2_2$ by the monotonicity properties of the arrow-relation. On the other hand, by Exercise 1.20, $2^\lambda \not\to (\lambda^+)^2_2$. A contradiction. □

Lemma 1.25. *The existence of an inaccessible cardinal is not provable in ZFC.*

Proof. Suppose there is an inaccessible cardinal. Let κ be the least inaccessible cardinal. It is not hard to check that V_κ satisfies ZFC. On the other hand, it is not hard to check that no cardinal is inaccessible in V_κ. □

Exercise 1.26. Show that if κ is inaccessible, then V_κ satisfies the Power Set Axiom (Potenzmengenaxiom).

Exercise 1.27. Show that if κ is inaccessible, then V_κ satisfies the Axiom of Replacement (Ersetzungsaxiom).

Corollary 1.28. *It is consistent with ZFC that there is no uncountable cardinal κ satisfying $\kappa \to (\kappa)^2_2$.*

2. Continuous Ramsey theory

2.1. Polish spaces.

Definition 2.1. X is a *Polish space* if it is a separable complete metric space.

Example 2.2. a) For every $n \in \omega$, $\mathbb{R}^n$ with the usual (euclidean) metric is a Polish space. Every separable Banach space is Polish.

b) 2^ω and ω^ω are Polish spaces. The metric is given by

$$d(x, y) = \begin{cases} 2^{-\Delta(x,y)}, & x \neq y \\ 0, & x = y, \end{cases}$$

where $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$ for $\{x, y\} \in [\omega^\omega]^2$.

Exercise 2.3. Verify that the space ω^ω with the metric defined in Example 2.2 is indeed a separable complete metric space.

Hint: A nice countable subset of ω^ω is the collection of all sequences that are eventually constant. Given two elements of ω^ω, what does it mean that there distance smaller than 2^{-n}?

Definition 2.4. Let X be a topological space and let $A \subseteq X$. A is *nowhere dense* if its closure $\mathrm{cl}(A)$ contains no nonempty open set. A is *meager* (*of first category*) if it is a union of countably many nowhere dense sets. A nonmeager set is of *second category.*

Theorem 2.5 (Baire category theorem)**.** *Every complete metric space is of second category (in itself).*

Proof. Let X be a space equipped with the complete metric d. Let $(N_n)_{n\in\omega}$ be a collection of nowhere dense subsets of X. By enlarging the N_n if necessary, we may assume that each N_n is closed. Notice that if A and B are closed and nowhere dense, then so is $A \cup B$. Hence, replacing N_n by $\bigcup_{k\leq n} N_k$ we may assume that $N_n \subseteq N_m$ if $n \leq m$.

We construct a sequence $(U_n)_{n\in\omega}$ of nonempty open subsets of X and a sequence $(x_n)_{n\in\omega}$ of points in X such that for all $n \in \omega$,

(1) U_n is disjoint from N_n,
(2) $\mathrm{cl}(U_{n+1}) \subseteq U_n$,
(3) the diameter of U_n is at most 2^{-n}, and
(4) $x_n \in U_n$.

Let $U_0 =$ be a nonempty subset of $X \setminus N_0$ of diameter at most 1. Since N_0 is closed an nowhere dense, U_0 is open and nonempty. Choose $x_0 \in U_0$. Suppose for some $n \in \omega$, U_n has been defined. Since N_n is nowhere dense, $U_n \not\subseteq N_{n+1}$. Let $x_{n+1} \in U_n \setminus N_{n+1}$. Since $U_n \setminus N_{n+1}$ is open, there is $\varepsilon > 0$ such that the ε-ball about x_{n+1} is contained in $U_n \setminus N_{n+1}$. We may choose $\varepsilon < 2^{-n-1}$. Finally let U_{n+1} be the $\varepsilon/2$-ball about x_{n+1}. This finishes the definition of the two sequences.

By (2), (3) and (4), $(x_n)_{n\in\omega}$ is a Cauchy-sequence. By the completeness of X, $(x_n)_{n\in\omega}$ converges to some $x \in X$. By (2), $x \in U_n$ for all $n \in \omega$. By (1), $x \notin \bigcup_{n\in\omega} N_n$. This shows that X is not the union of the sets N_n. $\square$

Definition 2.6. Let X be a topological space and let $A \subseteq X$. A point $x \in A$ is *isolated* in A if x has an open neighborhood U such that $A \cap U = \{x\}$. A nonempty set $P \subseteq X$ is *perfect* if P is closed and has no isolated points.

Theorem 2.7 (Cantor-Bendixson)**.** *Every closed uncountable subset of a Polish contains a perfect set. Every perfect set in a complete metric space contains a copy of 2^ω and therefore is of size at least $2^{\aleph_0}$.*

Proof. Let X be a Polish space. Fix a countable dense subset D of X. For each $x \in X$ and $\varepsilon > 0$ let $U_\varepsilon(x)$ denote the open ε-ball about x.

If $O \subseteq X$ is open and $x \in O$, then for some $n \in \omega$, $U_{2/n}(x) \subseteq O$. Choose $d \in D \cap U_{1/n}(x)$. Now $x \in U_{1/n}(d) \subseteq O$. It follows that there is a countable collection $\{O_n : n \in \omega\}$ of open subsets of X such that for every open set $O \subseteq X$ and all $x \in O$ there is $n \in \omega$ such that $x \in O_n \subseteq O$. The collection $\{O_n : n \in \omega\}$ is a *basis for the topology of* X. Note that every open set is the union of sets of the form O_n.

Now let $A \subseteq X$. Let A' denote the set of all points of A that are not isolated. A' is the *Cantor-Bendixson derivative of* A. Let $A^{(0)} = A$. If $A^{(\alpha)}$ has been defined for some ordinal α, let $A^{(\alpha+1)} = (A^{(\alpha)})'$. If α is a limit ordinal and $A^{(\beta)}$ has been defined for all $\beta < \alpha$, let $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$.

Note that A' is obtained by removing an open set from A, i.e., a union of sets of the form O_n, $n \in \omega$. By the countability of the collection $\{O_n : n \in \omega\}$, the decreasing sequence $(A^{(\alpha)})_{\alpha<\omega_1}$ has to stabilize after countably many steps. Let $\alpha < \omega_1$ be the least ordinal such that

$A^{(\alpha+1)} = A^{(\alpha+2)}$. The ordinal α is the *Cantor-Bendixson rank of* A. The set $A^{(\alpha+1)}$ has no isolated points. Since $A^{(\alpha+1)}$ is obtained from A by removing open sets, $A^{(\alpha+1)}$ is closed. It follows that $A^{(\alpha+1)}$ is either perfect or empty.

Notice that whenever x is an isolated point of some $A^{(\beta)}$, then there is some $n \in \omega$ such that $A^{(\beta)} \cap U_n = \{x\}$. It follows that $A \setminus A^{(\beta)}$ is countable. In particular, if A is uncountable, then so is $A^{(\alpha)}$. In this case, $A^{(\alpha)}$ is a perfect set.

We now prove the second statement of the theorem. Let $P \subseteq X$ be perfect. A family $(U_s)_{s\in 2^\omega}$ of open subsets of X is a *perfect scheme* if

(1) for all $s, t \in 2^{<\omega}$, if $s \subsetneq t$, then $\mathrm{cl}(U_t) \subseteq U_s$,
(2) for all $n \in \omega$, for all $s, t \in 2^n$ with $s \neq t$, $\mathrm{cl}(U_s) \cap \mathrm{cl}(U_t) = \emptyset$, and
(3) for all $n \in \omega$, for all $t \in 2^n$, the diameter of U_t is at most 2^{-n}.

By recursion on the wellfounded relation $\subsetneq$ on $2^{<\omega}$ we construct a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ along with a family $(x_s)_{s\in 2^{<\omega}}$ of points in P such that for all $s \in 2^{<\omega}$ we have $x_s \in U_s$.

Let $U_\emptyset$ be a nonempty subset of X of diameter at most 1 such that $U_\emptyset \cap P \neq \emptyset$. Let $x_\emptyset \in P \cap U_\emptyset$. Suppose for some $n \in \omega$ and $t \in 2^t$, U_t and x_t have been defined such that $x_t \in U_t \cap P$ Since P is perfect, $U_t \cap P$ contains at least two distinct points $x_{t^\frown 0}$ and $x_{t^\frown 1}$. Here $t^\frown i$ denotes the sequence that starts with t and has i as its next and last element.

Choose open balls $U_{t^\frown 0}$ and $U_{t^\frown 1}$ about $x_{t^\frown 0}$, respectively $x_{t^\frown 1}$ of diameter at most 2^{-n-1} such that

$$\mathrm{cl}(U_{t^\frown 0}) \cap \mathrm{cl}(U_{t^\frown 1}) = \emptyset$$

and

$$\mathrm{cl}(U_{t^\frown 0}), \mathrm{cl}(U_{t^\frown 1}) \subseteq U_t.$$

We now define a function $f : 2^\omega \to P$ as follows: For each $\eta \in 2^\omega$ let $f(\eta) = \lim_{n\in\omega} x_{\eta\restriction n}$. Note that the limit exists since $(x_{\eta\restriction n})_{n\in\omega}$ is a Cauchy sequence. Since the $x_{\eta\restriction n}$ are elements of P and P is closed, $f(\eta) \in P$.

It is easily checked that f is continuous and 1-1. Since 2^ω is compact, f is a homeomorphism onto its image. □

Corollary 2.8. *A closed subset of a Polish space is either countable or of size* $2^{\aleph_0}$.

Exercise 2.9. Show that for every countable ordinal α there is a subset of $\mathbb{R}$ of Cantor-Bendixson rank at least α.

Hint: Construct the examples by recursion on α. The argument for the successor steps and limit steps are similar but not identical. If α is a countable limit ordinal, then α is the supremum of set of ordinals of order type ω.

2.2. **The Baire property.**

Definition 2.10. Let X be a topological space. Then a set $A \subseteq X$ has the *Baire property*, if for some open set $O \subseteq X$ the *symmetric difference*

$$A \triangle O = (A \setminus O) \cup (O \setminus A)$$

is meager in X. We write $A =^* B$ if $A \triangle B$ is meager.

Lemma 2.11. *Let X be a topological space. The class of subsets of X with the Baire property is a σ-algebra that contains all open set.*

Proof. If $O \subseteq X$ is open, then $\mathrm{cl}(O) \setminus O$ is closed and nowhere dense. Therefore $\mathrm{cl}(O) \setminus O$ is meager. Similarly, if $F \subseteq X$ is closed, then $F \setminus \mathrm{int}(F)$ is meager.

Now, if $A \subseteq X$ has the Baire property and $A \triangle O$ is meager for some open set O, then $X \setminus A =^* X \setminus O =^* \mathrm{int}(X \setminus O)$, showing that $X \setminus A$ has the Baire property as well.

Finally, let $A_n \subseteq X$ have the Baire property for every $n \in \omega$. For each $n \in \omega$ choose an open set O_n such that $A_n =^* O_n$. Since countable unions of meager sets are again meager, $\bigcup_{n\in\omega} A_n =^* \bigcup_{n\in\omega} O_n$. This shows that $\bigcup_{n\in\omega} A_n$ has the Baire property. □

Lemma 2.12. *Let X be a Polish space without isolated points and suppose that $A \subseteq X$ is nonmeager and has the Baire property. Then A contains a perfect set.*

Corollary 2.13. *If X is Polish and $A \subseteq X$ has the Baire property, then A or $X \setminus A$ contains a perfect set.*

Corollary 2.14. *$\mathbb{R}$ has a subset without the Baire property.*

Proof. Every perfect set is closed. Every closed set is the complement of an open set. It follows that there are not more perfect subsets than open subsets of $\mathbb{R}$. Every open set is the union of sets from a fixed countable base for the topology of $\mathbb{R}$. A countable set has only $2^{\aleph_0}$ subsets. It follows that there are only $2^{\aleph_0}$ perfect subsets of $\mathbb{R}$. Let $(P_\alpha)_{\alpha<2^{\aleph_0}}$ be an enumeration of all perfect subsets of $\mathbb{R}$. Since each perfect set is of size $2^{\aleph_0}$, we can recursively choose sequences $(x_\alpha)_{\alpha<2^{\aleph_0}}$ and $(y_\alpha)_{\alpha<2^{\aleph_0}}$ such that the sets $A = \{x_\alpha : \alpha < 2^{\aleph_0}\}$ and $B = \{y_\alpha : \alpha < 2^{\aleph_0}\}$ are disjoint and for each $\alpha < 2^{\aleph_0}$, $x_\alpha, y_\alpha \subseteq P_\alpha$.

From the construction it follows that neither A nor $\mathbb{R} \setminus A$ contains a perfect set. Therefore A does not have the Baire property. □

A set $A \subseteq \mathbb{R}$ such that neither A itself nor the complement of A contain a perfect set is a *Bernstein set.*

Proof of Lemma 2.12. If A is nonmeager and has the Baire property, then there is some nonempty open set $O \subseteq X$ such that $A =^* O$. Let $(N_n)_{n\in\omega}$ be a sequence of closed nowhere dense sets such that

$$O \triangle A \subseteq \bigcup_{n\in\omega} N_n.$$

We may assume that for all $n \in \omega$, $N_n \subseteq N_{n+1}$.

We construct a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ of open subsets of O such that for all $n \in \omega$ and all $s \in 2^n$, $\mathrm{cl}(U_s) \cap N_n = \emptyset$. Along with $(U_s)_{s\in 2^{<\omega}}$ we choose a family $(x_s)_{2^{<\omega}}$ of points in O such that $x_s \in U_s$ for all $x \in 2^{<\omega}$. We start by choosing some open set $U_\emptyset \subseteq O \setminus N_n$ of diameter at most 1 and an arbitrary point $x_\emptyset \in U_\emptyset$.

If U_s has been chosen for some $s \in 2^{<\omega}$, we use the fact that X has no isolated points to find two distinct points $x_{s^\frown 0}, x_{s^\frown 1} \in U_s \setminus N_{n+1}$. Note that $U_s \setminus N_{n+1}$ is nonempty since N_{n+1} is nowhere dense. Choose open neighborhoods $U_{s^\frown 0}$ and $U_{s^\frown 1}$ of $x_{s^\frown 0}$ and $x_{s^\frown 1}$, respectively, such that the $U_{s^\frown i}$ are of diameter $< 2^{-n-1}$ and such that their closures are disjoint, disjoint from N_{n+1} and contained in U_s. As in the proof of Theorem 2.7, the function

$$f : 2^\omega \to X; \sigma \mapsto \lim_{n\to\infty} x_{\sigma\restriction n}$$

is a homeomorphism onto its image. Moreover,

$$f[2^\omega] = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} U_s = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \mathrm{cl}(U_s).$$

Since $\mathrm{cl}(U_s)$ is disjoint from N_n for every $s \in 2^n$, $f[2^\omega]$ is disjoint from $\bigcup_{n\in\omega} N_n$. Since $f[2^\omega] \subseteq \mathrm{cl}(U_\emptyset)$, we have $f[2^\omega] \subseteq O$. It follows that $f[2^\omega]$ is a perfect subset of A. □

Exercise 2.15. Let X be a complete metric space and let $(U_s)_{s\in 2^{<\omega}}$ be a perfect scheme of open subsets of X as defined in the proof of Theorem 2.7. For each $x \in 2^\omega$ let $f(x)$ be the unique element of $\bigcap_{n\in\omega} \mathrm{cl}(U_{x\restriction n})$. Show that f is actually well defined and that f is a homeomorphism from 2^ω to $f[2^\omega]$.

Definition 2.16. Let X and Y be topological spaces. Then $f : X \to Y$ is *Baire measurable* if for every open set $O \subseteq Y$, $f^{-1}[O]$ has the Baire property in X.

A set $A \subseteq X$ is *comeager* (in X) if $X \setminus A$ is meager.

Lemma 2.17. *Let X be a Polish space without isolated points. Let Y be a topological space with a countable base for the topology, i.e., let Y be a* second countable *space. Let $f : X \to Y$ be Baire measurable. Then there is a comeager set $A \subseteq X$ such that $f \restriction A : A \to Y$ is continuous.*

Proof. Let $\{O_n : n \in \omega\}$ be a base for the topology on Y. For each $n \in \omega$ let M_n be a meager set such that for some open set $U_n \subseteq X$, $f^{-1}[O_n] =^* U_n$. Then $M = \bigcup_{n\in\omega} M_n$ is meager. Let $A = X \setminus M$. Clearly, A is comeager in X. For each $n \in \omega$, $f^{-1}[O_n] \cap A = U_n \cap A$ and thus $(f \restriction A)^{-1}[O_n]$ is open in A. Since $\{O_n : n \in \omega\}$ is a base for the topology of Y this shows that $f \restriction A$ is continuous. □

2.3. **Galvin's theorem.**

Definition 2.18. Let X be a set and let $n \in \omega$ be at least 1. Let $(X)^n$ denote the set of all n-tuples from X with pairwise distinct entries.

Theorem 2.19 (Kuratowski, Mycielski)**.** *Let X be a Polish space without isolated points. Let $(n_i)_{i\in\omega}$ be a sequence of of natural numbers > 0 and let $(R_i)_{i\in\omega}$ be a sequence of relations on X such that for all $i \in \omega$, R_i is a comeager subset of X^{n_i}. Then there is a Cantor set $C \subseteq X$ such* *that for all $i \in \omega$, $(C)^{n_i} \subseteq R_i$. Here a* Cantor set *is a homeomorphic copy of 2^ω.*

Proof. For each $i \in \omega$ let $(M_{i,k})_{k\in\omega}$ be an increasing sequence of closed nowhere dense subsets of X^{n_i} such that

$$X^{n_i} \setminus \bigcup_{k\in\omega} M_{i,k} \subseteq R_i$$

As usual, we construct a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ of open subsets of X. During the construction we make sure that for all $i, k, m \in \omega$ there is $n > m$ such that for all pairwise distinct $s_1, \ldots, s_{n_i} \in 2^n$ the set $\mathrm{cl}(U_{s_1}) \times \cdots \times \mathrm{cl}(U_{s_i})$ is disjoint from $M_{i,k}$.

The perfect scheme $(U_s)_s \in 2^{<\omega}$ can be constructed by careful bookkeeping using the following argument:

Suppose we have chosen U_s for every $s \in 2^n$. Let $i, k \in \omega$ be given and suppose that the cardinal 2^{n+1} is at least n_i. Choose an enumeration $((s_1^j, \ldots, s_{n_i}^j))_{j<\ell}$ of the set of all n_i-tuples from 2^{n+1} with pairwise distinct entries.

For each $s \in 2^{n+1}$ choose V_s^0 so that $\mathrm{cl}(V_s^0)$ is a nonempty subset of $U_{s\restriction n}$ of diameter $< 2^{-n-1}$ and so that for distinct $s, t \in 2^{n+1}$, $\mathrm{cl}(V_s^0)$ is disjoint from $\mathrm{cl}(V_t^0)$. Suppose for some fixed $j < \ell$ and for all $s \in 2^{n+1}$, V_s^j has been defined. Since $M_{i,k}$ is a nowhere dense subset of X^{n_i}, there are nonempty open sets $V_s^{j+1} \subseteq V_s^j$, $s \in 2^{n+1}$, such that

$$\left(\mathrm{cl}\left(V_{s_1^j}^{j+1} j\right) \times \cdots \times \mathrm{cl}\left(V_{s_{n_i}^j}^{j+1}\right)\right) \cap M_{i,k} = \emptyset.$$

Finally, for each $s \in 2^{n+1}$ let $U_s = V_s^\ell$.

Having defined a perfect scheme $(U_s)_{s\in 2^{<\omega}}$ with the desired properties, we define $f : 2^\omega \to X$ by letting $f(x)$ be the unique element of $\bigcap_{n\in\omega} \mathrm{cl}(U_{x\restriction n})$ for every $x \in 2^\omega$. It is easily checked that $f[2^\omega]$ is a copy of 2^ω such that for every $i \in \omega$, $(C)^{n_i} \subseteq R_i$. □

Definition 2.20. Let X be a Hausdorff space. A subset A of $[X]^n$ is open if it is the union of sets of the form

$$[U_1, \ldots, U_n] = \{\{x_1, \ldots, x_n\} : x_1 \in U_1 \wedge \cdots \wedge x_n \in U_n\},$$

where $U_1, \ldots, U_n \subseteq X$ are open and disjoint.

Exercise 2.21. Show that for every metric space X and every $n > 0$ the space $[X]^n$ is metric as well.

Exercise 2.22. Let X be a Polish space, $n > 0$ and $K \subseteq [X]^n$. Show that K has the Baire property in $[X]^n$ iff the set $\{(x_1, \dots, x_n) : \{x_1, \dots, x_n\} \in K\}$ has the Baire property in X^n.

Definition 2.23. For $\{x, y\} \in [\omega^\omega]^2$ let

$$c_{\text{parity}}(x, y) = \Delta(x, y) \mod 2.$$

Let $c_{\min} = c_{\text{parity}} \restriction 2^\omega$.

Given a coloring $c : [X]^2 \to 2$ on a set X let $\mathfrak{hm}(c)$ denote the smallest size of a family $\mathcal{H}$ of c-homogeneous subsets of X such that $X = \bigcup \mathcal{H}$.

Exercise 2.24. Show that $\mathfrak{hm}(c_{\min})$ is uncountable.

Definition 2.25. Let X and Y be Hausdorff spaces and let $c : [X]^2 \to 2$ and $d : [Y]^2 \to 2$ be continuous colorings. We write $c \leq d$ if there is a continuous injection $e : X \to Y$ such that for all $\{x_0, x_1\} \in [X]^2$ we have $c(x_0, x_1) = d(e(x_0), e(x_1))$.

Clearly, if $c \leq d$, then $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$.

Lemma 2.26. *Let X be a Polish space and let $c : [X]^2 \to 2$ be continuous. Then either $\mathfrak{hm}(c)$ is countable or $c_{\min} \leq c$.*

Proof. For a set $A \subseteq X$ let

$$A' = A \setminus \bigcup \{O \subseteq X : O \text{ is open and } \mathfrak{hm}(c \restriction (A \cap O)) \text{ is countable}\}.$$

For an ordinal α we define $A^{(\alpha)}$ as in the case of Cantor-Bendixson derivatives. Since A' is obtained from A by removing an open set and since X is second countable, there is some $\alpha < \omega_1$ such that $X^{(\alpha)} = X^{(\alpha+1)}$. If $X^{(\alpha)}$ is empty, then $\mathfrak{hm}(c)$ is countable. Otherwise, $A = X^{(\alpha)}$ has the property that no open subset of A is c-homogeneous. This allows us to construct a perfect scheme $(U_s)_{s \in 2^{<\omega}}$ of open subsets of A such that for all $n \in \omega$, all distinct $s, t \in 2^n$, all $x \in \text{cl}(U_s)$ and all $y \in \text{cl}(U_t)$ we have $c(x, y) = \Delta(s, t) \mod 2$. Here $\Delta(s, t)$ is defined in the same way as for infinite sequence.

Now the function $f : 2^\omega \to A$ defined by letting $f(x)$ be the unique element of $\bigcap_{n \in \omega} \text{cl}(U_{x \restriction n})$ witnesses $c_{\min} \leq c$. $\square$

Theorem 2.27 (Galvin)**.** *Let X be a Polish space without isolated points, $n, m > 0$, and $c : [X]^2 \to m$ a Baire measurable coloring. Then there is a perfect c-homogeneous set $H \subseteq X$.*

Proof. For arbitrary m, the theorem follows by induction from the case $m = 2$. Hence we may assume that m is 2. Consider the map $d : X^2 \to 2$ defined by

$$d(x, y) = \begin{cases} c(x, y) & x \neq y \\ 1, & x = y. \end{cases}$$

Since the set $\{(x, x) : x \in X\}$ is closed in X^2 and thus has the Baire property and by Exercise 2.22, c' is Baire measurable. For $i \in 2$ choose an open set $U_i \subseteq X^2$ such that $d^{-1}(i) = *U_i$. Let M_i be meager such that $d^{-1}(i) \triangle U_i \subseteq M_i$.

By Theorem 2.19, there is a Cantor set $C \subseteq X$ such that $(C)^2$ is disjoint from $M_0 \cup M_1$. It follows that d is continuous on $(C)^2$. But this implies that c is continuous on $[C]^2$. We may therefore assume that c is continuous on $[X]^2$ to begin with.

Suppose now that $\mathfrak{hm}(c)$ is countable. Since c is continuous, the closure of a homogeneous set is again homogeneous. Hence X is covered by countably many closed c-homogeneous sets. By the Baire category theorem, at least one of these sets fails to be nowhere dense and therefore contains a nonempty open set. In other words, there is a nonempty open set that is c-homogeneous. But a nonempty open set in a Polish space without isolated points contains a copy of 2^ω.

If $\mathfrak{hm}(c)$ is uncountable, then $c_{\min} \leq c$. But 2^ω clearly contains a copy of 2^ω that is $c_{\min}$-homogeneous. It follows that X contains a copy of 2^ω that is c-homogeneous. □

Exercise 2.28. Let X be a perfect Polish space. Suppose that

$$[X]^2 = K_1 \cup \cdots \cup K_n,$$

where all the K_i have the Baire property. Show that there is a Cantor space $C \subseteq X$ such that for some i, $[C]^2 \subseteq K_i$.

2.4. **Covering the plane by functions.**

Definition 2.29. Let X be a set. A point $(x, y) \in X^2$ is *covered* by a function $f : X \to X$ if $f(x) = y$ or $f(y) = x$. A family $\mathcal{F}$ of functions from X to X covers X^2 each point of X^2 is covered by a function in $\mathcal{F}$.

Theorem 2.30 (Kuratowski)**.** *Let κ be an infinite cardinal. Then there is a family $\mathcal{F}$ of size κ of functions from κ^+ to κ^+ such that $\mathcal{F}$ covers $\kappa^+ \times \kappa^+$. No family of size $< \kappa$ covers $\kappa^+ \times \kappa^+$.*

Proof. For each $\alpha < \kappa^+$ choose a function g_α from κ onto $\alpha + 1$. For every $\gamma < \kappa$ let $f_\gamma : \kappa^+ \to \kappa^+$ be defined by letting $f_\gamma(\alpha) = g_\alpha(\gamma)$. Let $\mathcal{F} = \{f_\gamma : \gamma < \kappa\}$

We show that $\mathcal{F}$ covers $\kappa^+ \times \kappa^+$. Let $(\alpha, \beta) \in \kappa^+ \times \kappa^+$. Since the notion of being covered by a function is symmetric in the two coordinates, we may assume $\beta \leq \alpha$. Since $g_\alpha : \kappa \to \alpha + 1$ is onto, there is $\gamma < \kappa$ such that $g_\alpha(\gamma) = \beta$. But $\beta = g_\alpha(\gamma) = f_\gamma(\alpha)$. Hence (α, β) is covered by f_γ and $\mathcal{F}$ works for the theorem.

On the other hand, let X be an infinite set and $\mathcal{F}$ is a family of functions from X to X that covers X^2. Note that $\mathcal{F}$ has to be infinite. Hence, without changing the size of $\mathcal{F}$, we may assume that $\mathcal{F}$ is closed under composition of functions and contains the identity on X.

For $x, y \in X$ we let $x \leq_{\mathcal{F}} y$ if there is $f \in \mathcal{F}$ such that $x = f(y)$. Since $\mathcal{F}$ contains the identity, $\leq_{\mathcal{F}}$ is reflexive. Since $\mathcal{F}$ is closed under composition, $\leq_{\mathcal{F}}$ is transitive. Since $\mathcal{F}$ covers X^2, $\leq_{\mathcal{F}}$ is total in the sense that any two elements of X are comparable with respect to $\leq_{\mathcal{F}}$. In general, $\leq_{\mathcal{F}}$ will not be antisymmetric.

For all $x \in X$, the set $X \restriction x = \{y \in X : y \leq_{\mathcal{F}} x\}$ is of size at most $|\mathcal{F}|$. By recursion, choose a maximal, strictly $\leq_{\mathcal{F}}$-increasing sequence $(x_\alpha)_{\alpha<\delta}$ in X, indexed by some ordinal δ. Now

$$X = \bigcup_{\alpha<\delta} X \restriction x_\alpha.$$

Since $(x_\alpha)_{\alpha<\delta}$ is strictly $\leq_{\mathcal{F}}$-increasing, $(X \restriction x_\alpha)_{\alpha<\delta}$ is strictly $\subseteq$-increasing. Hence X is the union of a strictly increasing wellordered chain of sets of size $\leq |\mathcal{F}|$. But this implies that $|X| \leq |\mathcal{F}|^+$, finishing the proof of the theorem. $\square$

Definition 2.31. Let X be a metric space and let d denote the metric on X. For $c \in \mathbb{R}$ we say that a function $f : X \to X$ is *Lipschitz of class* $< c$ if for all $x_0, x_1 \in X$ with $x_0 \neq x_1$,

$$\left| \frac{d(f(x_0), f(x_1))}{d(x_0, x_1)} \right| < c.$$

We say that f is *Lipschitz of class* $\leq c$ if for all $x_0, x_1 \in X$ with $x_0 \neq x_1$,

$$\left| \frac{d(f(x_0), f(x_1))}{d(x_0, x_1)} \right| \leq c.$$

Lemma 2.32. *There is a family of Lipschitz functions of class* ≤ 1 *of size* $\mathfrak{hm}(c_{\text{parity}})$ *that covers* $(\omega^\omega)^2$. *Similarly, there is a family of Lipschitz functions of class* ≤ 1 *of size* $\mathfrak{hm}(c_{\min})$ *that covers* $(2^\omega)^2$.

Proof. We only show the lemma for ω^ω. For 2^ω the argument is the same.

For $x, y \in \omega^\omega$ let $x \otimes y = (x(0), y(0), x(1), y(1), \dots)$. The mapping $\otimes$ is a homeomorphism between $(\omega^\omega)^2$ and ω^ω.

If $H \subseteq \omega^\omega$ is c_{parity}-homogeneous of color 0, then for every $x \in \omega^\omega$ there is at most one $y \in \omega^\omega$ with $x \otimes y \in H$. If H is maximal homogeneous, then there is some y with $x \otimes y \in H$. Thus, a maximal c_{parity}-homogeneous set H of color 0 gives rise to a function $f_H : \omega^\omega \to \omega^\omega$ with $H = \{x \otimes f(x) : x \in \omega^\omega\}$.

Similarly, every maximal c_{parity}-homogeneous set H of color 1 gives rise to a function $f_H : \omega^\omega \to \omega^\omega$ with $H = \{f(x) \otimes x : x \in \omega^\omega\}$. A straight forward calculation shows that if H is of color 0, then f_H is Lipschitz of class ≤ 1 and if H is of color 1, then f_H is Lipschitz of class $\leq 1/2$.

If $x, y \in \omega^\omega$, $H \subseteq \omega^\omega$ is maximal c_{parity}-homogeneous of color 0 and $(x, y) \in H$, then $f_H(x) = y$. On the other hand, if H is maximal c_{parity}-homogeneous of color 1 and $(x, y) \in H$, then $f_H(y) = x$. Hence, if $\mathcal{H}$ is a family of maximal c_{parity}-homogeneous subsets of ω^ω such that $\omega^\omega = \bigcup \mathcal{H}$, then the corresponding family of Lipschitz functions covers $(\omega^\omega)^2$. □

Remark 2.33. We observe that this proof actually gives a little more information: there is a family $\mathcal{F}$ of size $\mathfrak{hm}(c_{\text{parity}})$ of Lipschitz functions such that for all $x, y \in \omega^\omega$ there is a Lipschitz function $f \in \mathcal{F}$ such that ($f(x) = y$ and f is of class ≤ 1) or ($f(y) = x$ and f is of class $\leq 1/2$).

Corollary 2.34. $2^{\aleph_0} \leq \mathfrak{hm}(c_{\min})^+$

Exercise 2.35. Show that at least $\mathfrak{hm}(c_{\min})$ Lipschitz functions from $\mathbb{R}$ to $\mathbb{R}$ of class ≤ 2 are needed to cover $\mathbb{R}^2$.

Hint: Construct an embedding $e : 2^\omega \to \mathbb{R}^2$ such that for every Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ of class ≤ 2, both $e^{-1}[\{(x, f(x)) : x \in \mathbb{R}\}]$ and $e^{-1}[\{(f(x), x) : x \in \mathbb{R}\}]$ are $c_{\min}$-homogeneous.

2.5. **Homogeneity numbers and other cardinal invariants.** We first observe that $\mathfrak{hm}(c_{\min})$ and $\mathfrak{hm}(c_{\text{parity}})$ are actually the same. Clearly, $c_{\min} \leq c_{\text{parity}}$ and therefore $\mathfrak{hm}(c_{\min}) \leq \mathfrak{hm}(c_{\text{parity}})$. Surpisingly, the converse is also true.

Lemma 2.36. $c_{\text{parity}} \leq c_{\min}$

Proof. We have to define an embedding $e : \omega^\omega \to 2^\omega$ witnessing $c_{\text{parity}} \leq c_{\min}$.

For $x \in \omega^\omega$, let $e(x)$ be the concatenation of the sequences b_n, $n \in \omega$, which are defined as follows.

If n is even, then let b_n be the sequence of length $2 \cdot x(n) + 2$ which starts with $2 \cdot x(n)$ zeros and then ends with two ones. If n is odd, let b_n be the sequence of length $2 \cdot x(n) + 2$ starting with $2 \cdot x(n) + 1$ zeros and ending with a single one.

It is clear that e is continuous and it is easy to check that e is an embedding witnessing $c_{\text{parity}} \leq c_{\min}$. $\square$

Corollary 2.37. $\mathfrak{hm}(c_{\min}) = \mathfrak{hm}(c_{\text{parity}})$

In the following we write $\mathfrak{hm}$ for $\mathfrak{hm}(c_{\min})$.

Definition 2.38. Let $\mathfrak{d}$ be the least size of a family of compact sets that covers ω^ω.

Lemma 2.39. $\mathfrak{d} \geq \aleph_1$

Proof. Let $C \subseteq \omega^\omega$ be compact and nonempty. For each $n \in \omega$, the function $p_n : \omega^\omega \to \omega : x \mapsto x(n)$ is continuous. It follows that $p_n[C]$ is compact and thus finite. Let $f_C : \omega \to \omega$ be defined by letting $f_C(n) = \max(p_n[C])$. Clearly, for all $n \in \omega$ and all $x \in C$ we have $x(n) \leq f_C(n)$. In other word, all of C is bounded by a single function.

But no open subset of ω^ω is bounded by a single function. It follows that C is nowhere dense. Hence, by the Baire category theorem, ω^ω is not covered by less than $\aleph_1$ compact sets. $\square$

Exercise 2.40. Show that $\mathfrak{d}$ is the least size of a family $\mathcal{F}$ of functions from ω to ω such that for all $g : \omega \to \omega$ there is $f \in \mathcal{F}$ such that for all $n \in \omega$, $g(n) \leq f(n)$.

Lemma 2.41. $\mathfrak{d} \leq \mathfrak{hm}$

Proof. By Lemma 2.32 there is a family $\mathcal{F}$ of size $\mathfrak{hm}$ of continuous functions from ω^ω to ω^ω that covers $(\omega^\omega)^2$. Let $C \subseteq \omega^\omega$ be a compact set of size $2^{\aleph_0}$, for example $C = 2^\omega$.

For the lemma we may assume that $\mathfrak{hm} < 2^{\aleph_0}$. We show that the family $\mathcal{C} = \{f[C] : f \in \mathcal{F}\}$ covers ω^ω. Namely, let $x \in \omega^\omega$. The set $A = \{f(x) : f \in \mathcal{F}\}$ is of size $\mathfrak{hm}$ and hence of size $< 2^{\aleph_0}$. It follows that there is $y \in C \setminus A$.

Since $y \notin A$, there is no $f \in \mathcal{F}$ with $f(x) = y$. Since $\mathcal{F}$ covers $(\omega^\omega)^2$, there is $f \in \mathcal{F}$ with $f(y) = x$. In particular, $x \in f[C]$. It follows that ω^ω can be covered by $\mathfrak{hm}$ compact sets and thus $\mathfrak{d} \leq \mathfrak{hm}$. $\square$

We are going to improve Lemma 2.41.

Definition 2.42. Let X be a set and let $I \subseteq \mathcal{P}(X)$. I is an *ideal* on X if $\emptyset \in I$, I is closed under finite unions and I is closed under taking subsets. I is a *σ-ideal* if additionally, I is closed under countable unions.

If C is a collection of subsets of X, then the *ideal generated by C* is the smallest ideal I on X such that $C \subseteq I$. The *σ-ideal generated by C* is the smallest σ-ideal I with $C \subseteq I$.

Examples of σ-ideals on $\mathbb{R}$ are the ideal `null` of measure zero sets and the ideal `meager` of meager subsets of $\mathbb{R}$. Examples of σ-ideals on ω^ω are the ideal of meager sets and the σ-ideal generated by all compact sets.

Definition 2.43. Given an ideal I on a space X, we define four cardinals that describe the combinatorial properties of the ideal.

(1) $\mathrm{add}(I) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq I \wedge \bigcup \mathcal{F} \notin I\}$, the *additivity* of I.
(2) $\mathrm{non}(I) = \min\{|A| \colon A \subseteq X \wedge A \notin I\}$, the *uniformity* of I.
(3) $\mathrm{cov}(I) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq I \wedge \bigcup \mathcal{F} = X\}$, the *covering number* of I.
(4) $\mathrm{cof}(I) = \min\{|\mathcal{F}| \colon \mathcal{F} \subseteq I \wedge \forall A \in I \exists B \in \mathcal{F}(A \subseteq B)\}$, the *cofinality* of I.

Lemma 2.44. *Let X be an uncountable set and let I be a σ-ideal on X such that $X \notin I$ and I contains all singletons. Then*

$$\aleph_1 \leq \mathrm{add}(I) \leq \mathrm{non}(I), \mathrm{cov}(I) \leq \mathrm{cof}(I) \leq 2^{|X|}.$$

Proof. Exercise. □

Definition 2.45. A *slalom* is a function $S:\omega\to[\omega]^{<\aleph_0}$ such that for all $n\in\omega$, $|S(n)|\leq 2^n$. For $x\in\omega^\omega$ we say that x *goes through* the slalom S if for all $n\in\omega$, $x(n)\in S(n)$.

Lemma 2.46. *Let I denote the σ-ideal on ω^ω generated by the sets*

$$\{x\in\omega^\omega:\ x \text{ goes through } S\},$$

where $S:\omega\to[\omega]^{<\aleph_0}$ is a slalom. Then $\mathrm{cof}(\mathtt{null})\leq\mathrm{cov}(I)$ *and* $\mathrm{add}(\mathtt{null})\geq\mathrm{non}(I)$.

Proof. Let λ denote the Lebesgue measure on $\mathbb{R}$. Let $\mathcal{S}$ be a family of slaloms such that every $x\in\omega^\omega$ goes through a slalom from $\mathcal{S}$.

Let $A\subseteq\mathbb{R}$ be of measure zero. Recall that if $A\subseteq\mathbb{R}$ is of measure zero, then for every sequence $(\varepsilon_n)_{n\in\omega}$ of positive real numbers there is a sequence $(U_n)_{n\in\omega}$ of finite unions of open intervals with rational endpoints such that $A\subseteq\bigcup_{n\in\omega}U_n$ and for all $n\in\omega$, $\lambda(U_n)<\varepsilon_n$.

Let $(O_n)_{n\in\omega}$ be an enumeration of all finite unions of open intervals with rational endpoints, let $e:\omega\times\omega\to\omega$ be a bijection, and fix a matrix $(\varepsilon_{ij})_{i,j\in\omega}$ of positive real numbers such that for all $i\in\omega$,

$$\sum_{j\in\omega}\varepsilon_{ij}<\frac{1}{2^i}.$$

Now for every $i\in\omega$ there is $f_i\in\omega^\omega$ such that $A\subseteq\bigcup_{j\in\omega}O_{f_i(j)}$ and for every $j\in\omega$,

$$2^{e(i,j)}\cdot\lambda(O_{f_i(j)})<\varepsilon_{ij}.$$

Consider the $r_A:\omega\to\omega$ defined by $r_A(n)=m$ iff $m=f_i(j)$ and $n=e(i,j)$. Now there is a slalom $S\in\mathcal{S}$ such that r_A goes through S. By the definition of r_A, A is a subset of

$$B_S=\bigcap_{i\in\omega}\bigcup_{j\in\omega}\bigcup\{O_m:m\in S(e(i,j))\wedge 2^{e(i,j)}\cdot\lambda(O_m)<\varepsilon_{ij}\}.$$

By the definition of a slalom, for all $i,j\in\omega$ the set $S(e(i,j))$ has at most $2^{e(i,j)}$ elements. It follows that the measure of $\bigcup\{O_m:m\in S(e(i,j))\wedge 2^{e(i,j)}\cdot\lambda(O_m)<\varepsilon_{ij}\}$ is not greater than ε_{ij}. Therefore, and by the choice of $(\varepsilon_{ij})_{i,j\in\omega}$, for every $i\in\omega$,

$$\lambda\Big(\bigcup_{j\in\omega}\bigcup\{O_m:m\in S(e(i,j))\wedge 2^{e(i,j)}\cdot\lambda(O_m)<\varepsilon_{ij}\}\Big)<\frac{1}{2^i}.$$

It follows that B_S is of measure zero.

It follows that `null` has a cofinal subset of size $|\mathcal{S}|$.

On the other hand, if $\mathcal{N}$ is a family of measure zero sets such that $\bigcup\mathcal{N}$ is not of measure zero, for each $A \in \mathcal{N}$ we can choose a function $r_A : \omega \to \omega$ such that for every slalom S such that r_A goes through S we have that $A \subseteq B_S$. We claim that $\{r_A : A \in \mathcal{N}\} \notin I$.

Looking for a contradiction, let $\mathcal{S}$ be a countable set of slaloms and suppose that each r_A, $A \in \mathcal{N}$, goes through a slalom from $\mathcal{S}$. For each $S \in \mathcal{S}$ the set B_S is of measure zero. If for some $A \in \mathcal{N}$, r_A goes through $S \in \mathcal{S}$, then $A \subseteq B_S$. It follows that $B = \bigcup_{S\in\mathcal{S}} B_S$ is of measure zero and contains $\bigcup\mathcal{N}$. This contradicts our assumption that $\bigcup\mathcal{N}$ is not of measure zero, showing that indeed, add(`null`) $\geq$ non(I). □

Theorem 2.47. cof(`null`) $\leq \mathfrak{hm}$

Proof. We reconsider the proof of Lemma 2.41. We may assume that $\mathfrak{hm} < 2^{\aleph_0}$. By the proof of Lemma 2.32, there is a family $\mathcal{F}$ of size $\mathfrak{hm}$ such that for all $x, y \in \omega^\omega$ there is $f \in \mathcal{F}$ such that (f Lipschitz function of class ≤ 1 from ω^ω to ω^ω and $f(x) = y$) or (f is a Lipschitz function from ω^ω to ω^ω of class $\leq 1/2$ and $f(y) = x$.

Claim 2.48. If $f : \omega^\omega \to \omega^\omega$ is Lipschitz of class $\leq 1/2$, then all the elements of $f[2^\omega]$ go through a single slalom.

For the proof of the claim, consider $S : \omega \to [\omega]^{<\aleph_0}$ defined by $S(n) = \{f(x)(n) : x \in 2^\omega\}$. Since f is Lipschitz of class $\leq 1/2$, $f(x)(n)$ depends only on the first n coordinates of x, i.e., $f(x)(n)$ only denpends on $x \upharpoonright n$. But there are only 2^n possibilities for $x \upharpoonright n$ if $x \in 2^\omega$. It follows that S is a slalom.

It follows from the definition of S that all elements of $f[2^\omega]$ go through S. This shows the claim.

Now let $x \in \omega^\omega$ be arbitrary. Since $\mathfrak{hm} < 2^{\aleph_0}$, there is $y \in 2^\omega$ such that for no $f \in \mathcal{F}$, $f(x) = y$. By the choice of $\mathcal{F}$, there is a Lipschitz function of class $\leq 1/2$ from ω^ω to ω^ω such that $f(y) = x$. In particular, $x \in f[2^\omega]$. It follows that the family

$$\{f[2^\omega] : f \in \mathcal{F},\ f : \omega^\omega \to \omega^\omega \text{ is Lipschitz of class } \leq 1/2\}$$

covers ω^ω.

By the claim, this implies that there is a family of $\mathfrak{hm}$ slaloms such that every element of ω^ω goes through a slalom from the family. By Lemma 2.46, this implies $\operatorname{cof}(\mathtt{null}) \leq \mathfrak{hm}$. $\square$

Cichoń's diagram shows the relations between $\mathfrak{b}$, $\mathfrak{d}$ and the cardinal invariants of the σ-ideals $\mathtt{null}$ and $\mathtt{meager}$ that can be proved in ZFC. Here an arrow "$\rightarrow$" stands for "$\leq$". The cardinal $\mathfrak{b}$ is $\operatorname{non}(\mathtt{compact})$, where $\mathtt{compact}$ is the σ-ideal generated by the compact subsets of ω^ω

$$\begin{tikzcd}
 & & & 2^{\aleph_0} \\
\operatorname{cov}(\mathtt{null}) \arrow[r] & \operatorname{non}(\mathtt{meager}) \arrow[r] & \operatorname{cof}(\mathtt{meager}) \arrow[r] & \operatorname{cof}(\mathtt{null}) \arrow[u] \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\operatorname{add}(\mathtt{null}) \arrow[uu] \arrow[r] & \operatorname{add}(\mathtt{meager}) \arrow[u] \arrow[r] & \operatorname{cov}(\mathtt{meager}) \arrow[u] \arrow[r] & \operatorname{non}(\mathtt{null}) \arrow[uu] \\
\aleph_1 \arrow[u] & & &
\end{tikzcd}$$

Moreover, it can be shown in ZFC that

$$\operatorname{add}(\mathtt{meager}) = \min(\operatorname{cov}(\mathtt{meager}), \mathfrak{b})$$

and

$$\operatorname{cof}(\mathtt{meager}) = \max(\operatorname{non}(\mathtt{meager}), \mathfrak{d}).$$

Exercise 2.49. Show that

$$\operatorname{non}(\mathtt{compact}) = \operatorname{add}(\mathtt{compact})$$

and

$$\operatorname{cov}(\mathtt{compact}) = \operatorname{cof}(\mathtt{compact}).$$

Exercise 2.50. Show that $\mathfrak{d} \leq \operatorname{cof}(\mathtt{null})$ and $\operatorname{add}(\mathtt{null}) \leq \mathfrak{b}$. You will have to use the fact that actually $\operatorname{cof}(\mathtt{null}) = \operatorname{cov}(I)$ and $\operatorname{add}(\mathtt{null}) = \operatorname{non}(I)$, where is is the ideal defined in Lemma 2.46.

Theorem 2.47 implies that $\mathfrak{hm}$ is at least as big as all the cardinals in Cichoń's diagram(not counting $2^{\aleph_0}$). It is consistent that $\mathfrak{hm}(c) < 2^{\aleph_0}$ for all continuous colorings $c : [X]^2 \to 2$ on a Polish space X.

2.6. **The special role of 2^ω and ω^ω.**

Definition 2.51. A topological space X is *zero-dimensional* if it has a basis of the topology consisting of sets that are both open and closed (*clopen*).

Example 2.52. The spaces $\{1/n : n > 0\} \cup \{0\}$ (the convergent sequence), 2^ω and ω^ω are zero-dimensional. Every subspace of a zero-dimensional space is zero-dimensional.

Lemma 2.53. *Let X be a compact space. Suppose $\mathcal{B}$ is collection of clopen subsets of X that is closed under complementation, unions and intersections. If $\mathcal{B}$* separates points, *in the sense that for all $x, y \in X$ with $x \neq y$ there is $A \in \mathcal{B}$ with $x \in A$ and $y \notin A$, then $\mathcal{B}$ is a basis for the topology on X.*

Proof. We have to show that for all open sets $O \subseteq X$ and all $x \in O$ there is $B \in \mathcal{B}$ such that $x \in B \subseteq O$. So, let $O \subseteq X$ be open and $x \in O$. Let $C = X \setminus O$. Then C is compact. For each $y \in C$ let $A_y \in \mathcal{B}$ be such that $y \in A_y$ and $x \notin A_y$. Now $C \subseteq \bigcup_{y \in C} A_y$.

Since C is compact, there is a finite set $F \subseteq C$ such that $C \subseteq \bigcup_{y \in F} A_y$. Now $B = X \setminus \bigcup_{y \in F} A_y \in \mathcal{B}$ and $x \in B \subseteq O$. □

Lemma 2.54. *Let X be a compact, second-countable, zero-dimensional space. Then X has a countable basis of the topology consisting of clopen sets.*

Proof. Let $\mathcal{A}$ be a countable basis for the topology on X. For every pair A, B of disjoint elements of $\mathcal{A}$ such that there is a clopen set $C \subseteq X$ with $A \subseteq C$ and $C \cap B = \emptyset$ choose one such clopen set $C_{A,B}$. Let $\mathcal{B}$ denote the collection of all Boolean combinations of sets of the form $C_{A,B}$. Then $\mathcal{B}$ is countable.

In order to show that $\mathcal{B}$ is a basis for the topology on X, by Lemma 2.53 it is enough to show that $\mathcal{B}$ separates points. Let $x, y \in X$ be such that $x \neq y$. Since X is zero-dimensional, there is a clopen set $D \subseteq X$ such that $x \in D$ and $y \notin D$. Since $\mathcal{A}$ is a basis, there are $A, B \in \mathcal{A}$ such that $x \in A \subseteq D$ and $y \in B \subseteq X \setminus D$. Now $C_{A,B} \in \mathcal{B}$ separates x and y. □

Lemma 2.55. *Every zero-dimensional, second countable, compact space X without isolated points is homeomorphic to 2^ω.*

Proof. Let $\mathcal{B} = \{O_n : n \in \omega\}$ be a countable basis for the topology on X consisting of clopen sets. We define a family $(U_\sigma)_{\sigma \in 2^\omega}$ of nonempty clopen subsets of X such that

(1) $U_\emptyset = X$,
(2) for all $\sigma \in 2^{<\omega}$, $U_\sigma = U_{\sigma^\frown 0} \cup U_{\sigma^\frown 1}$,
(3) for all $\sigma \in 2^{<\omega}$, $U_{\sigma^\frown 0} \cap U_{\sigma^\frown 1} = \emptyset$, and
(4) for all $n \in \omega$ and every $\sigma \in 2^{n+1}$, $U_\sigma \subseteq O_n$ or $U_\sigma \cap O_n = \emptyset$.

Suppose we have defined U_σ for some $\sigma \in 2^n$. If $U_\sigma \subseteq O_n$ or $U_\sigma \cap O_n = \emptyset$, we use the fact that X has no isolated points and is zero-dimensional to find two disjoint nonempty clopen sets $U_{\sigma^\frown 0}, U_{\sigma^\frown 1} \subseteq U_\sigma$ with $U_{\sigma^\frown 0} \cup U_{\sigma^\frown 1} = U_\sigma$. Otherwise let $U_{\sigma^\frown 0} = U_\sigma \cap U_n$ and $U_{\sigma^\frown 1} = U_\sigma \setminus U_n$.

Now let $s \in 2^\omega$. We claim that the set $D_s = \bigcap_{n \in \omega} U_{s \restriction n}$ has exactly one element. First of all, the set is nonempty since it is the intersection of a decreasing sequence of nonempty closed sets in a compact space. If $x \in D_s$ and $y \in X$ is different from x, there is $n \in \omega$ with $x \in O_n$ and $y \notin O_n$. By (4), either $U_{s \restriction (n+1)} \subseteq O_n$ or $U_{s \restriction (n+1)} \cap O_n = \emptyset$. It follows that not both x and y can be elements of D_s. Hence D_s has a unique element $f(s)$.

It is easily checked that $f : 2^\omega \to X$ is a homeomorphism. □

Definition 2.56. Let X be a topological space. A set $A \subseteq X$ is F_σ if it is the union of countably many closed sets. A is G_δ if it is the intersection of countably many open sets.

Exercise 2.57. Show that every open set in a metric space is F_σ.

Exercise 2.58. Use the previous exercise to show that Boolean combinations of open and closed sets are both F_σ and G_δ.

Theorem 2.59. *Every Polish space is a bijective continuous image of a closed subset of* ω^ω.

Proof. Let X be a Polish space without isolated points. We construct a socalled *Lusin scheme* $(U_\sigma)_{\sigma \in \omega^{<\omega}}$ of F_σ-subsets of X such that

(1) $U_\emptyset = X$,
(2) for all $n \in \omega$ and all $\sigma \in \omega^n$, $\operatorname{diam}(U_\sigma) < 2^{-n}$,
(3) for all $\sigma \in \omega^{<\omega}$ and all $n, m \in \omega$, if $n \neq m$, then $U_{\sigma^\frown n} \cap U_{\sigma^\frown m} = \emptyset$,
(4) for all $\sigma \in \omega^{<\omega}$, $U_\sigma = \bigcup_{m \in \omega} U_{\sigma^\frown m}$, and

(5) for all $\sigma \in \omega^{<\omega}$ and all $m \in \omega$, $\mathrm{cl}(U_{\sigma^\frown m}) \subseteq U_\sigma$.

Notice that some of the U_σ may be empty. Suppose U_σ has been defined for some $\sigma \in \omega^n$. Since U_σ is F_σ, there are closed sets $C_0 \subseteq C_1 \subseteq \dots$ such that $U_\sigma = \bigcup_{m\in\omega} C_m$.

Since X is separable, U_σ can be covered by countably many open sets O_m, $m \in \omega$, of diameter $< 2^{-n-1}$. Now $\{O_m \cap C_k : m, k \in \omega\}$ covers U_σ and is a countable collection of sets that are Boolean combinations of open and closed sets. It follows that there is a collection $(U_{\sigma^\frown m})_{m\in\omega}$ of pairwise disjoint sets of diameter $< 2^{-n-1}$ that are Boolean combinations of open and closed sets such that $U_\sigma = \bigcup_{m\in\omega} U_{\sigma^\frown m}$ and for all $m \in \omega$ there is $k \in \omega$ with $U_{\sigma^\frown m} \subseteq C_k$.

Now for each $m \in \omega$, $U_{\sigma^\frown m}$ is F_σ and $\mathrm{cl}(U_{\sigma^\frown m}) \subseteq U_\sigma$. This finishes the construction of the family $(U_\sigma)_{\sigma\in\omega^{<\omega}}$.

Let D the set of all $x \in \omega^\omega$ such that $\bigcap_{n\in\omega} U_{x\restriction n}$ is nonempty. Let x be such that $\bigcap_{n\in\omega} U_{x\restriction n}$ is empty. Since X is a complete metric space and for all $n \in \omega$ we have $\mathrm{cl}(U_{x\restriction n+1}) \subseteq U_{x\restriction n}$, $\bigcap_{n\in\omega} U_{x\restriction n} = \emptyset$ implies that for some n, $U_{x\restriction n} = \emptyset$. It follows that the open set of all extension of $x \restriction n$ is disjoint from D. Hence D is closed.

By (2), for all $x \in D$, the set $\bigcap_{n\in\omega} U_{x\restriction n}$ has exactly one element $f(x)$. It is easily checked that f is a continuous bijection from D onto X. □

Corollary 2.60. *Every Polish space is the union of at most $\mathfrak{d}$ Cantor spaces and singletons.*

Proof. Let X be a Polish space. Let $D \subseteq \omega^\omega$ be a closed set and let $f : D \to X$ be a continuous bijection. By our definition of $\mathfrak{d}$, there is a family $\mathcal{F}$ of compact sets such that $D \subseteq \bigcup \mathcal{F}$. Since D is closed, we can assume that the elements of $\mathcal{F}$ are subsets of D.

Since every compact subset of ω^ω can be written as the union of a countable set and a perfect compact set, we may assume that every element of $\mathcal{F}$ is a singleton or a perfect set. Every compact perfect subset of ω^ω is homeomorphic to 2^ω.

Since f is continuous and 1-1, for each $F \in \mathcal{F}$, $f[F]$ is a singleton or a homeomorphic copy of 2^ω. Clearly, $\bigcup\{f[F] : F \in \mathcal{F}\} = X$. □

2.7. **Constructing $c_{\max}$.** So far we have shown that $\mathfrak{hm}$, the smallest uncountable homogeneity number of a continuous coloring on a Polish space, is at least $\mathfrak{d}$ and that every Polish space X can be covered by $\mathfrak{d}$ sets that are singletons or Cantor spaces. (In fact, every perfect Polish space can be covered by $\mathfrak{d}$ Cantor spaces.) It follows that in order to understand large homogeneity numbers, it is enough to consider continuous colorings on 2^ω.

We will construct a continuous coloring on 2^ω with the maximal homogeneity number among all continuous colorings on a Polish space.

Definition 2.61. For a tree T and $t \in T$ let $\mathrm{succ}_T(t)$ be the set of immediate successors of t in T. Recall that if A is a subset of ω^ω, then $T(A)$ denotes the set of finite initial segments of the elements of A, a subtree of $\omega^{<\omega}$. If T is a subtree of $\omega^{<\omega}$, then $[T]$ denotes the set of all elements of ω^ω which have all their finite initial segments in T. $[T]$ is a closed subset of ω^ω. In this way closed subsets of ω^ω correspond to subtrees of $\omega^{<\omega}$ without finite maximal branches.

A natural way to construct continuous pair-colorings on a subset A of ω^ω is the following: To each $t \in T(A)$ assign a coloring $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$. Now for all $\{x, y\} \in [A]^2$ let t be the longest common initial segment of x and y and put

$$c(x, y) = c_t(x \restriction n + 1, y \restriction n + 1)$$

where $n = \mathrm{dom}(t)$. Clearly, c is continuous. We call a coloring which is defined in this way an *almost node-coloring*.

A *node-coloring* on A is obtained by assigning a color to every node $t \in T(A)$ and then defining the color of $\{x, y\} \in [A]^2$ to be the color of the longest common initial segment of x and y. Equivalently, a node-coloring is an almost node-coloring in which $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$ is constant for all $t \in T$.

Both $c_{\min}$ and c_{parity} are node-colorings. Not every continuous coloring of the two-element subsets of ω^ω is an almost node-coloring.

Exercise 2.62. Construct a continuous coloring $c : [2^\omega]^2 \to 2$ that is not an almost-node coloring. I assume that you will not have difficulties to extend your example to ω^ω.

However, the following holds:

Lemma 2.63. *Let $c : [2^\omega]^2 \to 2$ be continuous. Then there is a topological embedding $e : 2^\omega \to \omega^\omega$ such that for every c_{parity}-homogeneous* *set $H \subseteq e[2^\omega]$, the coloring $c^e \restriction H$ which is induced on H by c via e is an almost node-coloring.*

Proof. Let $n \in \omega$ and let $s, t \in 2^{n+1}$ be such that $\Delta(s,t) = n$. Let O_s and O_t denote the basic open subsets of 2^ω determined by s and t, respectively.

Since $O_s \times O_t$ is compact and c is continuous, there is $m > n$ such that for all $(x,y) \in O_s \times O_t$, $c(x,y)$ only depends on $x \restriction m$ and $y \restriction m$.

It follows that there is a function $f : \omega \to \omega$ such that for all $\{x,y\} \in [2^\omega]^2$, $c(x,y)$ only depends on $x \restriction f(\Delta(x,y))$ and $y \restriction f(\Delta(x,y))$. We can choose f strictly increasing and such that $f(0) \geq 1$. For $n \in \omega$ let $g(n) = f^n(0)$.

Identifying $2^{<\omega}$ and ω, we define the required embedding $e : 2^\omega \to \omega^\omega$ by letting $e(x) = (x \restriction g(0), x \restriction g(1), \dots)$. Let $E = e[2^\omega]$. c induces a continuous pair-coloring c^e on E via e. By the choice of f, for $\{u,v\} \in [E]^2$, $c^e(u,v)$ only depends on $u \restriction (\Delta(u,v)+2)$ and $v \restriction (\Delta(u,v)+2)$. This is because if $n = \Delta(u,v)$ and $x, y \in 2^\omega$ are such that $e(x) = u$ and $e(y) = v$, then $\Delta(x,y) < g(n)$ and thus $c(x,y)$ only depends on $x \restriction f(\Delta(x,y))$ and $y \restriction f(\Delta(x,y))$. But since f is strictly increasing, $f(\Delta(x,y)) < f(g(n)) = g(n+1)$.

Now let H be a c_{parity}-homogeneous subset of E. The c_{parity}-homogeneity of H implies that for all $\{u,v\} \in [H]^2$, the restrictions of u and v to $\Delta(u,v)+1$ uniquely determine the restrictions to $\Delta(u,v)+2$. Therefore, for all $\{u,v\} \in [H]^2$, $c^e(u,v)$ only depends on $u \restriction (\Delta(u,v)+1)$ and $v \restriction (\Delta(u,v)+1)$.

It follows that $c^e \restriction H$ is an almost node-coloring. $\square$

Corollary 2.64. *For every continuous coloring $c : [2^\omega]^2 \to 2$, there is an almost node-coloring d on some compact subset of ω^ω such that $\mathfrak{hm}(c) \leq \mathfrak{hm}(d)$.*

Proof. By the previous Lemma, 2^ω can be presented as a union of $\leq \mathfrak{hm}(c_{\min})$ sets on each of which c is reducible to an almost node-coloring. Now either $\mathfrak{hm}(c) \leq \mathfrak{hm}(c_{\min})$ and hence $d = c_{\min}$ works or or for some set H in the decomposition we have $\mathfrak{hm}(c \restriction H) > \mathfrak{hm}(c_{\min})$. But in the latter case $c \restriction H$ is reducible to some almost node-coloring d that lives on a compact subset of ω^ω. Since $\mathfrak{hm}(c_{\min})^+ \geq 2^{\aleph_0}$, we now have $\mathfrak{hm}(d) = 2^{\aleph_0}$. Hence $\mathfrak{hm}(c) = \mathfrak{hm}(d)$. $\square$

We shall now define a maximal almost node-coloring.

Definition 2.65. Let $E \subseteq [\omega]^2$. The graph (ω, E) is a *random graph* if for any two finite, disjoint sets $A, B \subseteq \omega$ there is $x \in \omega$ such that every point in A is connected with x by and edge and no point of B is connected with x by an edge.

Exercise 2.66. Show that if $G = (\omega, E)$ is random, then every countable graph embeds into G.

This exercise obviously shows that any random graph contains copies of every finite graph.

Exercise 2.67. Show that any two random graphs are isomorphic. Therefore we can call any random graph *the* random graph.

Hint: Recursively construct two sequences $(a_n)_{n\in\omega}$ and $(b_n)_{n\in\omega}$ such that for all $n \in \omega$, $\{(a_i, b_i) : i < n\}$ is a finite partial isomorphism between the two random graphs. If n is even, let a_n be the first vertex of the first random graph that is not among $\{a_i : i < n\}$ and choose a suitable vertex b_n of the second random graph. If n is odd, proceed as above, but the other way round. This procedure is called *back-and-forth*.

Definition 2.68. Let $\chi_{\text{random}} : [\omega]^2 \to 2$ be the (characteristic function of the) edge relation of the random graph. For $s, t \in \omega^{\leq\omega}$ write $\text{random}(s, t) = i$ iff $n = \Delta(s, t)$ exists and $i = \chi_{\text{random}}(s(n+1), t(n+1))$. Let $c_{\text{random}} : [\omega^\omega]^2 \to 2$ be defined by $c_{\text{random}}(x, y) = \text{random}(x, y)$. Finally, let

$$c_{\max} = c_{\text{random}} \restriction \prod_{n\in\omega}(n+1) \tag{2}$$

Clearly, c_{random} and $c_{\max}$ are almost node-colorings. Since $\prod_{n\in\omega}(n+1)$ is homeomorphic to 2^ω, we regard $c_{\max}$ as a coloring on 2^ω.

Lemma 2.69. *a) If c is an almost node-coloring on a subset of ω^ω, then $c \leq c_{\text{random}}$ via a level preserving embedding (isometry) of ω^ω into ω^ω.*

b) If c is an almost node-coloring on a compact subset of ω^ω, then $c \leq c_{\max}$.

Proof. Let us prove b) first. Suppose c is an almost node-coloring on a compact subset A of ω^ω. Then $T(A)$ is a finitely branching subtree of $\omega^{<\omega}$. For each $t \in T(A)$ fix a coloring $c_t : [\mathrm{succ}_{T(A)}(t)]^2 \to 2$ such that the c_t witnesses the fact that c is an almost node-coloring. For $s, t \in T$ let $\overline{c}(s,t) = c(x,y)$ if s and t are incomparable and $x, y \in [T]$ are such that $s \subseteq x$ and $t \subseteq y$. If s and t are comparable, then $\overline{c}(s,t)$ is undefined.

Let $T_k = \{t \in T(A) : |t| = k\}$. We construct a monotone (i.e., $\subseteq$-preserving) map $e : \bigcup_{k\in\omega} T_k \to T(\prod_{n\in\omega}(n+1))$ which induces the required embedding of A into $\prod_{n\in\omega}(n+1)$.

We argue by induction on k. Suppose that $e(s) \in \prod_{n\le n(k)}(n+1)$ is defined for all $s \in T_k$, and for all $s,t \in T_k$ we already have $\mathrm{random}(e(s), e(t)) = \overline{c}(s,t)$.

Find $n(k+1) > n(k)$ such that for all $s \in T_k$ there is $t \in \prod_{n<n(k+1)}(n+1)$ with $e(s) \subseteq t$ and $c_s \le \mathrm{random} \restriction \mathrm{succ}_{T(\prod_{n\in\omega}(n+1))}(t)$. Now it is obvious how to define e on T_{k+1} with images in $\prod_{n\le n(k+1)}(n+1)$.

a) is proved similarly, using the fact that every countable graph occurs as an induced subgraph of $(\mathrm{succ}_{\omega^{<\omega}}(s), \mathrm{random})$ for every $s \in \omega^{<\omega}$. $\square$

Corollary 2.70. *For every Polish space X and every continuous $c : [X]^2 \to 2$:*

$$\mathfrak{hm}(c) \le \mathfrak{hm}(c_{\max}).$$

Proof. Let c be an arbitrary continuous coloring on a Polish space X. Let $X = \bigcup_{\alpha<\mathfrak{d}} A_\alpha$, where each A_α is either a singleton or a Cantor space. Now

$$\mathfrak{hm}(c) \le \sum_{\alpha<\mathfrak{d}} \mathfrak{hm}(c \restriction A_\alpha) \le \mathfrak{d} \cdot \sup_{\alpha<\mathfrak{d}} \mathfrak{hm}(c \restriction A_\alpha).$$

For each $\alpha < \mathfrak{d}$, $\mathfrak{hm}(c \restriction A_\alpha) \le \mathfrak{hm}(c_{\max})$. Since $\mathfrak{d} \le \mathfrak{hm}(c_{\max})$, it follows that $\mathfrak{hm}(c) \le \mathfrak{hm}(c_{\max})$. $\square$

2.8. **Continuous n-colorings.** In the following we give a brief discussion of continuous n-colorings, $n > 2$. We restrict our attention colorings on 2^ω. 2^ω carries a natural linear order, namely the lexicographic order: for $x, y \in 2^\omega$ we have $x < y$ if $x \ne y$ and $x(\Delta(x,y)) = 0$ and $y(\Delta(x,y)) = 1$. In the following, when we consider $\{x_1, \dots, x_n\} \in [2^\omega]^n$, we always assume that $x_1 < \dots < x_n$.

Now let $a = \{x_1, \dots, x_n\} \in [2^\omega]^n$ be such that the set $D(a) = \{\Delta(x,y) : x, y \in a \wedge x \neq y\}$ is of size $n-1$. We define the *type* of a to be the unique permutation $\mathtt{type}(a) : n-1 \to n-1$ that maps k to ℓ if $\Delta(x_k, x_{k+1})$ is the ℓ-th element of $D(a)$.

Observe that $\mathtt{type}$ is continuous in the sense that whenever it is defined on $\{x_1, \dots, x_n\}$, then there are disjoint open sets $U_1, \dots, U_n \subseteq 2^\omega$ such that $\mathtt{type}$ is defined on all of $[U_1, \dots, U_n]$ and constant on that set. Also, every 3-element subset of 2^ω has a type.

Lemma 2.71. *Let $A \subseteq 2^\omega$ be uncountable. Then for each permutation σ on $n-1$ there is $\{x_1, \dots, x_n\} \in [A]^n$ with $\mathtt{type}(x_1, \dots, x_n) = \sigma$.*

Proof. After removing countably many points from A if necessary, we may assume that for every open set $O \subseteq 2^\omega$, if $O \cap A \neq \emptyset$, then $A \cap O$ is uncountable. Now the tree $T = T(A)$ of all finite initial segments of elements A is perfect in the sense that every node in T has two incomparable extensions in T.

We easily find n branches in the perfect tree T such that for the corresponding points $x_1, \dots, x_n$ in 2^ω we have $\mathtt{type}(x_1, \dots, x_n) = \sigma$. Being branches of T, the points $x_1, \dots, x_n$ are elements of the closure of A. By the continuity of $\mathtt{type}$, we can actually choose the branches so that $x_1, \dots, x_n \in A$. $\square$

Corollary 2.72. *There is a continuous coloring $c : [2^\omega]^3 \to 2$ such that every c-homogeneous subset of 2^ω is countable.*

Proof. Let $c(x,y,z) = \mathtt{type}(x,y,z)$. $\square$

However, Blass proved the following:

Theorem 2.73. *Let $c : [2^\omega]^n \to 2$ be continuous. Then there is a perfect set $P \subseteq 2^\omega$ every $a \in [P]^n$ has a type and on P, c only depends on the type of an n-element set.*

Exercise 2.74. Let $n > 2$. Show that every perfect set $P \subseteq 2^\omega$ has a perfect subset Q such that every $a \in [Q]^n$ has a type.

3. Open colorings

Definition 3.1. Let X be a Hausdorff space. A coloring $c : [X]^2 \to 2$ is *open*, if $c^{-1}(0)$ is an open subset of $[X]^2$.

Theorem 3.2 (Todorcevic)**.** *Let X be a Polish space and $c : [X]^2 \to 2$ an open coloring. Then either there is a perfect c-homogeneous set of color 0 or X is the union of countably many c-homogeneous sets of color 1.*

Proof. Suppose X is not the union of countably many c-homogeneous sets of color 1. For $A \subseteq X$ let

$$A' = X \setminus \bigcup\{O \subseteq X : O \text{ is open and } A \cap O \text{ is the union of countably many } c\text{-homogeneous sets of color } 1\}.$$

For every ordinal α let $A^{(\alpha+1)} = (A^{(\alpha)})'$ and let $A^{(\alpha)} = \bigcap_{\beta<\alpha} A^{(\beta)}$ if α is a limit ordinal.

Since X is second countable, there is a countable ordinal α such that $X^{(\alpha+1)} = X^{(\alpha)}$. Let $Y = X^{(\alpha)}$. Clearly, Y is closed in X and hence a Polish space. Since X is not the union of countably many c-homogeneous sets of color 1, Y is nonempty. Since $Y' = Y$, for every open set $O \subseteq X$, either $O \cap Y = \emptyset$ or $O \cap X$ is not the union of countably many c-homogeneous sets of color 1. In particular, no open subset of Y is c-homogeneous of color 1.

We use this information to construct a perfect scheme $(U_\sigma)_{\sigma\in 2^{<\omega}}$ of open subsets of Y such that for all $\sigma \in 2^{<\omega}$ and all $\{x, y\} \in [U_{\sigma^\frown 0}, U_{\sigma^\frown 1}]$ we have $c(x, y) = 0$. As usual, the perfect scheme induces an embedding $e : 2^\omega \to Y$. The image of this embedding is a perfect c-homogeneous subset of Y of color 0. $\square$

Definition 3.3. The *Open Coloring Axiom* by Todorcevic ($\mathrm{OCA}_{[\mathrm{T}]}$) states that for every open coloring c on a separable metric space X, either there is an uncountable c-homogeneous set of color 0 or X is the union of countably many c-homogeneous sets of color 1.

$\mathrm{OCA}_{[\mathrm{T}]}$ is consistent with the usual axioms of set theory (ZFC) and follows from the socalled *Proper Forcing Axiom* (PFA), which is a strengthening of Martin's Axiom (MA).

Note that $\mathrm{OCA}_{[\mathrm{T}]}$ implies that every open coloring on an uncountable, separable metric space has an uncountable homogeneous set.

Exercise 3.4. CH refutes $\mathrm{OCA}_{[\mathrm{T}]}$.

Hint: Use CH to construct an uncountable set $X \subseteq 2^\omega$ that intersects every nowhere dense subset of 2^ω in at most countably many points. Now consider the restriction of $c_{\min}$ to X.

We show some of the easier consequences of the Open Coloring Axiom.

Theorem 3.5. *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$.

a) Every uncountable subset of $\mathcal{P}(\omega)$ contains an uncountable chain or antichain. Here an antichain consists of pairwise incomparable elements.

b) From the conclusion of a) it follows that for every function from an uncountable subset X of $\mathbb{R}$ and every function $f : X \to \mathbb{R}$ there is an uncountable set $Y \subseteq X$ such that f is monotone on Y.

c) The conclusion of a) implies that $\mathfrak{b} \geq \aleph_2$.

Proof. a) Let $X \subseteq \mathcal{P}(\omega)$ be uncountable. Identifying every subset of ω with its characteristic function from ω to 2, X inherits a separable metric topology from 2^ω. We define a coloring $c : [X]^2 \to 2$ by letting $c(x, y) = 0$ if x and y are incomparable as subsets of ω.

If x and y are incomparable, then there are $n, m \in \omega$ such that $n \in x \setminus y$ and $m \in y \setminus x$. But with respect to the topology on X, the set of all $a \in X$ that contain a certain n and don't contain a certain m is open. It follows that incomparability is open in $[X]^2$, showing that c is an open coloring.

By the Open Coloring Axiom, there is an uncountable c-homogeneous set $H \subseteq X$. Depending on the color, H is either a chain or an antichain.

b) For each $x \in X$ let

$$q_x = \{y \in \mathbb{Q} : y \leq x\} \times \{z \in \mathbb{Q} : z \leq f(x)\}.$$

Observe that the map assigning x to q_x is 1-1. Hence $\{q_x : x \in X\}$ is an uncountable subset of $\mathcal{P}(\mathbb{Q} \times \mathbb{Q})$. By the conclusion of a), there is an uncountable set $Y \subseteq X$ such that $\{q_x : x \in Y\}$ is either a chain or an antichain.

If for $x, y \in X$ we have $q_x \subseteq q_y$, then $x \leq y$ and $f(x) \leq f(y)$. If q_x and q_y are incomparable, and $x < y$, then $f(x) > f(y)$. It follows that on Y, f is either monotonically increasing or strictly decreasing.

c) If $\mathfrak{b} = \aleph_1$, then there is a sequence $(f_\alpha)_{\alpha<\omega_1}$ of functions from ω to ω such that for no $g \in \omega^\omega$, $f_\alpha \leq^* g$ for all $\alpha < \omega_1$. Recall that $f \leq^* g$ if the set $\{n \in \omega : f(n) > g(n)\}$ is finite. We say that the sequence $(f_\alpha)_{\alpha<\omega_1}$ is unbounded.

We may assume that each f_α is strictly increasing. We may also assume that for all $\alpha, \beta < \omega_1$ with $\alpha < \beta$, we have $f_\alpha \leq^* f_\beta$ and $f_\beta \not\leq^* f_\alpha$. This is true because for every countable subset C of ω^ω there is a function $g : \omega \to \omega$ such that for all $f \in C$, $f \leq^* g$ and we can use this fact to recursively correct the sequence $(f_\alpha)_{\alpha<\omega_1}$ to be strictly $\leq^*$-increasing.

Claim 3.6. For every uncountable set $A \subseteq \omega_1$, there are $\alpha, \beta \in A$ such that $\alpha < \beta$ and $f_\alpha \leq f_\beta$.

For the proof of the claim, first observe that $(f_\alpha)_{\alpha\in A}$ is unbounded since A is cofinal in ω_1.

For each $s \in \omega^{<\omega}$ let α_s be the minimal $\alpha \in A$ with $s \subseteq f_\alpha$, provided there is such an α, and let α_s be the minimal element of A, otherwise. Since $\omega^{<\omega}$ is countable, there is $\beta_0 \in A$ such that $\alpha_s < \beta_0$ for all $s \in \omega^{<\omega}$.

Since $A \setminus \beta_0$ is uncountable, there is $n_0 \in \omega$ such that for an uncountable set $A_0 \subseteq A \setminus \beta_0$ and all $\alpha \in A_0$ we have

$$f_{\beta_0} \restriction \omega \setminus n_0 \leq f_\alpha \restriction \omega \setminus n_0.$$

By thinning out A_0 if necessary, we may assume that for some $s_0 \in \omega^{n_0}$ and all $\alpha \in A_0$ we have $s_0 \subseteq f_\alpha$.

Subclaim. There is $n \in \omega$ such that for all $\alpha < \omega_1$ and all $m \in \omega$ there is $\beta \in A_0$ such that $\alpha \leq \beta$ and $f_\beta(n) \geq m$.

Suppose not. Then for all $n \in \omega$ there are $\alpha_n < \omega_1$ and $m_n \in \omega$ such that for all $\beta \in A_0$, if $\alpha_n \leq \beta$, then $f_\beta(n) < m_n$. Let $\gamma = \sup_{n\in\omega} \alpha_n$. Then $\gamma < \omega_1$. For each $n \in \omega$ let $f(n) = m_n$. Now for all $\alpha \in A_0$ with $\alpha > \gamma$ we have $f_\alpha \leq f$, which contradicts the unboundedness of $(f_\alpha)_{\alpha<\omega_1}$. This proves the subclaim.

Now let $n_1 \in \omega$ be the minimal $n \in \omega$ witnessing the subclaim. Note that $n_1 \geq n_0$ since $s_0 \subseteq f_\beta$ for all $\beta \in A_0$. By the minimality of n_1, there are $s \in \omega^{n_1}$ and $\beta_1 < \omega_1$ such that $\beta_0 \leq \beta_1$ and for all $\beta \in A_0$ with $\beta_1 \leq \beta$, $f_\beta \restriction n_0 \leq s$.

Since $S = \{t \in \omega^{n_1} : t \leq s\}$ is finite, there are $s_1 \in S$ and an uncountable set $A_1 \subseteq A_0$ such that for all $\alpha \in A_1$, $s_1 \subseteq f_\alpha$. Let $n_2 \in \omega$ be such that $n_1 \leq n_2$ and

$$f_{\alpha_{s_1}} \restriction \omega \setminus n_2 \leq f_{\beta_0} \restriction \omega \setminus n_2$$

and choose $\alpha^* \in A_1$ such that $f_{\alpha^*}(n_1) \geq f_{\alpha_{s_1}}(n_2)$. Then $f_{\alpha_{s_1}} \leq f_{\alpha^*}$:

Let $n \in \omega$. If $n < n_1$, then $f_{\alpha_{s_1}}(n) = s_1(n) = f_{\alpha^*}(n)$. If $n \in n_2 \setminus n_1$, then

$$f_{\alpha_{s_1}}(n) \leq f_{\alpha_{s_1}}(n_2) \leq f_{\alpha^*}(n_1) \leq f_{\alpha^*}(n).$$

Finally, if $n \geq n_2$, then

$$f_{\alpha_{s_1}}(n) \leq f_{\beta_0}(n) \leq f_{\alpha^*}(n).$$

This finishes the proof of the claim.

Returning to the proof of c), for each $\alpha < \omega_1$, consider the set $A_\alpha = \{(n, m) \in \omega^2 : m \leq f_\alpha(n)\}$. By the conclusion of A, there is an uncountable set $X \subseteq \omega_1$ such that the A_α, $\alpha \in X$, are either pairwise comparable or pairwise incomparable. In the first case, the sequence $(f_\alpha)_{\alpha \in X}$ would have to be strictly increasing with respect to $\leq$ and thus the sequence $(A_\alpha)_{\alpha \in X}$ would have to be strictly increasing with respect to $\subseteq$. But this is impossible since all the A_α are subsets of the same countable set $\omega \times \omega$.

In the second case the f_α, $\alpha \in X$, would have to be pairwise $\leq$-incomparable. But that contradicts the claim. This shows that $\mathfrak{b} \geq \aleph_2$. $\square$

We mention some more consequences of the Open Coloring Axiom, but without proofs.

Theorem 3.7. *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$. *Then* $\mathfrak{b} = \aleph_2$.

Definition 3.8. a) Given two sets A and B let $A \triangle B$ denote the *symmetric difference* $(A \setminus B) \cup (B \setminus A)$ of A and B.

b) Let `fin` denote the collection of finite subsets of ω. Then `fin` is an *ideal* on ω, i.e., `fin` is closed under taking subsets and finite unions.

c) $\mathcal{P}(\omega)/\mathtt{fin}$ is the Boolean algebra obtained by identifying $A, B \in \mathcal{P}(\omega)/\mathtt{fin}$ if $A \triangle B \in \mathtt{fin}$. More precisely, let $\mathcal{P}(\omega)/\mathtt{fin}$ consist of all equivalence classes $A \triangle \mathtt{fin} = \{B \subseteq \omega : A \triangle B \in \mathtt{fin}\}$ of subsets A of ω. It is easily checked that the natural Boolean operations on $\mathcal{P}(\omega)/\mathtt{fin}$ are indeed well defined.

Exercise 3.9. Let $A, B \subseteq \omega$ be *cofinite*, i.e, such that $\omega \setminus A$ and $\omega \setminus B$ are finite. Let $f : A \to B$ be a bijection. Show that $F : \mathcal{P}(\omega)/\mathtt{fin} \to \mathcal{P}(\omega)/\mathtt{fin}$ defined by $F(C \triangle \mathtt{fin}) = f[C] \triangle \mathtt{fin}$ is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. These automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ are called trivial.

Theorem 3.10 (Veličković)**.** *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$ *together with Martin's Axiom for* $\aleph_1$ *dense sets. Then every automorphism of* $\mathcal{P}(\omega)/\mathtt{fin}$ *is trivial.*

It should be pointed out that under CH there is a non-trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$.

Exercise 3.11. It is easily checked that the trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ form a subgroup of the full automorphism group. Given a trivial automorphism F induced by a bijection $f : A \to B$ between cofinite sets, let $\mathrm{index}(F) = |\omega \setminus B| - |\omega \setminus A|$. Show that the index of a trivial automorphism is well defined and that the index map is a homomorphism from the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ to the group $(\mathbb{Z}, +)$.

Example 3.12. Let $A = \omega$ and $B = \omega \setminus 1$. Let $s : A \to B$ be defined by $s(n) = n + 1$. Clearly, s is a bijection between two cofinite subsets of ω. The *shift* S is the trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by s. S is of index 1.

It is an open problem whether the structure $(\mathcal{P}(\omega)/\mathtt{fin}, S)$ (where $\mathcal{P}(\omega)/\mathtt{fin}$ still carries its Boolean operations) can be isomorphic to $(\mathcal{P}(\omega)/\mathtt{fin}, S^{-1})$.

Theorem 3.13. *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$ *together with Martin's Axiom for* $\aleph_1$ *dense sets. Then* $(\mathcal{P}(\omega)/\mathtt{fin}, S)$ *is not isomorphic to* $(\mathcal{P}(\omega)/\mathtt{fin}, S^{-1})$*.*

Proof. Suppose there is an isomorphism f from $(\mathcal{P}(\omega)/\mathtt{fin}, S)$ to $(\mathcal{P}(\omega)/\mathtt{fin}, S^{-1})$. In particular, f is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. By Veličković's theorem, f is trivial and therefore has an index. For every $a \in \mathcal{P}(\omega)/\mathtt{fin}$,

$f(S(a)) = S^{-1}(f(a))$. In other words, $f \circ S = S^{-1} \circ f$. This can be written as $f \circ S \circ f^{-1} = S^{-1}$. Since the index map is a homomorphism, we have

$$-1 = \mathrm{index}(S^{-1}) = \mathrm{index}(f) + \mathrm{index}(S) - \mathrm{index}(f) = \mathrm{index}(S) = 1,$$

a contradiction. □

Definition 3.14. Let H be an infinite dimensional separable Hilbert space. Let $\mathcal{B}(H)$ be the C^*-algebra of all bounded operators on H. Let $\mathcal{K}(H)$ denote the ideal of compact operators. (An operator $\varphi : H \to H$ is compact if the closure of the image of the unit ball is compact.) The *Calkin algebra* $\mathcal{C}(H)$ is the quotient $\mathcal{B}(H)/\mathcal{K}(H)$.

$\mathcal{C}(H)$ can be considered as a non-commutative analog of $\mathcal{P}(\omega)/\mathtt{fin}$.

Definition 3.15. An automorphism φ of $\mathcal{C}(H)$ is *inner* if there is a unitary element $u \in \mathcal{C}(H)$ such that for all $a \in \mathcal{C}(H)$, $\varphi(a) = u * au$.

Theorem 3.16 (Farah)**.** *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$. *Then every automorphism of* $\mathcal{C}(H)$ *is inner.*

Again, we point out that CH implies the existence of an automorphism of $\mathcal{C}(H)$ that is not inner (Philips and Weaver).

Example 3.17. Fix an orthonormal basis $(e_n)_{n\in\omega}$ of the Hilbert space H. Let $s : H \to H$ be the operator that maps e_n to e_{n+1}. This operator turns out to be unitary modulo compact, i.e., its equivalence class S in $\mathcal{C}(H)$ is unitary.

Theorem 3.18. *Assume* $\mathrm{OCA}_{[\mathrm{T}]}$. *Then the structure* $(\mathcal{C}(H), S)$ *is not isomorphic to* $(\mathcal{C}(H), S^{-1})$. *Here* S *and* S^{-1} *are considered as constants.*

Proof. This proof is similar to the proof of the corresponding theorem for $\mathcal{P}(\omega)/\mathtt{fin}$, using the Fredholm index of a unitary operator. □

It is open whether $(\mathcal{C}(H), S)$ and $(\mathcal{C}(H), S^{-1})$ can be isomorphic.

4. Stone-Čech compactifications of discrete semigroups and Ramsey theory

We will use some simple facts about compact right-topological semigroups to prove highly non-trivial theorems in both finite and infinite Ramsey theory.

4.1. Compact right-topological semigroups.

Definition 4.1. a) Let S be a topological space and $\cdot : S \times S \to S$. Then $(S, \cdot)$ is a *right-topological semigroup* if $\cdot$ is associative and for all $s \in S$ the *right multiplication* $\rho_s : S \to S; t \mapsto t \cdot s$ is continuous. We often write st for $s \cdot t$.

b) A nonempty set $I \subseteq S$ is a *left ideal* (*right ideal*) if

$$SI = \{st : s \in S \wedge t \in I\} \subseteq I$$

(respectively $IS \subseteq I$). A left ideal is minimal if it has no proper subset that is a left ideal. $I \subseteq S$ is an ideal if it is both a left and a right ideal.

c) An element $e \in S$ is an idempotent if $ee = e$.

The following theorem collects all the facts about compact right-topological semigroups that we are going to use.

Theorem 4.2. *Let S be a compact right-topological semigroup. Then the following hold:*

1. *Every left ideal includes a minimal left ideal.*
2. *If L is a minimal left ideal of S and K an ideal, then $L \subseteq K$.*
3. *If L is a minimal left ideal of S and $p \in L$, then $L = Sp = \{sp : s \in S\}$.*
4. *Every minimal left ideal contains an idempotent.*
5. *If e is an idempotent and $x \in Se$, then $xe = e$.*

Proof. (1) We first observe that every left ideal L includes a compact left ideal. Namely, let $x \in L$. Then $Sx \subseteq L$ is a left ideal, and since $Sx = \rho_x[S]$, it is compact. It follows that every minimal left ideal is compact.

Now fix a left ideal L of S and consider the partial order of all compact left ideals $I \subseteq L$ ordered by reverse inclusion. If C is a chain in this partial order, then it has the finite intersection property, i.e., any finitely many elements of C have a nonempty intersection, and therefore $\bigcap C$ is nonempty. Clearly, $\bigcap C$ is a compact left ideal included in L. It follows that by Zorn's Lemma, the partial order has a minimal element I. Since all left ideals include a compact left ideal, I is indeed a minimal left ideal.

(2) Let L be a minimal left ideal and let K be an ideal of S. Let $x \in L$. Then $Sx \subseteq L$ and thus $Kx \subseteq L$. Since K is a right ideal, $Kx \subseteq K$. It follows that K and L have a nonempty intersection. Since K is a left ideal, $K \cap L$ is a left ideal as well. Since L is minimal, $K \cap L = L$ and thus $L \subseteq K$.

(3) If L is a minimal left ideal and $p \in L$, then $Sp \subseteq L$ and Sp is a left ideal. It follows that $Sp = L$.

(4) Let L be a minimal left ideal of S. In particular, L is a compact subsemigroup of S. Similar to the proof of (1), we use Zorn's Lemma to find a minimal compact subsemigroup A of L. Let $x \in A$. We show that $Ax = A$. Namely, let $B = Ax$. Then B is nonempty and compact. We have $BB = AxAx \subseteq AAAx \subseteq Ax = B$. It follows that B is a compact subsemigroup of L. By the minimality of A, $A = B$.

Let $C = \{y \in A : yx = x\}$. Since $Ax = A$ and $x \in A$, there is $y \in A$ such that $yx = x$. This shows that C is nonempty. Also, $C = A \cap \rho_x^{-1}(x)$. It follows that C is a closed subset of A and therefore compact.

Now let $y, z \in C$. Then $yz \in A$ and $yzx = yx = x$. It follows that $yz \in C$. Hence C is a subsemigroup of A. By the minimality of A, $C = A$. Since $x \in C$, we have $xx = x$.

(5) Let e be an idempotent in S and $x \in Se$. Choose $s \in S$ such that $x = se$. Now $xe = see = se = x$. □

4.2. Stone-Čech compactifications of discrete semigroups.

Definition 4.3. Let X be a set. $F \subseteq \mathcal{P}(X)$ is a *filter on* X if it is nonempty, closed under taking finite intersections and supersets and does not contain the emptyset.

A filter F is an *ultrafilter* if it is a maximal filter (with respect to set-theoretic inclusion).

Lemma 4.4. *Every family S of subsets of X with the finite intersection property can be extended to an ultrafilter.*

Proof. Let

$$F = \{A \subseteq X : \exists T \subseteq S(T \text{ is finite and } \bigcap T \subseteq A\}.$$

F is the smallest filter that includes S. Consider the partial order of all filters on X that extend F, ordered by set-theoretic inclusion. It is easily checked that the union of every chain of filters is again a filter on X. Hence, by Zorn's Lemma, the partial order has a maximal element, which is an ultrafilter that extends S. □

Lemma 4.5. *Let F be a filter on a set X. Then the following are equivalent:*

(1) *F is an ultrafilter.*
(2) *For all $A \subseteq X$, $A \in F$ iff $X \setminus A \notin F$.*
(3) *For all $A, B \subseteq X$, if $A \cup B \in F$, then $A \in F$ or $B \in F$.*

Proof. (1)⇒(2): Since F is a filter, it contains at most one of the sets A and $X \setminus A$. Suppose $X \setminus A \notin F$. Since F is closed under taking supersets, this implies that F contains no subset of $X \setminus A$. In other words, every element of F intersects A. Since F is closed under finite intersections, it follows that $F \cup \{A\}$ has the finite intersection property. Hence there is an ultrafilter G on X such that $F \cup \{A\} \subseteq G$. Since F is a maximal filter, $F = G$ and thus $A \in F$.

(2)⇒(3): Suppose neither A nor B are elements of F. By (2), $X \setminus A, X \setminus B \in F$ and therefore $X \setminus A \cap X \setminus B = X \setminus (A \cup B) \in F$. It follows that $A \cup B \notin F$.

(3)⇒(1): Let F be a filter satisfying (3). We show that F is maximal. Let $A \subseteq X$ be such that $F \cup \{A\}$ is contained in a filter, i.e., has the finite intersection property. Then $X \setminus A \notin F$. However, $(X \setminus A) \cup A = X \in F$ since F is nonempty and closed under taking supersets. By (3), $A \in F$. This shows the maximality of F. □

Lemma 4.6. *Let F be an ultrafilter on X. If $A_1, \dots, A_n \subseteq X$ and $A_1 \cup \dots \cup A_n \in F$, then at least one of the sets A_i, $i \in \{1, \dots, n\}$, is an element of F.*

Proof. This follows by induction from Lemma 4.5 (3). □

Definition 4.7. Let X be any set. We consider X as a topological space with the discrete topology. (Every subset of X is open.) The

Stone-Čech compactification βX of X is the set of all ultrafilters on X with the topology generated by all sets of the form

$$\hat{A} = \{p \in \beta X : A \in p\}$$

where $A \subseteq X$. We consider X as a subset of βX by identifying every $x \in X$ with the ultrafilter $\{A \subseteq X : x \in A\}$.

Lemma 4.8. *For every set X, βX is compact and X is dense in βX.*

Proof. We first show that X is dense in βX. Let $O \subseteq \beta X$ be nonempty and open. Then there is $A \subseteq X$ such that $\emptyset \neq \hat{A} \subseteq O$. For every $x \in A$ we have $x \in \hat{A}$.

We have to show that βX is Hausdorff. Let p and q be two distinct ultrafilters on X. We may assume that there is $A \in p \setminus q$. Since $A \notin q$, $X \setminus A \in q$. $\hat{A}$ and $\widehat{X \setminus A}$ are disjoint open neighborhoods of p and q, respectively.

Now let $\mathcal{U}$ be an open cover of βX. We may assume that each $U \in \mathcal{U}$ is of the form $\hat{A}$ for some $A \subseteq X$. Let S be the collection of all $A \subseteq X$ with $\hat{A} \in \mathcal{U}$. Suppose $\mathcal{U}$ has no finite subcover. Then for all $n \in \omega$ and all $A_1, \dots, A_n \in S$, $\hat{A}_1 \cup \dots \cup \hat{A}_n \neq \beta X$.

Now observe that an ultrafilter F on X is an element of $\hat{A}_1 \cup \dots \cup \hat{A}_n$ iff for some $i \in \{1, \dots, n\}$, $A_i \in F$. By Lemma 4.6, this is equivalent to $A_1 \cup \dots \cup A_n \in F$. Since $\mathcal{U}$ has no finite subcover and since X is an element of every ultrafilter on X, it follows that for all $n \in \omega$ and all $A_1, \dots, A_n \in S$ we have $A_1 \cup \dots \cup A_n \neq X$.

Passing to complements, this shows that $T = \{X \setminus A : A \in S\}$ has the finite intersection property. Hence there is an ultrafilter F extending T. Clearly, for all $A \in S$, $A \notin F$ and therefore $F \notin \hat{A}$. This shows that $\mathcal{U}$ is not an open cover of βX, a contradiction. $\square$

Observe that if F is an ultrafilter on X and $A \in F$, then $F \cap \mathcal{P}(A)$ is an ultrafilter on A. On the other hand, if F is an ultrafilter on $A \subseteq X$, then the collection of all sets $B \subseteq X$ such that $C \subseteq B$ for some $C \in F$ is an ultrafilter on X and an element of $\hat{A}$. In this way, every element of $\hat{A}$ corresponds to an element of βA. In fact, the two topological spaces are homeomorphic. The set $\hat{A}$ is the closure of the set A in βX.

The crucial property of the Stone-Čech compactification is stated in the following theorem.

Theorem 4.9. *Let X be a set and Y a compact space. Then every function $f : X \to Y$ has a unique continuous extension $\beta f : \beta X \to Y$.*

Proof. Let $p \in \beta X$. Since p is a filter, $\{f[A] : A \in p\}$ has the finite intersection property. It follows that $\{\mathrm{cl}(f[A]) : A \in p\}$ has the finite intersection property. Since Y is compact, $\bigcap\{\mathrm{cl}(f[A]) : A \in p\}$ is nonempty. We show that $\bigcap\{\mathrm{cl}(f[A]) : A \in p\}$ contains only a single point.

Suppose there are two distinct points $y_0, y_1 \in \bigcap\{\mathrm{cl}(f[A]) : A \in p\}$. Let $U_0, U_1 \subseteq Y$ be open and disjoint with $y_0 \in U_0$ and $y_1 \in U_1$. For every $A \in p$, since $y_0, y_1 \in \mathrm{cl}(f[A])$, $f[A]$ intersects both U_0 and U_1. Fix $A \in p$. Let $A_0 = \{x \in A : f(x) \in U_0\}$ and $A_1 = A \setminus A_0$. Since $A_0 \cup A_1 \in p$, either $A_0 \in p$ or $A_1 \in p$. But in either case there is $i \in 2$ such that $A_i \in p$ and $f[A_i]$ is disjoint from U_{1-i}. A contradiction. It follows that $\bigcap\{\mathrm{cl}(f[A]) : A \in p\}$ contains exactly one point y. Let $\beta f(p) = y$.

For every $x \in X$, $\beta f(x)$ is the unique element of $f[\{x\}]$ and thus $\beta f(x) = f(x)$. In other words, βf extends f. We have to show that βf is continuous. Uniqueness then follows from the fact that X is dense in βX.

Let $U \subseteq Y$ be open and let $p \in \beta X$ be such that $\beta f(p) \in U$. Choose an open neighborhood V of $f(p)$ with $\mathrm{cl}(V) \subseteq U$. As before, for every $A \in p$, $f[A] \cap V$ is nonempty. It follows that $B = f^{-1}[V]$ intersects every element of p. Hence $p \cup \{B\}$ has the finite intersection property and is contained in an ultrafilter q. By the maximality of p, $p = q$ and therefore $B \in p$.

Since $\mathrm{cl}(f[B]) \subseteq U$, we have $\beta f(r) \in U$ for every $r \in \hat{B}$. This shows that βf is continuous. $\square$

We now extend the multiplication on a discrete semigroup to its Stone-Čech compactification.

Definition 4.10. Let $(S, \cdot)$ be a semigroup. For each $s \in S$ the left multiplication $\lambda_s : S \to S; x \mapsto sx$ can be considered as a map from S to βS and therefore has a continuous extension $\beta\lambda_s : \beta S \to \beta S$. Now for each $x \in \beta S$ we have a map $\rho_x : S \to \beta X; s \mapsto \beta\lambda_s(x)$. This has a unique continuous extension $\beta\rho_x : \beta S \to \beta S$. For $x, y \in \beta S$ we define $x \cdot y = \beta\rho_y(x)$.

Lemma 4.11. *For every discrete semigroup $(S, \cdot)$, $(\beta S, \cdot)$ is a compact right-topological super-semigroup of $(S, \cdot)$.*

Proof. From the definition of $\cdot$ on βS it follows that $\cdot$ extends the multiplication on S. Also, for each $y \in \beta S$ the right multiplication $x \mapsto xy$ is just $\beta\rho_y$, which is continuous by definition.

For all $x, y, z \in \beta S$ we have

$$(xy)z = \beta\rho_z(xy) = \beta\rho_z(\beta\rho_y(x)) = (\beta\rho_z \circ \beta\rho_y)(x)$$

and $x(yz) = \beta\rho_{yz}(x)$. Hence, in order to show that the multiplication on βS is associative, we have to prove that $\beta\rho_z \circ \beta\rho_y = \beta\rho_{yz}$.

Clearly, $\beta\rho_z \circ \beta\rho_y$ is a continuous function from βS to βS. By the uniqueness of $\beta\rho_{yz}$, it is enough to show that $\beta\rho_z \circ \beta\rho_y$ agrees with ρ_{yz} on S. Let $s \in S$. Then $\rho_{yz}(s) = \beta\lambda_s(yz) = \beta\lambda_s(\beta\rho_z(y))$. On the other hand, $(\beta\rho_z \circ \beta\rho_y)(s) = \beta\rho_z(\beta\lambda_s(y))$. The two functions $\beta\lambda_s \circ \beta\rho_z$ and $\beta\rho_z \circ \beta\lambda_s$ are both continuous functions from βS to βS. In order to show that they are equal, it is enough to show that they agree on the dense subset S of βS.

Let $t \in S$. Then $(\beta\lambda_s \circ \beta\rho_z)(t) = \beta\lambda_s(\beta\lambda_t(z))$ and

$$(\beta\rho_z \circ \beta\lambda_s)(t) = \beta\rho_z(\lambda_s(t)) = \beta\rho_z(st) = \beta\lambda_{st}(z).$$

It remains to show that $\beta\lambda_s \circ \beta\lambda_t$ is the same as $\beta\lambda_{st}$. Again, it is enough to verify this on S.

Let $r \in S$. Then $(\beta\lambda_s \circ \beta\lambda_t)(r) = \beta\lambda_s(tr) = s(tr)$. On the other hand, $\beta\lambda_{st}(r) = (st)r$. Finally, $(st)r = s(tr)$ since $\cdot$ is associative on S. It follows that $\cdot$ is associative on βS. $\square$

4.3. **The theorems of Hales-Jewett and van der Waerden.** We use an abstract theorem of Koppelberg to deduce two classical theorems in Ramsey theory.

Definition 4.12. Let S be a semigroup and T a subsemigroup of S. We call T a *nice* subsemigroup if $R = S \setminus T$ is an ideal of S. Note that T is nice iff for all $x, y \in S$ we have $xy \in T$ iff $x \in T$ and $y \in T$.

A semigroup homomorphism $\sigma : S \to T$ is a *retraction* (from S to T) if $\sigma(t) = t$ for all $t \in T$.

Theorem 4.13 (Koppelberg)**.** *Let S be a semigroup and T a proper nice subsemigroup of S. Let Σ be a finite set of retractions from S to T*

and let $(B_1, \dots, B_n)$ be a partition of T. Then there are $j \in \{1, \dots, n\}$ and $r \in R = S \setminus T$ such that for all $\sigma \in \Sigma$, $\sigma(r) \in B_j$.

Proof. It is easily checked that $\hat{T}$ is isomorphic to a βT and therefore a subsemigroup of βS. Also, $\hat{R}$ is equal to $\beta S \setminus \hat{T}$ and is an ideal of βS. Finally, for each $\sigma \in \Sigma$, $\beta\sigma$ is a retraction from βS to $\mathcal{T}$.

Let L be a minimal left ideal of $\hat{T}$ and let $q \in L$ be an idempotent. Let I be a minimal left ideal in the left ideal $\beta S \cdot q$ of βS and choose an idempotent $i \in I$. Let $p = qi$. Now $p \in I$.

Note that $I \subseteq \hat{R}$ since $\hat{R}$ is an ideal of βS. It follows that $p \in \hat{R}$ and thus $R \in p$. Since $i \in I \subseteq \beta S \cdot q$ and q is an idempotent, $iq = i$. Now $qp = qqi = qi = p$, $pq = qiq = qi = p$ and $pp = qiqi = qii = qi = p$. Hence

$$(*) \qquad p = p^2 = pq = qp.$$

Let $\sigma \in \Sigma$ and $u = \beta\sigma(p)$. Clearly, $u \in \hat{T}$. Also $q \in \hat{T}$. We apply $\beta\sigma$ to equation $(*)$ and obtain

$$u = u^2 = uq = qu.$$

In particular, $u = uq \in L$. Since L is a minimal left ideal of $\hat{T}$, $L = \hat{T} \cdot u$. Hence $q \in \hat{T} \cdot u$. Note that u is an idempotent. It follows that $qu = q$. This shows that $\beta\sigma(p) = q$ for every $\sigma \in \Sigma$.

Recall that q is actually an ultrafilter on S such that $T \in q$. It follows that there is $j \in \{1, \dots, n\}$ such that $B_j \in q$. For every $\sigma \in \Sigma$ we have $B_j \in q = \beta\sigma(p)$. It follows that $\hat{B}_j$ intersects $\sigma[A]$ for every $A \in p$. In other words, for every $A \in p$, $\sigma[A]$ contains an ultrafilter that contains the set B_j. But $\sigma[A]$ consists of ultrafilters that correspond to elements of S. Identifying these ultrafilters with the corresponding elements of S, we see that for all $A \in p$, $\sigma[A]$ intersects B_j. Hence $\sigma^{-1}[B_j]$ intersects every set $A \in p$. Since p is an ultrafilter, this implies $\sigma^{-1}[B_j] \in p$.

Since $R \in p$, also the set $D = R \cap \bigcap_{\sigma \in \Sigma} \sigma^{-1}[B_j]$ is in p, and hence nonempty. Every $r \in D$ works for the theorem. $\square$

Theorem 4.14 (van der Waerden). *Assume $(A_1, \dots, A_n)$ is a partition of ω into finitely many pieces and $m \in \omega$. Then there are $j \in \{1, \dots, n\}$ and natural numbers a and $d > 0$ such that*

$$\{a, a + d, a + 2d, \dots, a + md\} \subseteq A_j,$$

i.e., A_j contains an arithmetic progression of length $m+1$.

Note that the theorem implies that given a partition of ω into finitely many pieces, one of the pieces contains arbitrarily long arithmetic progression.

Proof. Consider the semigroup $S = \omega \times \omega$ and let $T = \omega \times \{0\}$. Then T is a nice subsemigroup of S. For each $j \in \{1, \dots, n\}$ let $B_j = A_j \times \{0\}$. Now $(B_1, \dots, B_n)$ is a partition of T. For each $k \leq m$ and all $a, d \in \omega$ let $\sigma_k(a, d) = (a + kd, 0)$. Each σ_k is a retraction from S to T.

Hence, by Koppelberg's theorem there are $j \in \{1, \dots, r\}$ and $(a, d) \in S \setminus T$ such that for all $k \leq m$, $\sigma_k(a, d) \in B_j$. Now by the definition of σ_k and of B_j, for all $k \leq m$ we have $a + kd \in B_j$. This finishes the proof of the theorem. $\square$

Definition 4.15. Let M be a finite set, the *alphabet*. A *word* over M is a finite sequence of elements of M. M^* denotes the set of all words over M.

Let x be a variable. We assume that $x \notin M$. A *variable word* over M is a word over $M \cup \{x\}$ with at least one occurrence of x. Given a word w over $M \cup \{x\}$ and $u \in M$, let $w(u)$ denote the word over M obtained by replacing every occurence of x by u.

A *combinatorial line* over M is a set of the form $\{w(u) : u \in M\}$, where w is a variable word over M.

Theorem 4.16 (Hales-Jewett)**.** *Let M be a finite alphabet and let $(A_1, \dots, A_n)$ be a partition of M^*. Then there is $j \in \{1, \dots, n\}$ such that A_j includes a combinatorial line.*

Proof. Let S be the semigroup $(M \cup \{x\})^*$ with the concatenation of words as multiplication. Let $T = M^*$. Then T is a nice subsemigroup of S. For each $u \in M$ and $w \in S$ let $\sigma_u(w) = w(u)$. Each σ_u is a retraction from S to T.

Hence there are some $w \in S \setminus T$, i.e., a variable word over M, and some $j \in \{1, \dots, n\}$, such that for all $u \in M$, $w(u) \in A_j$. In other words, A_j includes the combinatorial line generated by w. $\square$

We derive a finite version of the Hales-Jewett theorem from the infinite version above. For $m \in \omega$, $M^{\leq m}$ denotes the set of all words over M of length at most m. M^m is the set of words of length m.

Theorem 4.17 (Hales-Jewett, finite version)**.** *Let M be a finite alphabet. For every $n \in \omega$ there is $m > 0$ such that whenever $C_1, \dots, C_n$ is a partition of M^m, then there is $j \in \{1, \dots, n\}$ such that C_j includes a combinatorial line.*

We derive the theorem from the following lemma.

Lemma 4.18. *Let M and n be as in Theorem 4.17. There is $m > 0$ such that whenever $C_1, \dots, C_n$ is a partition of $M^{\leq m}$, then there is $j \in \{1, \dots, n\}$ such that C_j includes a combinatorial line.*

Proof. Suppose there is no such m. Consider the collection T of all n-tuples $(C_1, \dots, C_n)$ such that for some $m \in \omega$, $C_1, \dots, C_n$ is a partition of $M^{\leq m}$ such that no C_j includes a combinatorial line. Note that for technical reasons we also consider partitions of M^0. For $A, B \in T$, $A = (A_1, \dots, A_n)$, $B = (B_1, \dots, B_n)$, let $A \sqsubset B$ if for all $j \in \{1, \dots, n\}$, $A_j \subsetneq B_j$.

Clearly, T is a tree. By our assumption, T is infinite. Moreover, whenever $(A_1, \dots, A_n) \in T$ is a partition of $M^{\leq m}$ and $k \leq m$, then $(A_1 \cap M^{\leq k}, \dots, A_n \cap M^{\leq k}) \in T$. It follows that each level of T consists of partitions of $M^{\leq m}$ for a fixed $m > 0$. Since for each m the set $M^{\leq m}$ is finite, each level of T is finite. Hence, by König's Lemma, T has an infinite branch $\mathcal{B}$. For each $j \in \{1, \dots, n\}$ let

$$C_j = \bigcup \{A : \exists (B_1, \dots, B_n) \in \mathcal{B}(A = B_j)\}.$$

Now $(C_1, \dots, C_n)$ is a partition of M^* such that no C_j includes a combinatorial line. This contradicts Theorem 4.16. $\square$

Proof of Theorem 4.17. Let $m > 0$ be a minimal witness of Lemma 4.18. Let $C_1, \dots, Cn$ be a partition of M^m. By the minimality of m, there is a partition $A_1, \dots, A_n$ of $M^{\leq m-1}$ such that no A_j includes a combinatorial line. Now $A_1 \cup C_1, \dots, A_n \cup C_n$ is a partition of $M^{\leq m}$. By the choice of m, there is $j \in \{1, \dots, n\}$ such that $C_j \cup A_j$ includes a combinatorial line. Since A_j does not include a combinatorial line, the combinatorial line included in $A_j \cup C_j$ consists of words of length m. Hence A_j includes a combinatorial line. $\square$

4.4. **Hindman's theorem.**

Definition 4.19. For a set $A \subseteq \omega$ let

$$FS(A) = \left\{ \sum_{a \in F} a : F \subseteq A \text{ is nonempty and finite} \right\}.$$

Theorem 4.20 (Hindman)**.** *Let $C_1, \dots, C_n$ be a partition of ω. Then there are $j \in \{1, \dots, n\}$ and an infinite set $B \subseteq \omega$ such that $FS(B) \subseteq C_j$.*

The proof of this theorem requires a closer analysis of the addition on $\beta\omega$.

Lemma 4.21. *a) Let S be a set and let $f : S \to S$ be a function. Then for each $p \in \beta S$, $\beta f(p) = \{A \subseteq S : f^{-1}[A] \in p\}$.*

b) Let $+$ be the extension of the addition on ω that turns $\beta\omega$ into a right-topological semigroup. Then for $p, q \in \beta\omega$,

$$p + q = \{A \subseteq \omega : \{n \in \omega : A - n \in q\} \in p\}.$$

Proof. a) Note that we have already used this in the proof of Theorem 4.13. Let $p \in \beta S$ and $q = \beta f(p)$. For each $B \subseteq \omega$, $\hat{B}$ is compact and thus, $\beta f[\hat{B}]$ is closed. Since B is dense in $\hat{B}$, $f[B]$ is dense in $\beta f[\hat{B}]$. It follows that $\beta f[\hat{B}] = \widehat{f[B]}$. Hence, if $B \in p$, then $f[B] \in q$. It follows that $\{A \subseteq \omega : f^{-1}[A] \in p\} \subseteq q$.

We are done if we can show that $r = \{A \subseteq \omega : f^{-1}[A] \in p\}$ is already an ultrafilter and hence must be equal to q. But this follows easily from the fact that $A \mapsto f^{-1}[A]$ is a Boolean homomorphism from $\mathcal{P}(\omega)$ to $\mathcal{P}(\omega)$.

b) For $n \in \omega$ let $\lambda_n(m) = n + m$. For $A \subseteq n$, by $A - n$ we denote $(\lambda_n)^{-1}[A] = \{m - n : m \in A \wedge m \geq n\}$. By a), for each $q \in \beta\omega$,

$$\beta\lambda_n(q) = \{A \subseteq \omega : A - n \in q\}$$

Now for $q \in \beta\omega$ and $n \in \omega$ we let $\rho_q(n) = \beta\lambda_n(q)$. The function $\beta\rho_q$ is the unique continuous extension of ρ_q to $\beta\omega$ and $p + q = \beta\rho_q(p)$.

We define another function $\sigma_q : \beta\omega \to \beta\omega$ by letting

$$\sigma_q(p) = \{A \subseteq \omega : \{n \in \omega : A - n \in q\} \in p\}.$$

We first show that $r = \{A \subseteq \omega : \{n \in \omega : A - n \in q\} \in p\}$ actually is an ultrafilter.

If $A, B \subseteq \omega$ and $A \subseteq B$, then

$$\{n \in \omega : A - n \in q\} \subseteq \{n \in \omega : B - n \in q\}$$

and therefore r is closed under taking supersets. If $A, B \in r$, then $\{n \in \omega : A - n \in q\}, \{n \in \omega : B - n \in q\} \in p$ and therefore

$$\{n \in \omega : A - n \in q\} \cap \{n \in \omega : B - n \in q\} \in p.$$

But

$$\begin{aligned} &\{n \in \omega : A - n \in q\} \cap \{n \in \omega : B - n \in q\} \\ &\quad = \{n \in \omega : (A - n) \cap (B - n) \in q\} = \{n \in \omega : (A \cap B) - n \in q\}. \end{aligned}$$

It follows that r is closed under intersection. Since the empty set is not in p or q, $\emptyset \notin r$. Now assume $A \subseteq \omega$ is not an element of r. Then $\{n \in \omega : A - n \in q\} \notin p$ and thus

$$\begin{aligned} \omega \setminus \{n \in \omega : A - n \in q\} &= \{n \in \omega : \omega \setminus (A - n) \in q\} \\ &= \{n \in \omega : (\omega \setminus A) - n \in q\} \in p. \end{aligned}$$

Hence $\omega \setminus A \in r$. It follows that r is indeed an ultrafilter.

Now for b) it is enough to show that σ_q is continuous and agrees with ρ_q on ω. Let $m \in \omega$ and let p be the ultrafilter generated by $\{m\}$. Then

$$\begin{aligned} \sigma_q(m) &= \{A \subseteq \omega : \{n \in \omega : A - n \in q\} \in p\} \\ &= \{A \subseteq \omega : m \in \{n \in \omega : A - n \in q\}\} \\ &= \{A \subseteq \omega : A - m \in q\} = \beta\lambda_m(q) \end{aligned}$$

It remains to show the continuity of σ_q. Let $p \in \beta\omega$. Let $U \subseteq \beta\omega$ be an open set such that $\sigma_q(p) \in U$. Then there is $A \subseteq \omega$ such that $\sigma_q(p) \in \hat{A} \subseteq U$. We have $A \in r = \sigma_q(p)$ and therefore $\{n \in \omega : A - n \in p\} \in q$. □

Proof of Theorem 4.20. Observe that $\widehat{\omega \setminus \{0\}}$ is a compact subsemigroup of $\beta\omega$ and therefore contains an idempotent p. Clearly, 0 is the only idempotent of ω. Hence $p \notin \omega$ and therefore p contains no finite sets. Since p is an ultrafilter, there is $j \in \{1, \ldots, n\}$ such that $C_j \in p$.

By Lemma 4.21 b) we have

$$\{A \subseteq \omega : \{m \in \omega : A - m \in p\} \in p\} = p + p = p. \tag{$**$}$$

Let $A \in p$. By $(**)$, $\{m \in \omega : A - m \in p\} \in p$. Hence

$$A \cap \{m \in \omega : A - m \in p\} \in p.$$

Now let $A_0 = C_j \in p$.

It follows that there is $m_1 \in A_0$ such that $A_1 = A \cap (A_0 - m_1) \in p$. Iterating this procedure, we can choose $m_2 \in A_1$, $m_2 > m_1$, such that

$$\begin{aligned} A_2 &= A_1 \cap (A_1 - m_2) \\ &= A_0 \cap (A_0 - m_1) \cap (A_0 - m_2) \cap (A_0 - m_1 - m_2) \in p. \end{aligned}$$

Note that by the choice of m_2, m_2 can be written as $a - m_1$ for some $a \in A$. Now $m_1 + m_2 = m_1 + a - m_1 = a$. In particular, $m_1 + m_2 \in A$.

Continuing in this fashion, we obtain a sequence $A_0 \supseteq A_1 \supseteq \dots$ of elements of p and a sequence $m_1 < m_2 < \dots$ of elements of A_0 such that for every $i \in \omega$, $m_{i+2} \in A_{i+1} = A_i \cap (A_i - m_{i+1})$. To finish the proof of the theorem, it remains to show

Claim 4.22. Let $B = \{m_i : i > 0\}$. Then $FS(B) \subseteq C_j$.

By induction we see that for every $i \in \omega$,

$$A_{i+1} = \bigcap_{F \subseteq \{m_1, \dots, m_{i+1}\}} \left(A_0 - \sum_{a \in F} a \right).$$

Since $A_{i+1} \in p$, no set of the form $A_0 - \sum_{a \in F} a$, $F \subseteq \{m_1, \dots, m_{i+1}\}$, is empty. Let $F \subseteq \{m_1, \dots, m_{i+1}\}$. Since $m_{i+2} \in A_{i+1}$, there is $b \in A_0$ such that $m_{i+1} = b - \sum_{a \in F} a$. Hence $m_{i+1} + \sum_{a \in F} a = b \in A_0$. This shows that for each finite, nonempty set $F \subseteq B$, $\sum_{a \in F} a \in A_0$ and the claim follows. $\square$

5. Metric Ramsey theory

We prove analogs of Ramsey's theorem and Galvin's theorem for metric spaces.

5.1. Embedding sequences into $\mathbb{R}$.

Definition 5.1. a) Let $f : X \to Y$ be an injection between metric spaces (X, d_X) and (Y, d_Y). For a real constant $K \geq 1$, f is a K-*embedding* if for all $x, y \in X$ with $x \neq y$ we have

$$\frac{1}{K} \leq \frac{d_Y(f(x), f(y))}{d_X(x, y)} \leq K.$$

b) A metric space (X, d) is K-*linear* if it K-embeds into $\mathbb{R}$.

Note that for K-embeddings $f : X \to Y$ and $g : Y \to Z$, $g \circ f : X \to Z$ is a K^2-embedding. Obviously, 1-embeddings are just isometric embeddings.

In this subsection we show that sequences in metric spaces are can be K-embedded into $\mathbb{R}$ if they either diverge or converge sufficiently fast. We interpolate the embeddings into $\mathbb{R}$ by embeddings into ultrametric spaces.

Definition 5.2. A metric space (X, d_X) is *ultrametric* if for all $x, y, z \in X$ we have $d_X(x, z) \leq \max(d_X(x, y), d_X(y, z))$.

Definition 5.3. A sequence $(x_n)_{n\in\omega}$ is *anti-Cauchy* with respect to a metric d if for every $k \in \omega$ there is $n_0 \in \omega$ such that for all $n, m \in \omega$ with $n_0 \leq n < m$ we have $k \leq d(x_n, x_m)$.

Lemma 5.4. *Let $K > 1$. Suppose $(x_n)_{n\in\omega}$ is a sequence that is anti-Cauchy with respect to some metric d. Then $X = \{x_n : n \in \omega\}$ has an infinite subset that is K-linear.*

The proof of this lemma is based on

Lemma 5.5. *Let $K > 1$ and $\varepsilon = 1 - \frac{1}{K}$. Suppose $(x_n)_{n\in\omega}$ is a sequence without repetitions such that, with respect to some metric d, the following holds:*

For every $n \in \omega$ and all $i, j < n$,

$$d(x_i, x_j) \leq \varepsilon \cdot d(x_i, x_n).$$

We define an ultrametric by letting

$$d_{\mathrm{ultra}}(x_i, x_j) = d(x_0, x_{\max(i,j)})$$

for all $i, j \in \omega$ *with* $i \neq j$.

Then the identity map on $\{x_n : n \in \omega\}$ *is a* K*-embedding with respect to* d *and* d_{ultra}. *Moreover,* $(\{x_n : n \in \omega\}, d_{\mathrm{ultra}})$ *is* K*-linear.*

Proof. We first show that d_{ultra} is indeed an ultrametric. Observe that the sequence $(d(x_0, x_n))_{n\in\omega}$ is increasing and hence $d_{\mathrm{ultra}}(x_i, x_k) \leq d_{\mathrm{ultra}}(x_j, x_\ell)$ if $i < k$, $j < \ell$ and $k \leq \ell$. Now let $i, j, k \in \omega$ be pairwise distinct. If $\max(i, j, k) = j$, then $d_{\mathrm{ultra}}(x_i, x_k) \leq d_{\mathrm{ultra}}(x_j, x_k)$. If $\max(i, j, k) \in \{i, k\}$, then $d_{\mathrm{ultra}}(x_i, x_k) = d_{\mathrm{ultra}}(x_i, x_j)$ or $d_{\mathrm{ultra}}(x_i, x_k) = d_{\mathrm{ultra}}(x_j, x_k)$. In any case we have

$$d_{\mathrm{ultra}}(x_i, x_k) \leq \max(d_{\mathrm{ultra}}(x_i, x_j), d_{\mathrm{ultra}}(x_j, x_k)).$$

In order to show that the identity map is a K-embedding with respect to d and d_{ultra} let $i, j \in \omega$ be such that $i < j$. Then

$$\begin{aligned}\frac{d_{\mathrm{ultra}}(x_i, x_j)}{d(x_i, x_j)} = \frac{d(x_0, x_j)}{d(x_i, x_j)} \leq \frac{d(x_0, x_i) + d(x_i, x_j)}{d(x_i, x_j)}& \\ = 1 + \frac{d(x_0, x_i)}{d(x_i, x_j)} &\leq 1 + \varepsilon \leq K.\end{aligned}$$

On the other hand,

$$\begin{aligned}\frac{d_{\mathrm{ultra}}(x_i, x_j)}{d(x_i, x_j)} = \frac{d(x_0, x_j)}{d(x_i, x_j)} \geq \frac{d(x_i, x_j) - d(x_0, x_i)}{d(x_i, x_j)}& \\ = 1 - \frac{d(x_0, x_i)}{d(x_i, x_j)} &\geq 1 - \varepsilon \geq \frac{1}{K}.\end{aligned}$$

Finally, consider the embedding

$$e : \{x_n : n \in \omega\} \to \mathbb{R}; x_n \mapsto d(x_0, x_n).$$

For $i, j \in \omega$ with $i < j$ we have

$$\frac{|e(x_i) - e(x_j)|}{d_{\mathrm{ultra}}(x_i, x_j)} = \frac{d(x_0, x_j) - d(x_0, x_i)}{d(x_0, x_j)} \leq 1$$

and

$$\frac{|e(x_i) - e(x_j)|}{d_{\mathrm{ultra}}(x_i, x_j)} = \frac{d(x_0, x_j) - d(x_0, x_i)}{d(x_0, x_j)} \geq 1 - \frac{d(x_0, x_i)}{d(x_0, x_j)} \geq 1 - \varepsilon \geq \frac{1}{K}.$$

This shows that e is a K-embedding with respect to d_{ultra} and the usual metric on $\mathbb{R}$. □

Proof of Lemma 5.4. If $(x_n)_{n\in\omega}$ is anti-Cauchy, then it can easily be thinned out to a sequence as in Lemma 5.5 for the constant $\sqrt{K}$. Lemma 5.4 now follows by the remark after Definition 5.1. □

Observe that a metric space X contains an anti-Cauchy sequence if and only if its set of distances is unbounded. Therefore Lemma 5.4 implies

Corollary 5.6. *Let $K > 1$. Then every metric space X with an unbounded set of distances has an infinite subset that is K-linear.*

For Cauchy sequences we have the following analog of Lemma 5.4:

Lemma 5.7. *Let $K > 1$. Suppose $(x_n)_{n\in\omega}$ is a sequence without repetitions that is Cauchy with respect to some metric d. Then $\{x_n : n \in \omega\}$ has an infinite subset that is K-linear.*

The proof of Lemma 5.7 uses

Lemma 5.8. *Let $K > 1$ and $\varepsilon = 1 - \frac{1}{K}$. Suppose $(x_n)_{n\in\omega}$ is a sequence without repetitions such that, with respect to some metric d, the following holds:*

For every $n \in \omega$ and all $i, j, k > n$,

$$d(x_i, x_j) \leq \varepsilon \cdot d(x_n, x_k).$$

We define an ultrametric by letting

$$d_{\text{ultra}}(x_i, x_j) = \inf_{k>i} d(x_i, x_k)$$

for all $i, j \in \omega$ with $i < j$.

Then the identity map on $\{x_n : n \in \omega\}$ is a K-embedding with respect to d and d_{ultra}. Moreover, $(\{x_n : n \in \omega\}, d_{\text{ultra}})$ is K-linear.

Proof. We show that d_{ultra} is an ultrametric. First observe that the distance $d_{\text{ultra}}(x_i, x_j)$ only depends on the smaller one of the indices. Moreover, the sequence $(d_{\text{ultra}}(x_i, x_{i+1}))_{i\in\omega}$ is decreasing since for all $j > i + 1$ and all $k > i$ we have $d(x_{i+1}, x_j) \leq \varepsilon \cdot d(x_i, x_k)$ and hence

$$d_{\text{ultra}}(x_{i+1}, x_{i+2}) = \inf_{j>i+1} d(x_{i+1}, x_j) \leq \inf_{k>i} \varepsilon \cdot d(x_i, x_k) = \varepsilon \cdot d_{\text{ultra}}(x_i, x_{i+1}).$$

If $i, j, k \in \omega$ are pairwise distinct, then either $j = \min(i, j, k)$ or $\min(i, j, k) \in \{i, k\}$. In the first case $d_{\text{ultra}}(x_i, x_k) \leq d_{\text{ultra}}(x_i, x_j)$. In the second case $d_{\text{ultra}}(x_i, x_k) = d_{\text{ultra}}(x_i, x_j)$ or $d_{\text{ultra}}(x_i, x_k) = d_{\text{ultra}}(x_j, x_k)$. In any case we have

$$d_{\text{ultra}}(x_i, x_k) \leq \max(d_{\text{ultra}}(x_i, x_j), d_{\text{ultra}}(x_j, x_k)).$$

Now let $i, j \in \omega$ with $i < j$. Then

$$\frac{d_{\text{ultra}}(x_i, x_j)}{d(x_i, x_j)} = \frac{\inf_{k>i} d(x_i, x_k)}{d(x_i, x_j)} \leq 1.$$

On the other hand,

$$\frac{d_{\text{ultra}}(x_i, x_j)}{d(x_i, x_j)} = \frac{\inf_{k>i} d(x_i, x_k)}{d(x_i, x_j)} \geq \frac{d(x_i, x_j) - \sup_{k>i} d(x_j, x_k)}{d(x_i, x_j)} \geq 1 - \varepsilon = \frac{1}{K}.$$

It follows that the identity map is a K-embedding with respect to d and d_{ultra}.

Finally consider the embedding

$$e : \{x_n : n \in \omega\} \to \mathbb{R}; x_n \mapsto d_{\text{ultra}}(x_n, x_{n+1}).$$

For all $i, j \in \omega$ with $i < j$ we have

$$\frac{|x_i - x_j|}{d_{\text{ultra}}(x_i, x_j)} = \frac{d_{\text{ultra}}(x_i, x_{i+1}) - d_{\text{ultra}}(x_j, x_{j+1})}{d_{\text{ultra}}(x_i, x_{i+1})} \leq 1$$

and

$$\begin{aligned}\frac{|x_i - x_j|}{d_{\text{ultra}}(x_i, x_j)} &= \frac{d_{\text{ultra}}(x_i, x_{i+1}) - d_{\text{ultra}}(x_j, x_{j+1})}{d_{\text{ultra}}(x_i, x_{i+1})} \\ &= 1 - \frac{\inf_{k>j} d(x_j, x_k)}{\inf_{\ell>i} d(x_i, x_\ell)} \geq 1 - \sup_{k>j, \ell>i} \frac{d(x_j, x_k)}{d(x_i, x_\ell)} \geq 1 - \varepsilon = \frac{1}{K}.\end{aligned}$$

It follows that e is a K-embedding with respect to d_{ultra} and the usual metric on $\mathbb{R}$. □

Proof of Lemma 5.7. Since $(x_n)_{n\in\omega}$ has no repetitions, we may assume, after removing a point from the sequence, that $(x_n)_{n\in\omega}$ does not converge to any of the x_n. For each $n \in \omega$ the sequence $(d(x_n, x_i))_{i\in\omega}$ is Cauchy in $\mathbb{R}$ since $(x_n)_{n\in\omega}$ is Cauchy with respect to d. Let $d_n = \lim_{i\to\infty} d(x_n, x_i)$. Note that $d_n > 0$.

Let $\varepsilon = 1 - \frac{1}{\sqrt{K}}$. By recursion on $m \in \omega$ we choose a strictly increasing sequence $(n_m)_{m\in\omega}$ in ω such that for all $m \in \omega$ and all $i, j, k \geq n_{m+1}$ we have

$$d(x_i, x_j) \leq \frac{\varepsilon}{2} \cdot d_{n_m}$$

and

$$\frac{1}{2} \cdot d_{n_m} \leq d(x_{n_m}, x_k).$$

Now if $i, j, k, m \in \omega$ are such that $i, j, k > m$, then

$$d(x_{n_i}, x_{n_j}) \leq \frac{\varepsilon}{2} \cdot d_{n_m} \leq \varepsilon \cdot d(x_{n_m}, x_{n_k}).$$

In other words, the sequence $(x_{n_m})_{m\in\omega}$ satisfies the requirements in Lemma 5.8 for the constant $\sqrt{K}$. Lemma 5.7 now easily follows by the remark after Definition 5.1. □

If X is an infinite subset of $\mathbb{R}^n$, then either it is unbounded and therefore contains an anti-Cauchy sequence or its closure is compact and therefore X contains a Cauchy sequence. From Lemma 5.4 and Lemma 5.7 we now easily obtain

Corollary 5.9. *Let $K > 1$. Then every infinite set $X \subseteq \mathbb{R}^n$ has an infinite subset Y that is K-linear.*

5.2. **Metric spaces with a set of non-zero distances bounded from below and above.** We show that every infinite metric space that neither has distinct points of very small nor of very large distance has an infinite subsets where any two distinct points have nearly the same distance.

Definition 5.10. A metric space X is *uniform* if there is a constant D such that any two distinct points in X have distance D. X is *K-uniform* if it K-embeds into a uniform metric space.

Clearly, a uniform metric space is ultrametric.

Observe that if the non-zero distances in a metric space X only vary by a factor of at most K, then X is K-uniform. Just choose any $D > 0$ that occurs as a distance in X and replace the metric on X by the uniform metric with distance D.

On the other hand, if X is K-uniform, then the non-zero distances in X only vary by a factor of at most K^2.

Lemma 5.11. *Let $K > 1$. Let (X, d) be an infinite metric space and assume that there are $\varepsilon > 0$ and $N \in \omega$ such that for all $x, y \in X$ with $x \neq y$ we have $\varepsilon \leq d(x, y) < N$.*

Then X has an infinite subset Y that is K-uniform.

Proof. For every $n \in \omega$ let $c_n = \varepsilon \cdot K^n$. Let $M \in \omega$ be maximal with $c_M < N$. For all $x, y \in X$ with $x \neq y$ let $c(x, y)$ be the unique $i \in \{0, \dots, M\}$ such that $d(x, y) \in [c_i, c_{i+1})$. By the infinite Ramsey Theorem, there is an infinite set $Y \subseteq X$ such that for some $i \in \{0, \dots, M\}$ for all $x, y \in Y$ with $x \neq y$ we have $c(x, y) = i$. Now for all $a, b, x, y \in Y$ with $a \neq b$ and $x \neq y$ we have

$$c_i \leq d(a, b) < c_{i+1} = K \cdot c_i \leq K \cdot d(x, y).$$

By the remark after Definition 5.10, this shows that Y is K-uniform. □

5.3. **A Ramsey-type theorem for infinite metric spaces.**

Theorem 5.12 (Matoušek)**.** *Let X be an infinite metric space and $K > 1$. Then there is an infinite set $Y \subseteq X$ that is either K-linear or K-uniform.*

Proof. Let d denote the metric on X. By Corollary 5.6 we may assume that the set of distances in X is bounded (from above). Fix $n > 0$. For all $x, y \in X$ with $x \neq y$ let

$$c_n(x, y) = \begin{cases} 0 & \text{if } d(x, y) < \frac{1}{n}, \\ 1 & \text{if } d(x, y) \geq \frac{1}{n} \end{cases}$$

By recursion on n we construct a decreasing sequence $(H_n)_{n \in \omega}$ of infinite subsets of X as follows:

Let $H_0 = X$. Assume we have constructed H_n. By the infinite Ramsey Theorem, H_n has an infinite subset H_{n+1} such that for some $i \in \{0, 1\}$ for all $x, y \in H_{n+1}$ with $x \neq y$ we have $c_{n+1}(x, y) = i$. We say that i is the *color* of H_{n+1}.

Observe that if for some $n > 0$ the set H_n is of color 1, then for every $m > n$ the set H_m is of color 1. If H_n is of color 0, then for every $m > n$, H_m is of color 0 or 1.

We are left with two cases.

(1) For every $n > 0$ the color of H_n is 0.

(2) There is $m > 0$ such that for all $n \geq m$ the color of H_n is 1.

In Case (1) we choose a sequence $(x_n)_{n\in\omega}$ without repetitions such that for every $n \in \omega$, $x_n \in H_n$. It is easily checked that the sequence is Cauchy. It now follows from Lemma 5.7 that $\{x_n : n \in \omega\}$ has an infinite subset Y that is K-linear.

In Case (2) it follows from Lemma 5.11 that H_m has an infinite subset Y that is K-uniform. $\square$

Since the K-embeddings in $\mathbb{R}$ in the proof of Theorem 5.12 all factor through low distortion embeddings into ultrametric spaces, we actually have the following slightly more explicit theorem:

Theorem 5.13. *Let $K > 1$. Then every infinite metric space X has an infinite subset Y that K-embeds into an ultrametric space that is either K-linear or uniform.*

5.4. **A Ramsey-type theorem for complete metric spaces.** We now prove a metric analog of Galvin's theorem. Again we interpolate between a large subset Y of a given metric space X and the real line using an ultrametric space.

Definition 5.14. For $f, g \in 2^\omega$ let $\mathrm{lci}(f, g)$ denote the longest common initial segment of f and g. We have $\mathrm{lci}(f, g) \in 2^{<\omega}$ if and only if f and g are distinct.

Lemma 5.15. *Let $K > 1$ and $\varepsilon = 1 - \frac{1}{K}$. Let $\Delta : 2^{<\omega} \to [0, \infty)$ be such that for all $s \in 2^{<\omega}$ we have*

$$\Delta(s^\frown 0), \Delta(s^\frown 1) \leq \frac{\varepsilon}{2} \cdot \Delta(s).$$

We define a metric on 2^ω by letting $d_{\mathrm{ultra}}(f, g) = \Delta(\mathrm{lci}(f, g))$.

Then d_{ultra} is an ultrametric and $(2^\omega, d_{\mathrm{ultra}})$ is K-linear.

Proof. In order to verify that d_{ultra} is an ultrametric, let f, g and h be pairwise distinct elements of 2^ω. Let $s = \mathrm{lci}(f, g)$. If s is an initial segment of h, then s is also an initial segment of $\mathrm{lci}(f, h)$ and hence $d_{\mathrm{ultra}}(f, h) \leq \Delta(s) = d_{\mathrm{ultra}}(f, g)$. If s is not an initial segment of h, then $\mathrm{lci}(f, h) = \mathrm{lci}(g, h)$ and hence $d_{\mathrm{ultra}}(f, h) = \Delta(\mathrm{lci}(g, h)) = d_{\mathrm{ultra}}(g, h)$. In both cases we have

$$d_{\mathrm{ultra}}(f, h) \leq \max(d_{\mathrm{ultra}}(f, g), d_{\mathrm{ultra}}(g, h)),$$

showing that d_{ultra} indeed is an ultrametric.

We define an embedding of 2^ω into $\mathbb{R}$ by letting

$$e(f) = \sum_{n=0}^{\infty} (-1)^{f(n)} \cdot \frac{\Delta(f \restriction n)}{2}$$

for every $f \in 2^\omega$. The series $e(f)$ converges for every f since $(\Delta(f \restriction n))_{n\in\omega}$ decreases sufficiently fast. More precisely, for every $m \in \omega$,

$$\left| e(f) - \sum_{n=0}^{m} (-1)^{f(n)} \cdot \frac{\Delta(f \restriction n)}{2} \right| = \left| \sum_{n=m+1}^{\infty} (-1)^{f(n)} \cdot \frac{\Delta(f \restriction n)}{2} \right|$$
$$\leq \frac{\varepsilon}{2} \cdot \Delta(f \restriction m) \cdot \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\varepsilon}{2} \cdot \Delta(f \restriction m).$$

It follows that if $f, g \in 2^\omega$ are distinct and $s = \mathrm{lci}(f,g)$ then

$$\frac{1}{K} \leq 1 - \varepsilon \leq \frac{(1-\varepsilon)\cdot d_{\text{ultra}}(f,g)}{d_{\text{ultra}}(f,g)}$$
$$= \frac{(1-\varepsilon)\cdot \Delta(s)}{d_{\text{ultra}}(f,g)} \leq \frac{|e(f) - e(g)|}{d_{\text{ultra}}(f,g)} \leq \frac{(1+\varepsilon)\cdot \Delta(s)}{d_{\text{ultra}}(f,g)}$$
$$= \frac{(1+\varepsilon)\cdot d_{\text{ultra}}(f,g)}{d_{\text{ultra}}(f,g)} \leq 1 + \varepsilon \leq K.$$

Therefore e is a K-embedding. □

Using Lemma 5.15 it is now easy to prove

Theorem 5.16. *Let $K > 1$. Let (X, d) be a complete metric space without isolated points. Then X has a perfect subset Y that K-embeds into an ultrametric space that is K-linear.*

Proof. The proof of this theorem is a straight forward construction of a Cantor space using a tree of open sets.

Let $\varepsilon = 1 - \frac{1}{K}$. We choose a family $(x_s)_{s\in 2^{<\omega}}$ of points in X and a family $(O_s)_{s\in 2^{<\omega}}$ of open subsets of X such that the following conditions are satisfied:

(1) For all $s \in 2^{<\omega}$, $x_s \in O_s$.
(2) If $t \in 2^{<\omega}$ is a proper extension of $s \in 2^{<\omega}$, then $\mathrm{cl}(O_t) \subseteq O_s$.
(3) For all $s \in 2^{<\omega}$ the diameters of $U_{s^\frown 0}$ and $U_{s^\frown 1}$ are at most $\frac{\varepsilon}{4} \cdot \Delta(s)$ where $\Delta(s) = d(x_{s^\frown 0}, x_{s^\frown 1})$.

Since $\varepsilon < 1$, (3) implies

(4) If $s,t \in 2^{<\omega}$ are distinct sequences of the same length, then $\mathrm{cl}(U_s)$ and $\mathrm{cl}(U_t)$ are disjoint.

The families $(x_s)_{s\in 2^{<\omega}}$ and $(O_s)_{s\in 2^{<\omega}}$ can be chosen by recursion on the length of s since X has no isolated points and therefore every non-empty open subset of X is infinite.

By (1)–(3), for every $f : \omega \to 2$ the sequence $(x_{f\restriction n})_{n\in\omega}$ is Cauchy. Since X is complete, $x_f = \lim_{n\to\infty} x_{f\restriction n}$ exists. By (4), if $f \neq g$, then $x_f \neq x_g$. If follows that $e : 2^\omega \to X; f \mapsto x_f$ is 1-1. It is easily checked that $Y = e[2^\omega]$ is a perfect set. In fact, Y is a homeomorphic copy of the Cantor set.

Note that by (1)–(3), Δ satisfies the requirements of Lemma 5.15. Let d_{ultra} be the ultrametric on 2^ω defined from Δ. By Lemma 5.15, $(2^\omega, d_{\mathrm{ultra}})$ is K-linear.

It remains to show that e is a K-embedding with respect to d_{ultra} and d.

Let $f,g \in 2^\omega$ be distinct. Let $s = \mathrm{lci}(f,g)$. Then $d_{\mathrm{ultra}}(f,g) = \Delta(s)$. We may assume that $s^\frown 0$ is an initial segment of f and $s^\frown 1$ of g.

By (2), $x_f \in U_{s^\frown 0}$ and $x_g \in U_{s^\frown 1}$. Now by (3) we have

$$\frac{1}{K} \leq \frac{(1-\varepsilon)\cdot d(x_{s^\frown 0}, x_{s^\frown 1})}{d(x_{s^\frown 0}, x_{s^\frown 1})} \leq \frac{d(x_f, x_g)}{d_{\mathrm{ultra}}(x_f, x_g)} \leq \frac{(1+\varepsilon)\cdot d(x_{s^\frown 0}, x_{s^\frown 1})}{d(x_{s^\frown 0}, x_{s^\frown 1})} \leq K.$$

This shows that e indeed is a K-embedding. $\square$

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