THE SHIFT ON $\mathcal{P}(\omega)/\mathtt{fin}$

STEFAN GESCHKE

1. Introduction

The map $n \mapsto n+1$ induces an automorphism of the Boolean algebra $\mathcal{P}(\omega)/\mathtt{fin}$, which we call the *shift* and denote by s. In this note we prove a number of results that are related to the question whether the two structures $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ and $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ can be isomorphic. Here we consider $\mathcal{P}(\omega)/\mathtt{fin}$ with the usual Boolean operations $\cap$, $\cup$, and complementation.

Observe that an isomorphism $\psi : (\mathcal{P}(\omega)/\mathtt{fin}, s) \to (\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ is the same as an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $s^{-1} = \psi \circ s \circ \psi^{-1}$ (see Lemma 3.3 below). Hence $(\mathcal{P}(\omega)/\mathtt{fin}, s)$ is isomorphic to $(\mathcal{P}(\omega)/\mathtt{fin}, s^{-1})$ iff s is conjugated to s^{-1} in the automorphism group of $\mathcal{P}(\omega)/\mathtt{fin}$.

2. preliminaries

As usual, $\triangle$ denotes symmetric difference, i.e., $A \triangle B = (A \setminus B) \cup (B \setminus A)$. We write $A =^* B$ if $A \triangle B$ is finite and $A \subseteq^* B$ if $A \setminus B$ is finite. The collection of finite subsets of ω is denoted by $\mathtt{fin}$. In the following, elements of $\mathcal{P}(\omega)/\mathtt{fin}$ are denoted by small letters from the beginning of the alphabet, while subsets of ω are denoted by capital letters from the beginning of the alphabet. For each $A \subseteq \omega$, $a = A \triangle \mathtt{fin} = \{A \triangle B : B \in \mathtt{fin}\}$ is the corresponding element of $\mathcal{P}(\omega)/\mathtt{fin}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ and let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism. If $b : A \to B$ is a bijection between cofinite subsets of ω, then b *induces* ψ if for all $a \in \mathcal{A}$ and all $A \in a$, $\psi(a) = b[A] \triangle \mathtt{fin}$. An automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ that is induced by a bijection between two cofinite subsets of ω is *trivial*.

$S : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is the right shift, i.e., the map that maps every set $A \subseteq \omega$ to the set $A + 1 = \{n + 1 : n \in A\}$. Let s be the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by S. Similarly, let $S_{-1} : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ is the left shift, i.e., the map that maps every set $A \subseteq \omega$ to the set $A - 1 = \{n - 1 : n \in A \setminus \{0\}\}$. The automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by S_{-1} is just the inverse s^{-1} of s. For each $k \in \mathbb{Z}$ we define $S_k : \mathcal{P}(\omega) \to \mathcal{P}(\omega)$ as follows:

(1) If $k = 0$, let S_k be the identity on $\mathcal{P}(\omega)$.
(2) If $k > 0$, let $S_k = S^k$.
(3) If $k = -\ell$ for some $\ell > 0$, let $S_k = (S_{-1})^\ell$.

Clearly, for each $k \in \mathbb{Z}$, the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by S_k is just s^k.

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2.1. **The shift versus its inverse.** Van Douwen [3] showed that the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ has a nontrivial normal subgroup, the group of automorphisms induced by permutations of ω.

More precisely, given a trivial automorphism ψ and a bijection $b : A \to B$ between cofinite subsets of ω that induces ψ, we assign an *index* to ψ by putting

$$\mathrm{index}(\psi) = |\omega \setminus B| - |\omega \setminus A|.$$

Van Douwen showed that the index is a homomorphism from the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ into $(\mathbb{Z}, +)$. The trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ of index 0 are precisely those automorphisms that are induced by a permutation of ω. The shift s has index 1 and s^{-1} has index -1.

The following lemma seems to be folklore.

Lemma 2.1. *There is no trivial automorphism ψ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $s^{-1} = \psi \circ s \circ \psi^{-1}$.*

Proof. If ψ is trivial, then

$$\mathrm{index}(\psi\circ s\circ\psi^{-1}) = \mathrm{index}(\psi)+\mathrm{index}(s)+\mathrm{index}(\psi^{-1}) = \mathrm{index}(\psi)+1-\mathrm{index}(\psi) = 1$$

while $\mathrm{index}(s^{-1}) = -1$. Hence $\psi \circ s \circ \psi^{-1} \neq s^{-1}$. □

3. The algebra of periodic elements

We call a subalgebra $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ that is closed under s and s^{-1} *shift-closed*. An easy Löwenheim-Skolem argument shows that every countable subset of $\mathcal{P}(\omega)/\mathtt{fin}$ is contained in a shift-closed countable atomless subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. As it turns out, there is a particularly nice countable atomless subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is shift-closed.

Let $\mathtt{Per}$ denote the set of all *periodic* elements of $\mathcal{P}(\omega)/\mathtt{fin}$, i.e., let

$$\mathtt{Per} = \{a \in \mathcal{P}(\omega)/\mathtt{fin} : \{s^k(a) : k \in \mathbb{Z}\} \text{ is finite}\}.$$

It is easily checked that $\mathtt{Per}$ is a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and that it is shift-closed. For each $n > 0$ let $\mathtt{Per}_n$ be the set of all elements of $\mathcal{P}(\omega)/\mathtt{fin}$ that are fixed points of s^n. Again, each $\mathtt{Per}_n$ is a shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. Clearly, $\mathtt{Per} = \bigcup_{n=1}^{\infty} \mathtt{Per}_n$.

Lemma 3.1. *a) $\mathtt{Per}_n \subseteq \mathtt{Per}_m$ iff $n|m$.*

b) $a \in \mathtt{Per}_n$ iff there is $B_a \subseteq n$ such that $a = \left(\bigcup_{k\in\omega} S^{kn}(B_a)\right) \triangle\, \mathtt{fin}$. Moreover, B_a is unique and the map $f_n : \mathtt{Per}_n \to \mathcal{P}(n); a \mapsto B_a$ is an isomorphism.

c) $\mathtt{Per}$ is atomless.

Proof. a) Suppose $n|m$. Let $a \in \mathtt{Per}_n$. Then $s^n(a) = a$. Now clearly $s^m(a) = a$. Hence $a \in \mathtt{Per}_m$. It follows that $\mathtt{Per}_n \subseteq \mathtt{Per}_m$.

On the other hand, suppose $\mathtt{Per}_n \subseteq \mathtt{Per}_m$. Let $A = \{k \in \omega : n|k\}$ and let $a = A \triangle \mathtt{fin}$. Then $a \in \mathtt{Per}_n$ and hence, by our assumption, $a \in \mathtt{Per}_m$. In particular, $s^m(a) = a$. But this can only happen if $n|m$.

b) Let $a \in \mathtt{Per}_n$ and choose $A \in a$. Then there is m_0 such that for all $m \geq m_0$,

$$(A \cap [nm, n(m+1))) + n = A \cap [n(m+1), n(m+2)).$$

Let $B_a = [A \cap [nm_0, n(m_0+1))] - nm_0$. Then $A =^* \left(\bigcup_{k\in\omega} S^{kn}(B_a)\right)$ and thus $a = \left(\bigcup_{k\in\omega} S^{kn}(B_a)\right) \triangle \mathtt{fin}$. It is easily checked that f_n is an isomorphism.

c) Let $a \in \mathtt{Per}$ but $a > 0$. Fix $n \in \omega$ such that $a \in \mathtt{Per}_n$. Let f_n and f_{2n} be as in b). Since $a > 0$, $f_n(a)$ has at least one element k. By the definition of f_{2n}, we have $k, k+n \in f_{2n}(a)$. Hence $\{k\}$ and $\{k+n\}$ are disjoint elements of $\mathcal{P}(2n)$ below $f_{2n}(a)$. Since f_{2n} is an isomorphism, a is not an atom of $\mathtt{Per}_{2n}$ and the therefore it is not an atom of $\mathtt{Per}$. $\square$

Lemma 3.2. *There is an automorphism φ of $\mathtt{Per}$ such that $s^{-1} \restriction \mathtt{Per} = \varphi \circ s \circ \varphi^{-1}$.*

Proof. On $\mathcal{P}(n)$ we consider the automorphism $S_{\mathrm{mod}\, n}$ that maps every $A \subseteq n$ to the set $\{(k+1) \mod n : k \in A\}$. Clearly, $S_{\mathrm{mod}\, n} = f_n \circ s \circ f_n^{-1}$ and for each $A \subseteq n$ we have $S^{-1}_{\mathrm{mod}\, n}(A) = \{(k-1) \mod n : k \in A\}$.

For each $n > 0$ let Φ_n be the automorphism of $\mathcal{P}(n)$ that maps every $A \subseteq n$ to the set $\{n-k-1 : k \in A\}$. Clearly, $\Phi_n^{-1} = \Phi_n$. Since for all k we have

$$n - ((n-k-1)+1) - 1 = n - n + k - 1 = k - 1,$$

it holds that $S^{-1}_{\mathrm{mod}\, n} = \Phi_n \circ S_{\mathrm{mod}\, n} \circ \Phi_n^{-1}$.

Now let $m > 0$ and consider the embedding $e_n^{mn} : \mathcal{P}(n) \to \mathcal{P}(mn)$ that maps each $A \subseteq n$ to the set

$$e_n^{mn}(A) = \bigcup_{i\in m} (A + in) = \{k + in : k \in A \wedge i \in m\}.$$

It is easily checked that $e_n^{mn} = f_{mn} \circ f_n^{-1}$.

For all $n > 0$ let $\varphi_n = f_n^{-1} \circ \Phi_n \circ f_n$. Now φ_n is an automorphism of $\mathtt{Per}_n$ such that $s^{-1} \restriction \mathtt{Per}_n = \varphi_n \circ s \circ \varphi_n^{-1}$.

We show that $e_n^{mn} \circ \Phi_n = \Phi_{mn} \circ e_n^{mn}$. Let $A \subseteq n$. Then

$$\begin{aligned}\Phi_{mn}(e_n^{mn}(A)) &= \{mn-k-1 : k \in e_n^{mn}(A)\} = \{mn-(k+in)-1 : k \in A \wedge i \in m\}\\ &= \{(mn-in)-k-1 : k \in A \wedge i \in m\} = \{jn-k-1 : k \in A \wedge j \in \{1,\dots,m\}\}\\ &= \{(n-k-1)+in : k \in A \wedge i \in m\} = e_n^{mn}(\Phi_n(A)).\end{aligned}$$

It follows that for all $m, n > 0$, φ_{mn} is an extension of φ_n. Hence $(\varphi_n)_{n>0}$ is a directed system in the sense that for all $m, n > 0$, φ_m and φ_n have a common extension, namely φ_{mn}. Since

$$\mathtt{Per} = \bigcup_{n=1}^{\infty} \mathtt{Per}_n = \bigcup_{n=1}^{\infty} \operatorname{dom}\varphi_n = \bigcup_{n=1}^{\infty} \operatorname{ran}\varphi_n,$$

$\varphi = \bigcup_{n>0} \varphi_n$ is an automorphism of $\mathtt{Per}$ such that $s^{-1} = \varphi \circ s \circ \varphi^{-1}$. $\square$

Lemma 3.3. *Let X and Y be sets and let $p : X \to X$ and $q : Y \to Y$ bijections. A bijection $\psi : X \to Y$ is an isomorphism between the structures (X, p) and (Y, q) if and only if $q = \psi \circ p \circ \psi^{-1}$.*

Proof. The map ψ is an isomorphism between (X,p) and (Y,q) if and only if $\psi\circ p = q\circ\psi$. But the latter statement is equivalent to $q = \psi\circ p\circ\psi^{-1}$. $\square$

Lemma 3.4. *Let $n>0$. Then the powers of $S_{\mathrm{mod}\,n}$ are the only automorphisms of the structure $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$. Here $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$ is understood to be the Boolean algebra $\mathcal{P}(n)$ with the additional unary operation $S_{\mathrm{mod}\,n}$.*

Proof. Clearly, for all $k\in\mathbb{Z}$ we have $S^k_{\mathrm{mod}\,n}\circ S_{\mathrm{mod}\,n}\circ S^{-k}_{\mathrm{mod}\,n} = S_{\mathrm{mod}\,n}$. Now using Lemma 3.3 it follows that $S^k_{\mathrm{mod}\,n}$ is an automorphism of $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$.

On the other hand, $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$ is generated by the atom $\{0\}$. Hence, an automorphism T of $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$ is already determined by the image of $\{0\}$. But $T(\{0\})$ is an atom $\mathcal{P}(n)$, i.e., $T(\{0\}) = \{k\}$ for some $k\in n$. It follows that $T = S^k_{\mathrm{mod}\,n}$. $\square$

Lemma 3.5. *Let ψ be an automorphism of* $\mathtt{Per}$ *such that $s^{-1}\restriction\mathtt{Per} = \psi\circ s\circ\psi^{-1}$. Let φ be the automorphism of* $\mathtt{Per}$ *constructed in Lemma 3.2.*

a) For all $n>0$, $\psi_n = \psi\restriction\mathtt{Per}_n$ is an automorphism of $\mathtt{Per}_n$.

b) For all $k\in\mathbb{Z}$, if $\rho = s^k\circ\psi$, then $s^{-1}\restriction\mathtt{Per} = \rho\circ s\circ\rho^{-1}$.

c) For each $n>0$ let $\Psi_n = f_n\circ\psi_n\circ f_n^{-1}$ where f_n is the isomorphism between $\mathtt{Per}_n$ and $\mathcal{P}(n)$ from Lemma 3.1. Moreover, let $\varphi_n = \varphi\restriction\mathtt{Per}_n$ and $\Phi_n = f_n\circ\varphi_n\circ f_n^{-1}$ as in the proof of Lemma 3.2. Then there is some $k\in\mathbb{Z}$ such that $\Psi_n = S^k_{\mathrm{mod}\,n}\circ\Phi_n$. In other words, $\psi_n = s^k\circ\varphi_n$.

Proof. a) Since $s^n(a) = a$ if and only if $s^{-n}(a) = a$, we obtain the same set $\mathtt{Per}_n$ if in the definition of $\mathtt{Per}_n$, s is replaced by s^{-1}. In other words, applying the $(\mathtt{Per}, s)$-definition of $\mathtt{Per}_n$ in the structure $(\mathtt{Per}, s^{-1}\restriction\mathtt{Per})$ yields the same set. Since ψ is an isomorphism between $(\mathtt{Per}, s\restriction\mathtt{Per})$ and $(\mathtt{Per}, s^{-1}\restriction\mathtt{Per})$ by Lemma 3.3, $\psi\restriction\mathtt{Per}_n$ is an isomorphism between $(\mathtt{Per}_n, s\restriction\mathtt{Per}_n)$ and $(\mathtt{Per}_n, s^{-1}\restriction\mathtt{Per}_n)$. In particular, ψ is an automorphism of $\mathtt{Per}_n$.

b) Let $k\in\mathbb{Z}$. Clearly, $s^k\circ s^{-1}\circ s^{-k} = s^{-1}$. Hence, by Lemma 3.3, s^k is an automorphism of $(\mathtt{Per}, s^{-1}\restriction\mathtt{Per})$. Since ψ is an isomorphism between $(\mathtt{Per}, s\restriction\mathtt{Per})$ and $(\mathtt{Per}, s^{-1}\restriction\mathtt{Per})$, ρ is an isomorphism between $(\mathtt{Per}, s\restriction\mathtt{Per})$ and $(\mathtt{Per}, s^{-1}\restriction\mathtt{Per})$. Again by Lemma 3.3, $s^{-1}\restriction\mathtt{Per} = \rho\circ s\circ\rho^{-1}$.

c) Let $n>0$. Using Lemma 3.3 it is easily checked that $\Psi_n\circ\Phi_n^{-1}$ is an automorphism of $(\mathcal{P}(n), S_{\mathrm{mod}\,n})$. By Lemma 3.4 there is some $k\in\mathbb{Z}$ such that $\Psi_n\circ\Phi_n^{-1} = S^k_{\mathrm{mod}\,n}$. Now $\Psi_n = S^k_{\mathrm{mod}\,n}\circ\Phi_n$ $\square$

The following lemma is a special case of a result of Bella et al. [2] saying that every automorphism of a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ is induced by a permutation on ω and therefore extends to all of $\mathcal{P}(\omega)/\mathtt{fin}$. Bella et al. use a forcing argument for the proof of their theorem. We will explicitly construct a permutation that induces the automorphism φ of $\mathtt{Per}$.

Lemma 3.6. *The automorphism φ of* $\mathtt{Per}$ *constructed in Lemma 3.2 is induced by a permutation b of ω.*

Proof. Let $I_0 = \{0\}$. For all $n > 0$ let $I_n = [n!, (n+1)!)$. If $n > 0$ and $k \in I_n$, let

$$b(k) = (n+1)! - 1 - (k - n!).$$

Let $b(0) = 0$. Note that for all $n \in \omega$, $b \restriction I_n$ is a permutation of I_n. Hence, b is a permutation of ω. For every $a \in \mathtt{Per}$ choose $A \in a$ and let $\psi(a) = b[A] \bigtriangleup \mathtt{fin}$.

Now, whenever $m \geq n > 0$ and $k \in I_m$, then

$$b(k) \bmod n = ((m+1)! - 1 - (k - m!)) \bmod n = (-1-k) \bmod n = (n-1-k) \bmod n.$$

It follows that $f_n \circ \psi \circ f_n^{-1} = \Phi_n$ where Φ_n is the automorphism of $\mathcal{P}(n)$ defined in the proof of Lemma 3.2, i.e., $\Phi_n = f_n \circ \varphi \circ f_n^{-1}$. Hence $\psi \restriction \mathtt{Per}_n = \varphi \restriction \mathtt{Per}_n$. Since this holds for every $n > 0$, we have $\psi = \varphi$ and thus, φ is induced by b. □

A slight modification of the proof of Lemma 3.6 shows that there are in fact $2^{\aleph_0}$ permutations of ω that all induce φ on $\mathtt{Per}$. We will prove a more general result in the next section.

4. Isomorphisms between countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$

We prove a slight generalization of the result of Bella et al. that every automorphism of a countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ is induced by a permutation of ω. Our proof is rather explicit and does not use forcing.

Theorem 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ and let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism. Then there is a permutation of ω that induces ψ.*

In order simplify the proof of this theorem a bit, we first prove a lemma that implies that we can restrict our attention to isomorphisms between atomless subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. The lemma is intuitively clear, but writing down an exact proof turns out to be slightly technical.

Lemma 4.2. *Let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism between subalgebras of an atomless Boolean algebra $\mathcal{C}$. Then there are atomless subalgebras $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ of $\mathcal{C}$ and an isomorphism $\overline{\psi}$ between them such that $\mathcal{A} \subseteq \overline{\mathcal{A}}$, $\mathcal{B} \subseteq \overline{\mathcal{B}}$, and $\psi \subseteq \overline{\psi}$. $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ can be chosen of size $|\mathcal{A}| + \aleph_0$.*

Proof. Since $\mathcal{C}$ is atomless, for each atom $a \in \mathcal{A}$ the Boolean algebra $\mathcal{C} \restriction a = \{c \in \mathcal{C} : c \leq a\}$ is atomless. In particular, $\mathcal{C} \restriction a$ has a countable atomless subalgebra $\mathcal{A}_a$. Since a is an atom of $\mathcal{A}$, $\psi(a)$ is an atom of $\mathcal{B}$. Choose a countable atomless subalgebra $\mathcal{B}_{\psi(a)}$ of $\mathcal{C} \restriction \psi(a)$. Since any two countable atomless Boolean algebras are isomorphic, there is an isomorphism $\psi_a : \mathcal{A}_a \to \mathcal{B}_{\psi(a)}$.

Note that every atom b of $\mathcal{B}$ is of the form $\psi(a)$ for some atom a of $\mathcal{A}$. Let $\overline{\mathcal{A}}$ be the subalgebra of $\mathcal{C}$ generated by

$$\mathcal{D} = \mathcal{A} \cup \bigcup \{\mathcal{A}_a : a \text{ is an atom of } \mathcal{A}\}$$

and let $\overline{\mathcal{B}}$ be the subalgebra of $\mathcal{C}$ generated by

$$\mathcal{B} \cup \bigcup \{\mathcal{B}_b : b \text{ is an atom of } \mathcal{B}\}.$$

Consider the set $\mathcal{E}$ of all elements of $\mathcal{C}$ of the form $a_0 \vee c_1 \vee \cdots \vee c_n$ where $a_0 \in \mathcal{A}$ and the c_i are pairwise disjoint elements of $\bigcup\{\mathcal{A}_a : a \text{ is an atom of } \mathcal{A}\} \setminus \mathcal{A}$ such that for each atom a of $\mathcal{A}$ at most one c_i lies in $\mathcal{A}_a$. We explicitly allow $n = 0$, so that $\mathcal{A} \subseteq \mathcal{E}$. Note that for $d \in \mathcal{E}$ the representation as $a_0 \vee c_1 \vee \cdots \vee c_n$ as above is unique up to the order of the c_i. It is straightforward to check that $\mathcal{E}$ is a subalgebra of $\mathcal{C}$. Since $\mathcal{D} \subseteq \mathcal{E}$ and all the elements of $\mathcal{E}$ are Boolean combinations of elements of $\mathcal{D}$, $\mathcal{E} = \overline{\mathcal{A}}$.

Now let $d \in \overline{\mathcal{A}}$ and let $a_0 \vee c_1 \vee \cdots \vee c_n$ be its unique representation as above. For each $i \in \{1, \ldots, n\}$ let a_i be the unique atom of $\mathcal{A}$ such that $c_i \in \mathcal{A}_{a_i}$. Define

$$\overline{\psi}(d) = \psi(a_0) \vee \psi_{a_1}(c_1) \vee \cdots \vee \psi_{a_n}(c_n).$$

It is easily checked that $\overline{\psi}$ is an isomorphism between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. By definition, $\overline{\psi}$ extends ψ. $\square$

Proof of Theorem 4.1. Let $\mathcal{A}$ and $\mathcal{B}$ be countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ and let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism. By Lemma 4.2 we may assume that $\mathcal{A}$ and $\mathcal{B}$ are atomless. Being countable and atomless, $\mathcal{A}$ is a free Boolean algebra over countably many generators. Let $(a_n)_{n\in\omega}$ be an independent family generating $\mathcal{A}$. For every $n \in \omega$ let $\mathcal{A}_n$ be the subalgebra of $\mathcal{A}$ generated by $\{a_i : i < n\}$. Now $\mathcal{A}_n$ is isomorphic to $\mathcal{P}(2^n)$ and in passing from n to $n+1$, every atom of $\mathcal{A}_n$ is split into two atoms of $\mathcal{A}_{n+1}$.

We define to sequences $(P_n)_{n\in\omega}$, $(Q_n)_{n\in\omega}$ of finite partitions of ω into infinite sets. Let 2^n denote the set of all binary sequences of length n. Let $P_0 = \{A_\emptyset\}$ and $Q_0 = \{B_\emptyset\}$ where $A_\emptyset = B_\emptyset = \omega$. Now suppose we have defined $P_n = \{A_\sigma : \sigma \in 2^n\}$ and $Q_n = \{B_\sigma : \sigma \in 2^n\}$ so that

(1) $\{A_\sigma \triangle \mathtt{fin} : \sigma \in 2^n\}$ is the set of atoms of $\mathcal{A}_n$,
(2) $\{B_\sigma \triangle \mathtt{fin} : \sigma \in 2^n\}$ is the set of atoms of $\mathcal{B}_n$, and
(3) for all $\sigma \in 2^n$, $\psi(A_\sigma \triangle \mathtt{fin}) = B_\sigma \triangle \mathtt{fin}$.

For each $\sigma \in 2^n$ and $i \in 2$ let $\sigma^\frown i$ denote the sequence σ extended by the single digit i. Now choose $A_{\sigma^\frown 0}, A_{\sigma^\frown 1}, B_{\sigma^\frown 0}, B_{\sigma^\frown 1} \subseteq \omega$ such that

(a) $A_{\sigma^\frown 0}$ and $A_{\sigma^\frown 1}$ are disjoint with union A_σ,
(b) $B_{\sigma^\frown 0}$ and $B_{\sigma^\frown 1}$ are disjoint with union B_n^σ,
(c) $A_{\sigma^\frown 0} \triangle \mathtt{fin}$ and $A_{\sigma^\frown 1} \triangle \mathtt{fin}$ are the two atoms of $\mathcal{A}_{n+1}$ below $A_\sigma \triangle \mathtt{fin}$,
(d) $B_{\sigma^\frown 0} \triangle \mathtt{fin}$ and $B_{\sigma^\frown 1} \triangle \mathtt{fin}$ are the two atoms of $\mathcal{B}_{n+1}$ below $B_\sigma \triangle \mathtt{fin}$, and
(e) for all $i \in 2$, $\psi(A_{\sigma^\frown i} \triangle \mathtt{fin}) = B_{\sigma^\frown i} \triangle \mathtt{fin}$.

By recursion on n, we define a sequence $(b_n)_{n\in\omega}$ of injections from finite subsets of ω to finite subsets of ω. Let $b_0 = \{(0,0)\}$. Suppose b_n has been defined. For each $\sigma \in 2^{n+1}$ let x_σ be the first element of $A_\sigma \setminus \mathrm{dom}(b_n)$ and let y_σ be the first element of $B_\sigma \setminus \mathrm{ran}(b_n)$. Since the A_σ, $\sigma \in 2^{n+1}$, are pairwise disjoint, the x_σ are pairwise distinct. Similarly, the y_σ are pairwise distinct. It follows that $b = \bigcup_{n\in\omega} b_n$ is a bijection between two infinite subsets of ω.

Claim 4.3. The bijection b is a permutation of ω, i.e., $\mathrm{dom}(b) = \mathrm{ran}(b) = \omega$.

We only show $\mathrm{dom}(b) = \omega$ since the proof of $\mathrm{ran}(b) = \omega$ is practically identical. Suppose $\mathrm{dom}(b) \neq \omega$ and let x be the smallest element of $\omega \setminus \mathrm{dom}(b)$. Let n be minimal with $x \subseteq \mathrm{dom}(b_n)$. Since $P_{n+1} = \{A_\sigma : \sigma \in 2^{n+1}\}$ is a partition of ω, there is some $\sigma \in 2^{n+1}$ such that $x \in A_\sigma$. But now x is the smallest element of $A_\sigma \setminus \mathrm{dom}(b_n)$. Hence $x = x_\sigma \in \mathrm{dom}(b_{n+1}) \subseteq \mathrm{dom}(b)$, a contradiction. This proves the claim.

Claim 4.4. For all $n \in \omega$ and all $\sigma \in 2^{n+1}$, $b[A_\sigma \setminus \mathrm{dom}(b_n)] = B_\sigma \setminus \mathrm{ran}(b_n)$.

Let $x \in A_\sigma \setminus \mathrm{dom}(b_n)$. Then $x = x_\tau$ for some $\tau \in 2^{<\omega} \setminus 2^n$. But since $x_\tau \in A_\tau$ and A_τ intersects A_σ only if σ and τ are comparable, $\sigma \subseteq \tau$. It follows that $b(x) = y_\tau \in B_\tau \subseteq B_\sigma$. But by the choice of y_τ, $y_\tau \notin \mathrm{ran}(b_n)$. Hence

$$b[A_\sigma \setminus \mathrm{dom}(b_n)] \subseteq B_\sigma \setminus \mathrm{ran}(b_n).$$

A similar argument shows that

$$b^{-1}[B_\sigma \setminus \mathrm{ran}(b_n)] \subseteq A_\sigma \setminus \mathrm{dom}(b_n)$$

and the claim follows.

By Claim 4.3, b induces an automorphism $\overline{\psi}$ of $\mathcal{P}(\omega)/\mathtt{fin}$. If for some $n \in \omega$, a is an atom of $\mathcal{A}_n$, then for some $\sigma \in 2^n$, $a = A_\sigma \triangle \mathtt{fin}$. By Claim 4.4,

$$\overline{\psi}(a) = b[A_\sigma] \triangle \mathtt{fin} = B_\sigma \triangle \mathtt{fin} = \psi(a).$$

Since $\mathcal{A}$ is generated by the union of the sets of atoms of the $\mathcal{A}_n$, $n \in \omega$, we have $\overline{\psi} \restriction \mathcal{A} = \psi$ and thus ψ is induced by b. $\square$

Corollary 4.5. *Every isomorphism between two countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ extends to $2^{\aleph_0}$ automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ that are induced by permutations of $\mathcal{P}(\omega)/\mathtt{fin}$.*

Proof. Let $\psi : \mathcal{A} \to \mathcal{B}$ be an isomorphism between two countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$. It is well-known that for every countable subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ there is an element of $\mathcal{P}(\omega)/\mathtt{fin}$ that is independent over the subalgebra. Iterating this argument we obtain subalgebras $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\overline{\mathcal{A}}$ is a free product of $\mathcal{A}$ and a countable free Boolean algebra $\mathcal{F}_\mathcal{A} \subseteq \mathcal{P}(\omega)/\mathtt{fin}$ and likewise, $\overline{\mathcal{B}}$ is the free product of $\mathcal{B}$ and a countable free Boolean algebra $\mathcal{F}_\mathcal{B} \subseteq \mathcal{P}(\omega)/\mathtt{fin}$. There are $2^{\aleph_0}$ isomorphisms between $\mathcal{F}_\mathcal{A}$ and $\mathcal{F}_\mathcal{B}$ and each of them extends to an isomorphism between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$ that agrees with ψ on $\mathcal{A}$. In other words, ψ has $2^{\aleph_0}$ extensions to isomorphisms between $\overline{\mathcal{A}}$ and $\overline{\mathcal{B}}$. By Theorem 4.1, each of these extensions of ψ is induced by a permutation of ω. But if an isomorphism between subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ is induced by a permutation b of ω, then it clearly extends to all of $\mathcal{P}(\omega)/\mathtt{fin}$, namely to the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ that is induced b. $\square$

Remark 4.6. If every isomorphism between two subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ of size $\aleph_1$ is induced by a bijection between cofinite subsets of ω, then $2^{\aleph_0} = 2^{\aleph_1}$.

Proof. Let $\mathcal{F}$ be an independent family in $\mathcal{P}(\omega)/\mathtt{fin}$ of size $\aleph_1$ and let $\mathcal{A}$ be the subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ generated by $\mathcal{F}$. There are $2^{\aleph_1}$ permutations of $\mathcal{F}$ and each of them extends to an automorphism of $\mathcal{A}$. If each of these $2^{\aleph_1}$ automorphisms of $\mathcal{A}$ is induced by a bijection between cofinite subsets of ω, then $2^{\aleph_1} \leq 2^{\aleph_0}$ since there are only $2^{\aleph_0}$ bijections between cofinite subsets of ω. $\square$

It was pointed out by Farah that it is consistent with MA+$\neg$CH that there is an automorphism of a subalgebra $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ of size $\aleph_1$ that does not extend to an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. This for example follows from the next remark together with the fact that MA+$2^{\aleph_0} = \aleph_2$ is consistent with the non-existence of non-trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ (see [7]).

Remark 4.7. If $2^{\aleph_0} = \aleph_2$, MA holds, and every automorphism of every subalgebra $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ of size $\aleph_1$ extends to all of $\mathcal{P}(\omega)/\mathtt{fin}$, then $\mathcal{P}(\omega)/\mathtt{fin}$ has a nontrivial automorphism.

Proof. Let $(\varphi_\alpha)_{\alpha<\omega_2}$ list all the trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. Let $(a_\alpha)_{\alpha<\omega_2}$ be an enumeration of $\mathcal{P}(\omega)/\mathtt{fin}$. Let $\mathcal{A}_0$ be the trivial subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ and let ψ_0 be the identity on $\mathcal{A}_0$.

Suppose for some $\alpha < \omega_2$, we have constructed a subalgebra $\mathcal{A}_\alpha$ of size $\leq \aleph_1$ and an automorphism ψ_α of $\mathcal{A}_\alpha$. By MA, let $a \in \mathcal{P}(\omega)/\mathtt{fin}$ be independent over $\mathcal{A}_\alpha$. Now ψ_α has two different extensions to the subalgebra $\mathcal{A}_\alpha(a)$ generated by $\mathcal{A}_\alpha$ and a, namely one mapping a to itself and one mapping a to $\neg a$. One of these two extensions is not a restriction of φ_α to $\mathcal{A}_\alpha(a)$. Since every automorphism of $\mathcal{A}_\alpha(a)$ extends to all of $\mathcal{P}(\omega)/\mathtt{fin}$, there are a subalgebra $\mathcal{A}_{\alpha+1}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ of size $\aleph_1$ and an automorphism $\psi_{\alpha+1}$ of $\mathcal{A}_{\alpha+1}$ such that $a, a_\alpha \in \mathcal{A}_{\alpha+1}$, $\mathcal{A}_\alpha \subseteq \mathcal{A}_{\alpha+1}$, $\psi_\alpha \subseteq \psi_{\alpha+1}$ and $\psi_{\alpha+1} \neq \varphi_\alpha \restriction \mathcal{A}_{\alpha+1}$.

If $\beta \leq \omega_2$ is a limit ordinal and $\mathcal{A}_\alpha$ and ψ_α have been defined for all $\alpha < \beta$, let $\mathcal{A}_\beta = \bigcup_{\alpha<\beta} \mathcal{A}_\alpha$ and $\psi_\beta = \bigcup_{\alpha<\beta} \psi_\alpha$. By our construction, $\mathcal{A}_{\omega_2} = \mathcal{P}(\omega)/\mathtt{fin}$ and ψ_{ω_2} is an automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ that is different from all trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$. $\square$

5. The shift and its inverse on small subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$

Let $\mathrm{Sym}(\omega)$ denote the group of all permutations of ω.

Lemma 5.1. *For every $p \in \mathrm{Sym}(\omega)$, p and p^{-1} are conjugate in $\mathrm{Sym}(\omega)$.*

Proof. By Lemma 3.3, it is enough to show that the structures (ω, p) and (ω, p^{-1}) are isomorphic. We construct an isomorphism $I : (\omega, p) \to (\omega, p^{-1})$ as follows:

Let $\sim$ denote the orbit equivalence relation on ω induced by p, i.e., for $n, m \in \omega$ let $n \sim m$ if there is $k \in \mathbb{Z}$ such that $p^k(n) = m$. Let $\omega/\sim$ denote the collection of $\sim$-classes. For each $\sim$-class $a \in \omega/\sim$ choose $n_a \in a$. Now, if $n \in \omega$, let a be the $\sim$-class of n. Choose $k \in \mathbb{Z}$ such that $n = p^k(n_a)$ and let $I(n) = p^{-k}(n_a)$. It is easily checked that I is well defined and an isomorphism between (ω, p) and (ω, p^{-1}). $\square$

Theorem 5.2. *Let $\mathcal{A}$ and $\mathcal{B}$ be countable subalgebras of $\mathcal{P}(\omega)/\mathtt{fin}$ and $t : \mathcal{A} \to \mathcal{B}$ an isomorphism. Then there are automorphisms τ and ψ of $\mathcal{P}(\omega)/\mathtt{fin}$ such that τ extends t and $\psi \circ \tau \circ \psi^{-1} = \tau^{-1}$.*

Proof. By Theorem 4.1, t is induced by by a permutation b of ω. By Lemma 5.1, there is a permutation p of ω such that $b^{-1} = p \circ b \circ p^{-1}$. Let τ be the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by b and let ψ be the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by p. Then clearly, τ extends t and $\psi \circ \tau \circ \psi^{-1} = \tau^{-1}$. □

The weakness of this theorem is that even if $\mathcal{A} = \mathcal{B}$ and hence $\mathcal{A}$ is closed under τ and τ^{-1}, there is no reason to assume that $\mathcal{A}$ is closed under the automorphism ψ of $\mathcal{P}(\omega)/\mathtt{fin}$ that conjugates τ and τ^{-1}.

We address this issue in case of the shift and its inverse. In this particular case, for a given sufficiently small subalgebra $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ there is an explicit way to construct a permutation of ω that induces the respective automorphism on $\mathcal{A}$.

Definition 5.3. Let $\sigma = (n_k)_{k\in\omega}$ be a strictly increasing sequence in ω and put $n_{-1} = -1$. We define two permutations b_σ and p_σ of ω as follows: For $m \in \omega$ let

$$b_\sigma(m) = \begin{cases} m+1, & \text{if } m \notin \{n_k : k \in \omega\}, \\ n_{k-1}+1, & \text{if } m = n_k \text{ for some } k \in \omega \end{cases}$$

and

$$p_\sigma(m) = (n_k - m) + n_{k-1} + 1$$

where $k \in \omega$ is minimal with $n_k \geq m$.

Lemma 5.4. *Let B, σ, b_σ and p_σ be as in Definition 5.3.*

a) $p_\sigma^{-1} = p_\sigma$

b) $p_\sigma \circ b_\sigma \circ p_\sigma^{-1} = b_\sigma^{-1}$

Proof. a) Let $m \in \omega$ and let $k \in \omega$ be minimal with $n_k \geq m$. Now $m \in (n_{k-1}, n_k]$ and $p_\sigma(m) = (n_k - m) + n_{k-1} + 1$, which is at least $n_{k-1} + 1$ and at most n_k. It follows that $(n_{k-1}, n_k]$ is invariant under p_σ. Let $z = n_k - m$. Now $m = n_k - z$ and $p_\sigma(m) = n_{k-1} + 1 + z$. Clearly,

$$p_\sigma(p_\sigma(m)) = p_\sigma(n_{k-1} + 1 + z) = (n_k - (n_{k-1} + 1 + z)) + n_{k-1} + 1 = n_k - z = m.$$

Since m was arbitrary, it follows that $p_\sigma^{-1} = p_\sigma$.

b) We show that $p_\sigma \circ b_\sigma = b_\sigma^{-1} \circ p_\sigma$. Let $m \in \omega$ and let $k \in \omega$ be minimal with $m \leq n_k$. First assume that $m < n_k$. Now $b_\sigma(m) = m + 1$ and

$$(p_\sigma \circ b_\sigma)(m) = p_\sigma(m+1) = (n_k - (m+1)) + n_{k-1} + 1 = (n_k - m) + n_{k-1}.$$

On the other hand, $p_\sigma(m) = (n_k - m) + n_{k-1} + 1 > n_{k-1} + 1$ and

$$(b_\sigma^{-1} \circ p_\sigma)(m) = b_\sigma^{-1}((n_k - m) + n_{k-1} + 1) = (n_k - m) + n_{k-1} = (p_\sigma \circ b_\sigma)(m).$$

Now assume that $m = n_k$. In this case $b_\sigma(m) = n_{k-1} + 1$ and

$$(p_\sigma \circ b_\sigma)(m) = p_\sigma(n_{k-1} + 1) = (n_k - (n_{k-1} + 1)) + n_{k-1} + 1 = n_k.$$

On the other hand, $p_\sigma(m) = (n_k - n_k) + n_{k-1} + 1 = n_{k-1} + 1$ and

$$(b_\sigma^{-1} \circ p_\sigma)(m) = b_\sigma^{-1}(n_{k-1} + 1) = n_k = (p_\sigma \circ b_\sigma)(m).$$

Since m was arbitrary, it follows that $p_\sigma \circ b_\sigma = b_\sigma^{-1} \circ p_\sigma$ □

Lemma 5.5. *Let B be an infinite subset of ω and $\sigma = (n_k)_{k\in\omega}$ its increasing enumeration. Then $S(B) =^* p_\sigma[B]$.*

Proof. Let $k \in \omega$. Then

$$p_\sigma(n_k) = (n_k - n_k) + n_{k-1} + 1 = n_{k-1} + 1.$$

If $k > 0$, $n_{k-1} + 1 \in S(B)$. It follows that $p_\sigma[B] \subseteq^* S(B)$. On the other hand, if $m \in S(B)$, then $m = n_{k-1}+1$ for some $k > 1$ and $n_{k-1}+1 = (n_k - n_k)+n_{k-1}+1 = p_\sigma(n_k) \in p_\sigma[B]$ and thus $S(B) \subseteq p_\sigma[B]$. □

Lemma 5.6. *Let $\mathcal{A}$ be a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. Suppose $b \in \mathcal{P}(\omega)/\mathtt{fin}$ is such that for all $a \in \mathcal{A}$ either $b \leq a$ or $b \leq \neg a$. We say that b* diagonalizes *$\mathcal{A}$. Choose $B \in b$ and let $\sigma = (n_k)_{k\in\omega}$ be the increasing enumeration of B. Then on $\mathcal{A}$, the shift s is induced by b_σ.*

Proof. Let $a \in \mathcal{A}$. Choose $A \in a$. If $b \leq \neg a$, then for almost all $k \in \omega$, $n_k \notin A$. Hence for almost all $m \in A$, $b_\sigma(m) = m + 1$, and therefore $b_\sigma[A] =^* S[A]$.

Now suppose that $b \leq a$. In this case for almost all $k \in \omega$, $n_k \in A$. If $m \in A \setminus B$, then $b_\sigma(m) = m + 1$. For almost all $k \in \omega$ we have $n_k, n_{k-1} \in A$ and $b_\sigma(n_k) = n_{k-1} + 1$. It follows that $b_\sigma[A] =^* S[A]$.

This shows that on $\mathcal{A}$, the shift s is induced by b_σ. □

Lemma 5.7. *Let $b \in \mathcal{P}(\omega)/\mathtt{fin}$ and let*

$$\mathcal{D}(b) = \{a \in \mathcal{P}(\omega)/\mathtt{fin} : \forall z \in \mathbb{Z}(b \leq s^z(a) \vee b \leq \neg s^z(a))\}.$$

Then $\mathcal{D}(b)$ is a shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. In fact, $\mathcal{D}(b)$ is the maximal shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is diagonalized by b.

Proof. It is clear that $\mathcal{D}(b)$ is closed under s and s^{-1}.

Now let $a_0, a_1 \in \mathcal{D}(b)$ and $z \in \mathbb{Z}$. If $b \leq s^z(a_0)$, then $b \leq \neg s^z(\neg a_0)$. If additionally $b \leq s^z(a_1)$, then $b \leq s^z(a_0 \wedge a_1)$. If $b \not\leq s^z(a_0)$, then $b \leq \neg s^z(a_0)$ and thus $b \leq \neg s^z(a_0 \wedge a_1)$. Similarly, if $b \not\leq s^z(a_1)$, then $b \leq \neg s^z(a_0 \wedge a_1)$.

Since z was arbitrary, it follows that $\mathcal{D}(b)$ is closed under $\neg$ and $\wedge$ and hence a subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$. From the definition of $\mathcal{D}(b)$ it is clear that it is the maximal shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ that is diagonalized by b. □

Theorem 5.8. *For every countable subset T of $\mathcal{P}(\omega)/\mathtt{fin}$ there are an automorphism ψ of $\mathcal{P}(\omega)/\mathtt{fin}$ and a shift-closed subalgebra $\mathcal{A}$ of $\mathcal{P}(\omega)/\mathtt{fin}$ that contains T such that $\mathcal{A}$ is closed under ψ and ψ^{-1} and $(\psi \circ s \circ \psi^{-1}) \restriction \mathcal{A} = s^{-1} \restriction \mathcal{A}$.*

Under MA_κ this extends to all subsets T of $\mathcal{P}(\omega)/\mathtt{fin}$ of size at most κ.

Proof. Let $\mathcal{B}$ be a shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ of size at most κ such that $T \subseteq \mathcal{B}$. Let P be an ultrafilter of $\mathcal{B}$. It is well-known that MA_κ implies the existence of $b \in \mathcal{P}(\omega)/\mathtt{fin}$ such that $b > 0$ and $b \leq a$ for all $a \in P$. (See for instance [6, Ch. II, Theorem 2.15].) Let $\mathcal{A} = \mathcal{D}(b)$. By Lemma 5.7, $\mathcal{A}$ is a shift-closed subalgebra of $\mathcal{P}(\omega)/\mathtt{fin}$ such that $\mathcal{B} \subseteq \mathcal{A}$. In particular, $T \subseteq \mathcal{A}$. Moreover, $\mathcal{A}$ is diagonalized by b.

Choose $B \in b$ and let $\sigma = (n_k)_{k\in\omega}$ be the increasing enumeration of B. Let $n_{-1} = -1$. By Lemma 5.6, the shift is induced by b_σ on $\mathcal{A}$. Let ψ be the automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ induced by p_σ. By Lemma 5.4 b), $p_\sigma \circ b_\sigma \circ p_\sigma^{-1} = b_\sigma^{-1}$. Hence the theorem follows if we can prove

Claim 5.9. $\mathcal{A}$ is closed under ψ.

Let $a \in \mathcal{A}$ and $z \in \mathbb{Z}$. We have to show that $b \leq s^z(\psi(a))$ or $b \leq \neg s^z(\psi(a))$. By the definition of $\mathcal{D}(b)$, $b \leq s^{-z-1}(a)$ or $b \leq \neg s^{-z-1}(a)$.

First assume that $b \leq s^{-z-1}(a)$. Then $\psi(b) \leq (\psi \circ s^{-z-1})(a)$. By Lemma 5.4 b), $\psi \circ s^{-z-1} = s^{z+1} \circ \psi$. By Lemma 5.5, $\psi(b) = s(b)$. It follows that $s(b) \leq (s^{z+1} \circ \psi)(a)$ and thus, $b \leq s^z(\psi(a))$.

Similarly, if $b \leq \neg s^{-z-1}(a) = s^{-z-1}(\neg a)$, then $b \leq s^z(\psi(\neg a)) = \neg s^z(\psi(a))$. Since z was arbitrary, this shows $\psi(a) \in \mathcal{D}(b) = \mathcal{A}$. □

6. The shift as an automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$

Let us briefly consider a structure that is in some sense a non-commutative variant of $\mathcal{P}(\omega)/\mathtt{fin}$. Namely, let $\mathtt{FS}$ denote the normal subgroup of $\mathrm{Sym}(\omega)$ consisting of all permutations of ω that move only finitely many elements. It is well-known that $\mathtt{FS}$ is the largest proper normal subgroup of $\mathrm{Sym}(\omega)$, and hence the quotient $\mathrm{Sym}(\omega)/\mathtt{FS}$ is simple, i.e., has no non-trivial normal subgroup.

Let $A, B \subseteq \omega$ be cofinite and let $F : A \to B$ be a bijection. Then F induces an automorphism f of $\mathrm{Sym}(\omega)/\mathtt{FS}$ by conjugation in the following way:

For $\sigma \in \mathrm{Sym}(\omega)$ let $\overline{\sigma}$ denote the $\mathtt{FS}$-coset of σ. Now fix $\sigma \in \mathrm{Sym}(\omega)$. By passing to a different representative of the $\mathtt{FS}$-coset of σ if necessary, we may assume that σ does not move any element of $\omega \setminus B$. Let

$$F(\sigma) = \begin{cases} n, \text{ if } n \in \omega \setminus A, \text{ and} \\ (F \circ \sigma \circ F^{-1})(n), \text{ if } n \in A. \end{cases}$$

Now let $f(\overline{\sigma}) = \overline{F(\sigma)}$. This definition does not depend on the choice of σ within its $\mathtt{FS}$-coset. It is easily checked that f is an automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$.

Redefining notation from the previous sections, let $S : \omega \to \omega$ be the successor function that maps each $n \in \omega$ to $n + 1$. Regarding S as a bijection between two cofinite subsets of ω, S induces an automorphism s of $\mathrm{Sym}(\omega)/\mathtt{FS}$, which we call the *shift* on $\mathrm{Sym}(\omega)/\mathtt{FS}$.

$\mathrm{Sym}(\omega)$ acts in a natural way on $\mathcal{P}(\omega)/\mathtt{fin}$. Let $h : \mathrm{Sym}(\omega) \to \mathrm{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ be the homomorphism given by this action. The kernel of this homomorphism is

$\mathtt{FS}$. Hence $\mathrm{Sym}(\omega)/\mathtt{FS}$ acts on $\mathcal{P}(\omega)/\mathtt{fin}$. The image of h is the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$ of index 0. Conjugation of elements of $h[\mathrm{Sym}(\omega)]$ in $\mathrm{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ by the shift on $\mathcal{P}(\omega)/\mathtt{fin}$ corresponds to the shift s on $\mathrm{Sym}(\omega)/\mathtt{FS}$.

The subgroup of $\mathrm{Aut}(\mathcal{P}(\omega)/\mathtt{fin})$ generated by the trivial automorphisms of index 0 together with the shift on $\mathcal{P}(\omega)/\mathtt{fin}$ is precisely the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. Every trivial automorphism of $\mathcal{P}(\omega)/\mathtt{fin}$ gives rise to an automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$, as described above.

Alperin, Covington, and Macpherson [1] studied the group of automorphisms of $\mathrm{Sym}(\omega)/\mathtt{FS}$ and showed that it is generated by all the inner automorphisms of $\mathrm{Sym}(\omega)/\mathtt{FS}$ together with the shift s. In particular, $\mathrm{Aut}(\mathrm{Sym}(\omega)/\mathtt{FS})$ is isomorphic to the group of trivial automorphisms of $\mathcal{P}(\omega)/\mathtt{fin}$. It follows that the shift s on $\mathrm{Sym}(\omega)/\mathtt{FS}$ is not conjugated to s^{-1} in the automorphism group of $\mathrm{Sym}(\omega)/\mathtt{FS}$. This immediately gives the following theorem:

Theorem 6.1. *The structure* $(\mathrm{Sym}(\omega)/\mathtt{FS}, \mathtt{s})$ *is not isomorphic to* $(\mathrm{Sym}(\omega)/\mathtt{FS}, \mathtt{s}^{-1})$.

We derive the following example from the result of Alperin, Covington, and Macpherson:

Example 6.2. $\mathrm{Sym}(\omega)/\mathtt{FS}$ has two isomorphic countable subgroups such that no isomorphism between the two extends to all of $\mathrm{Sym}(\omega)/\mathtt{FS}$.

Proof. Let $\sigma \in \mathrm{Sym}(\omega)$ be such that σ has only finite orbits, but arbitrarily large finite orbits. Then for every $\tau \in \mathrm{Sym}(\omega)$ with $\overline{\sigma} = \overline{\tau}$, τ has no infinite orbits, either. For every $\varphi \in \mathrm{Sym}(\omega)$, $\varphi \circ \sigma \circ \varphi^{-1}$ has no infinite orbits. Also, $S(\sigma)$ has no infinite orbits. Replacing σ by another element of $\overline{\sigma}$ if necessary, we may assume that $\sigma(0) = 0$. Now there is $\tau \in \mathrm{Sym}(\omega)$ such that $S(\tau) = \sigma$. Just as σ, τ has no infinite orbits. Since the automorphism group of $\mathrm{Sym}(\omega)/\mathtt{FS}$ is generated by the inner automorphism and the shift, for every automorphism φ of $\mathrm{Sym}(\omega)/\mathtt{FS}$ and every $\tau \in \mathrm{Sym}(\omega)$ with $\varphi(\overline{\sigma}) = \overline{\tau}$, τ has no infinite orbits.

Finally, let $\tau \in \mathrm{Sym}(\omega)$ have an infinite orbit. Now both $\overline{\sigma}$ and $\overline{\tau}$ generate infinite cyclic subgroups of $\mathrm{Sym}(\omega)/\mathtt{FS}$. By the argument above, no automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$ maps $\overline{\sigma}$ to $\overline{\tau}$ or to $\overline{\tau}^{-1}$. Hence $\mathrm{Sym}(\omega)/\mathtt{FS}$ has two isomorphic countable subgroups such that no isomorphism between the two groups extends to an automorphism of $\mathrm{Sym}(\omega)/\mathtt{FS}$. □

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(Geschke) Hausdorff Center for Mathematics, Endenicher Allee 62, 53115 Bonn, Germany

E-mail address: geschke@math.boisestate.edu