CONVEX DECOMPOSITIONS IN THE PLANE, MEAGRE IDEALS AND CONTINUOUS PAIR COLORINGS OF THE IRRATIONALS

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Abstract. We address the structure of nonconvex closed subsets of the Euclidean plane. A closed subset $S \subseteq \mathbb{R}^2$ which is not presentable as a countable union of convex sets satisfies the following dichotomy:

1. There is a perfect nonempty $P \subseteq S$ so that $|C \cap P| < 3$ for every convex $C \subseteq S$. In this case covering S by convex subsets of S is equivalent to covering P by finite subsets, hence no nontrivial convex covers of S can exist.
2. There exists a continuous pair coloring $f : [\mathcal{N}]^2 \to \{0, 1\}$ of the space $\mathcal{N}$ of irrational numbers so that covering S by convex subsets is equivalent to covering $\mathcal{N}$ by f-monochromatic sets. In this case is it consistent that S has a convex cover of cardinality strictly smaller than the continuum $\mathfrak{c}$ in some forcing extension of the universe.

We also show that if $f : [\mathcal{N}]^2 \to \{0, 1\}$ is a continuous coloring of pairs, and no open subset of $\mathcal{N}$ is f-monochromatic, then the least number κ of f-monochromatic sets required to cover $\mathcal{N}$ satisfies that $\kappa^+ \geq \mathfrak{c}$. Consequently, a closed subset of $\mathbb{R}^2$ that cannot be covered by countably many convex subsets, cannot be covered by any number of convex subsets other than the continuum or the immediate predecessor of the continuum. The analogous fact is false for closed subset of $\mathbb{R}^3$.

1. Introduction

Since the 50s there has been continuous interest in classifying nonconvexity of closed subsets of Euclidean spaces. A coarse measurement of nonconvexity of a set S is its convexity number $\gamma(S)$: the least number of convex subsets required to cover S. Sets with infinite γ, with which we are concerned here, are divided to the countably convex sets — those that can be presented as a countable union of convex sets — and to the uncountably convex ones.

Let S be a closed subset of a Banach space. If S is a countably convex, it has an ordinal rank $\varrho(S) < \omega_1$ which measures the complexity of countable convex covers of S, and which bears geometric information about S. For example, a countable compact Hausdorff space K is completely determined by the rank of the unit sphere in the space $C(K)$ with the supremum norm [17]. The class of closed sets with $\gamma = \aleph_0$ is classified by ϱ into uncountably many types.

If, on the other hand, S is uncountably convex then after removing from S its maximal open countably convex subset, a perfect nonempty subset $K(S)$ remains

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with the property that for every convex $C \subseteq S$ the set $C \cap K(S)$ is nowhere dense in $K(S)$ [17]. Let $\mathcal{I}(S)$ be the σ-ideal which is generated on $K(S)$ by the intersections $C \cap K(S)$ for all convex $C \subseteq S$. $\mathcal{I}(S)$ is contained in the σ-ideal of meagre subsets of $K(S)$. It follows that to cover $K(S)$ or, equivalently, to cover S itself, by convex subsets of S is at least as difficult as covering a Polish space by its meagre subsets[1].

Let us quote two examples of such ideals. The extreme example is when the set S in question is contained in $\mathbb{R}^1$. In this case it is easy to observe that $|C \cap K(S)| \leq 2$ for any convex $C \subseteq S$. Put differently, for an uncountably convex 1-dimensional closed set, $\mathcal{I}(S)$ is just the trivial ideal of countable sets; thus, covering $K(S)$ by convex subsets of S is strictly more difficult than covering a Polish space by meagre sets: A Polish space can consistently be coverable by fewer than continuum meagre sets, but one can never cover the continuum by fewer than continuum countable sets.

In $\mathbb{R}^3$ the existence of a simply connected compact S has been known for which $\mathcal{I}(S)$ is isomorphic to the ideal over the space $3^{\mathbb{N}}$ which is generated by all subset which do not contain an equilateral triangle under the usual metrics on this space. It was shown that this ideal may have a cover of an arbitrary regular cardinality below the continuum [15, 17]. In other words, the continuum may be very big, but S may still be coverable by a small (uncountable) number of convex subsets.

The correspondence $S \mapsto \mathcal{I}(S)$ between closed sets and sub-ideals of the meagre ideal facilitates a mutual and fruitful connection between convex geometry and cardinal invariants of the continuum. On the one hand, one can classify uncountably convex sets S in terms of set-theoretic properties of $\mathcal{I}(S)$, using the vast literature on cardinal invariants of the continuum; on the other hand, new cardinal invariants of the continuum may be discovered by looking at $\mathcal{I}(S)$ for various closed S.

Our present investigation of uncountably convex closed subsets of the plane yields results in both directions. The geometric investigation reveals a new type of meagre ideal which, apart from the trivial ideal of countable sets, is the only type of ideal which can occur as $\mathcal{I}(S)$ for a closed planar S. The set theoretic properties of this type of ideals are quite unusual, and when translated back to geometry give a complete classification of uncountable convexity in closed planar sets. Let us describe these ideals briefly. Let $\mathcal{N}$ denote the space of the irrationals. Suppose that $f : [\mathcal{N}]^2 \to \{0, 1\}$ is a continuous pair coloring of $\mathcal{N}$, so that no open subset of $\mathcal{N}$ is monochromatic with respect to f. Each nontrivial ideal $\mathcal{I}(S)$ that can occur in the plane is isomorphic to the ideal which is generated by the monochromatic sets of such colorings.

Although the most natural context for studying covering the properties of those ideals is set theory of the reals, it in this geometric context that they appear and are being studied for the first time. We show that these ideals can admit non-trivial covers — but only barely: the successor of the covering number of such an ideal must always be greater than or equal to the continuum! To be more concrete, if the continuum were $\aleph_{100}$ then no uncountably convex closed planar set could be covered by fewer than $\aleph_{99}$ convex subsets; but at the same time, some uncountably convex closed subsets of $\mathbb{R}^3$ could be covered by just $\aleph_1$ convex subsets.

Covering numbers of continuous pair coloring are, thus, new cardinal invariants of the continuum at the high end of the spectrum of cardinal invariants, and can

[1]The number of meagre sets needed to cover a Polish space is, as is well known, identical in all Polish spaces.

assume, in any universe of set theory, at most one value below the continuum — the immediate predecessor of the continuum (which does not always exist).

Combining the geometric results with the set theoretic ones, we obtain a complete classification of infinite convex covers of closed planar sets. A closed planar set S contains a perfect P with $|P \cap C| < 3$ for all convex $C \subseteq S$ if and only if it may have a nontrivial convex cover in a forcing extension.

1.1. **Shelah's theorem.** No closed uncountably convex closed subsets of $\mathbb{R}^2$ were previously known which could have a nontrivial convex cover. In fact, for 10 years it was believed to be impossible to have such sets in the plane, because Theorem 2.2 in [15], due to Saharon Shelah, asserted that every uncountably convex closed subset of the Euclidean plane contained a perfect set $P \subseteq S$ with the property that $|P \cap C| \leq 2$ for all convex $C \subseteq S$.

The starting point of the present paper is the discovery, made by W. Kubiś when he was trying to extend Shelah's Theorem to G_δ sets, of a counter-examples. Several key ideas from Shelah's proof have been adopted in our proof of the Decomposition Theorem in Section 3.

1.2. **Organization of the paper.**

Section 2. An uncountably convex closed subset of $\mathbb{R}^2$ which does not contain uncountable cliques is constructed.

Section 3. The convexity number of the set from Section 2 is shown to be equal to the covering number of the Cantor space by c-monochromatic sets for a particular continuous pair coloring $c : [2^\omega]^2 \to \{0, 1\}$. Then it is shown that for any Polish space and any continuous pair coloring $c : [X]^2 \to 2$, if countably many c-homogeneous sets do not suffice to cover X, then also κ c-homogeneous sets do not suffice if $\kappa^+ < \mathfrak{c}$.

Section 4. The main decomposition theorem for closed planar sets is proved in this section: every closed planar set which does not contain a perfect 3-clique is the union of countably many convex sets and countably many graphs of continuously differentiable functions on each of which $\mathcal{I}(S)$ is isomorphic to the ideal generated by monochromatic sets for some continuous pair coloring. Several consequences are drawn from the decomposition theorem. At the end of the Section the decomposition theorem is extended to G_δ subsets of the plane.

Section 5. It is proved that in the Sacks model, in which $\mathfrak{c} = \aleph_2$, for every Polish space X and a continuous coloring $c : [X]^2 \to 2$, X is covered by $\leq \aleph_1$ c-homogeneous sets. As a corollary, in the Sacks model the convexity number of every closed planar set which does not contain a perfect 3-clique is $\aleph_1$. Then a model is constructed in which the least number of monochromatic sets required to cover $\mathcal{N}$ is continuum for every continuous pair coloring with no open monochromatic sets, and the continuum is strictly larger than all cardinals in Cichoń's diagram. In this model, every closed planar set is either countably convex or has $\gamma = \mathfrak{c}$, but some closed $S \subseteq \mathbb{R}^3$ satisfies $\gamma(S) = \aleph_1 < \mathfrak{c}$.

Section 7. Several open problems are listed.

The set in Section 2 was discovered by W. Kubiś, who also proved the decomposition theorem in Section 4. The independence results in Section 4 are due to

S. Geschke and R. Schipperus. The proof in Section 3 that $\mathfrak{hm}^+ \geq \mathfrak{c}$ is due to S. Geschke.

1.3. **Notation and Preliminaries.** A subset C of a linear space is *convex* if for any two points $p_1, p_2 \in C$ the line segment $[p_1, p_2]$ is contained in C. For a subset X, $\operatorname{conv}(X)$ denotes the *convex hull* of X, and $[x_1, x_2, x_3]$ is also used to denote the convex hull of $\{x_1, x_2, x_3\}$. A subset $X \subseteq S$ is called *defected in* S if $\operatorname{conv}(X) \nsubseteq S$. The *convexity number* $\gamma(S)$ of a set S is the least cardinality of a collection of convex sets whose union is S. S is *countably convex* if $\gamma(S) \leq \aleph_0$ and is *uncountably convex* otherwise. A subset $P \subseteq S$ is a *k-clique* in S, $k \geq 2$, if every k-element subset of P is defected in S. By $[S]^k$ the collection of all k-element subsets of S is denoted.

For two points $a = (x_1, y_1)$, $b = (x_2, y_2)$ in the plane, $\operatorname{dir}(a, b)$ is defined if $x_1 \neq x_2$ and is equal to $(y_2 - y_1)/(x_2 - x_1)$. By $B_1 < B_2$ for $B_1, B_2 \subseteq \mathbb{R}^2$ it is meant that every point in $\operatorname{proj}(B_1)$, projection of B_1 to the x-axis, is smaller than every point in $\operatorname{proj}(B_2)$. If $B_1 < B_2$ we write $\operatorname{dir}(B_1, B_2)$ for $\{\operatorname{dir}(a, b) : a \in B_1, b \in B_2\}$.

A *function* is a set or ordered pairs, and a *sequence* is a function defined on an initial segment (not necessarily proper) of ω, the set of natural numbers. Finite sequences are usually denoted by s, t, r, infinite sequences by σ, η, τ, ν. For two sequences s and t, $s \subseteq t$ means that s is an initial segment of t. A natural number n is identified with the set $\{0, 1, \ldots, n-1\}$.

The set of *unordered pairs* from A is denoted by $[A]^2$. For a natural number k, the *Ramsey number* $R(k)$ of k is the least n which satisfies $n \to (k)^2_2$, namely, for every pair coloring $c : [n]^2 \to 2$ there exists a k-element subset of n on which c is constant.

A subset $P \subseteq S$ for a subset S of a topological vector space is called a *semi-clique* if for all $p \in P$ and open neighborhood $u \ni p$, $P \cap u$ is defected in S. The union of all semi-cliques in S is a closed and maximal semi-clique, called the *convexity radical* of S and is denoted by $K(S)$. A closed subset S of a Polish linear space is countably convex if and only if $K(S) = \emptyset$.

A subset of a topological space is G_δ if it is an intersection of countably many open sets. If X is a Polish space, a subset $S \subseteq X$ is *completely metrizable*, that is, its induced topology is generated by some complete metric, if and only if S is G_δ in X.

By ω^ω or by $\mathcal{N}$ the **Baire space** of all infinite sequences of natural numbers is denoted, and 2^ω, the space of all infinite sequences over $\{0, 1\}$ is the **Cantor space**. For two different sequences η, ν of the same length denote by $\operatorname{dif}(\eta, \nu)$ the first number n for which $\eta(n) \neq \nu(n)$.

We denote by $<_{lx}$ the lexicographic order on either $2^{\leq\omega}$ or on $\omega^{\leq\omega}$. Concatenation of sequences is denoted by $^\frown$.

For an infinite cardinal κ the symbol κ^+ denotes the *successor cardinal* of κ, namely, if $\kappa = \aleph_\alpha$, then $\kappa^+ = \aleph_{\alpha+1}$. The symbol $\mathfrak{c}$ denotes the cardinality of the real line.

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2. A closed uncountably convex subset of $\mathbb{R}^2$ with no uncountable clique

This Section is devoted to:

Theorem 1. *There exists an uncountably convex closed set $S \subseteq \mathbb{R}^2$ which does not contain an uncountable clique.*

Proof. We begin by constructing a certain function $f \subseteq \mathbb{R}^2$, defined on a Cantor set. Let $\{I_s\}_{s\in 2^{<\omega}}$ be the standard tree of subintervals of $[-1,1]$ producing the Cantor set, namely $I_\emptyset = [-1,1]$ and for every $s \in 2^{<\omega}$, $I_{s^\frown 0}$ is the closed bottom third of I_s, $I_{s^\frown 1}$ the closed top third of I_s. For each $s \in 2^{<\omega}$ let J_s be the closed *middle* subinterval of I_s of length $1/9$ the length of I_s.

Let $\varphi\colon 2^{\leq\omega} \to 2^{\leq\omega}$ be defined by $\mathrm{dom}(\varphi(s)) = \mathrm{dom}(s)$, $\varphi(s)(k) = s(k)$ if k is odd and $\varphi(s)(k) = 1 - s(k)$ if k is even. Note that $\varphi(s) \subseteq \varphi(t)$ whenever $s \subseteq t$ and that φ is continuous.

Now define a tree $\{B_s\}_{s\in 2^{<\omega}}$ of open balls in $\mathbb{R}^2$ with the following properties:

(a) each B_s is a nonempty open ball of radius $\leq 2^{-\,\mathrm{length}(s)}$ and $\mathrm{cl}B_{s^\frown i} \subseteq B_s$ for all s and $i < 2$.
(b) $\mathrm{cl}B_{s^\frown 0} < \mathrm{cl}B_{s^\frown 1}$;
(c) $\mathrm{dir}(B_{s^\frown 0}, B_{s^\frown 1}) \subseteq J_{\varphi(s)}$.

Let $B_\emptyset$ be the open unit ball. Start by picking in B_0 two points $x_0 < x_1$ with $\mathrm{dir}(x_0, x_1) \in \mathrm{int}\, J_\emptyset$ and enlarging them to open sets B_0, B_1 preserving conditions (a),(b) and (c). The inductive step is similar: to split B_s find $x, y \in B_s$ with $x < y$ and $\mathrm{dir}(x,y) \in \mathrm{int}\, J_{\varphi(s)}$, next enlarge them to open balls $B_{s^\frown 0}$, $B_{s^\frown 1}$ preserving (a), (b) and (c).

Let $f = \bigcap_{n\in\omega} \bigcup_{s\in 2^n} \mathrm{cl}B_s$. For every $\sigma \in 2^\omega$ let $t_\sigma = \bigcap_n \mathrm{proj}(B_{\sigma\restriction n})$. By (a)+(b), f is a function with domain $C = \{t_\sigma \colon \sigma \in 2^\omega\}$, a Cantor set in $\mathbb{R}$. Put $x_\sigma = (t_\sigma, f(t_\sigma))$. Let d_σ be the unique point in $\bigcap_{n\in\omega} I_{\varphi(\sigma\restriction n)}$.

Fix $\sigma, \tau \in 2^\omega$ with $\sigma <_{lx} \tau$ and $n = \mathrm{dif}(\sigma,\tau)$. Let $s = \sigma \restriction n = \tau \restriction n$. By (c) we have $\mathrm{dir}(x_\sigma, x_\tau) \in J_{\varphi(s)}$. It follows that $\mathrm{dir}(x_\sigma, x_\tau)$ tends to d_σ when τ tends to σ. Hence $f'(t_\sigma) = d_\sigma$ and consequently f is continuously differentiable (since φ is continuous). Now suppose that n is even. Then $\varphi(s^\frown i) = \varphi(s)^\frown i$ and $x_\sigma \restriction (n+1) = s^\frown 0$, $x_\tau \restriction (n+1) = s^\frown 1$ so $f'(t_\sigma) < \mathrm{dir}(x_\sigma, x_\tau) < f'(t_\tau)$. Similarly, if n is odd then $f'(t_\sigma) > \mathrm{dir}(x_\sigma, x_\tau) > f'(t_\tau)$. We have thus proved:

Lemma 2. *$f : C \to \mathbb{R}$ is continuously differentiable and for all $\sigma <_{lx} \tau$ in 2^ω,*

1. $\mathrm{dif}(\sigma,\tau) \equiv 0 \mod 2 \Rightarrow f'(t_\sigma) < \mathrm{dir}(x_\sigma, x_\tau) < f'(t_\tau)$.
2. $\mathrm{dif}(\sigma,\tau) \equiv 1 \mod 2 \Rightarrow f'(t_\sigma) > \mathrm{dir}(x_\sigma, x_\tau) > f'(t_\tau)$.

Hence, no three points of f are collinear.

For convenience, let us write $f'(x_\sigma)$ instead of $f'(t_\sigma)$. We say that a pair x_σ, x_τ is *in configuration* $\sqcap$ (resp. $\sqcup$) if $x_\sigma < x_\tau$ and $f'(x_\sigma) > \mathrm{dir}(x_\sigma, x_\tau) > f'(x_\tau)$ (resp. $f'(x_\sigma) < \mathrm{dir}(x_\sigma, x_\tau) < f'(x_\tau)$).

If $x, y, z \in f$ are distinct and all pairs in $\{x, y, z\}$ are in the same configuration, call $[x, y, z]$ *decided*; Otherwise we say that $[x, y, z]$ is *undecided*. We write $x < y$ whenever $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $x_1 < y_1$.

Define

$$S = f \cup \bigcup \Big\{ [x,y,z] : \ x,y,z \in f \text{ and } [x,y,z] \text{ has decided configuration} \Big\} \tag{1}$$

Claim 3. *For each $x, y \in f$ we have $[x,y] \subseteq S$.*

Proof. Find $z \in f$ such that the triangle $[x,y,z]$ has decided configuration. Then $[x,y] \subseteq [x,y,z] \subseteq S$. $\square$

Claim 4. *If $x, y \in f$, $x < y$ and x, y are in configuration $\sqcup$ then all points of f between x and y lie below the segment $[x,y]$, and symmetrically for $\sqcap$.*

Proof. Suppose that $x < z < y$ and z lies above the segment $[x,y]$. Then $\operatorname{dir}(z,y) < \operatorname{dir}(x,y) < \operatorname{dir}(x,z)$ and $f'(x) < \operatorname{dir}(x,y) < f'(y)$. On the other hand, $\operatorname{dir}(x,z) < f'(z)$ because $f'(x) < \operatorname{dir}(x,z)$. Similarly $f'(z) < \operatorname{dir}(z,y)$; a contradiction. $\square$

Claim 5. *The set S is closed.*

Proof. It is enough to observe that the set of all pairs $(x,y) \in f \times f$ such that either $x = y$ or (x,y) is in configuration $\sqcup$, is closed in $f \times f$. Thus, if $v_n \in [a_n, b_n, c_n] \subseteq S$ and $v = \lim_{n\to\infty} v_n$ then, using the compactness of f, we may assume that the sequence (a_n, b_n, c_n) converges to (a,b,c) and each pair from a_n, b_n, c_n is in the same configuration. Then either $[a,b,c]$ is a decided triangle or $|\{a,b,c\}| < 3$. In both cases $v \in [a,b,c] \subseteq S$ (in the second case we use Claim 3). $\square$

Claim 6. *The set $S \setminus f$ is countably convex.*

Proof. Let D consist of all eventually constant sequences in 2^ω. Thus D is countable and dense in 2^ω. Let $X = \{x_\sigma \colon \sigma \in D\}$. We show that $S \setminus f = \bigcup\{[a,b,c] \setminus \{a,b,c\} \colon a,b,c \in X \ \& \ [a,b,c] \text{ is a decided triangle}\}$.

Fix $p \in S \setminus f$. If $p \in \operatorname{int}[a,b,c]$ for some decided triangle $[a,b,c]$ then in sufficiently small neighborhoods of a,b,c we can find vertices $a', b', c' \in X$ of a triangle with the same configuration as $[a,b,c]$ and such that $p \in [a',b',c']$. So assume that $p \notin \operatorname{int}[a,b,c]$ for any decided triangle $[a,b,c]$. Then $p \in [a,b]$ for some $a,b \in f$. We show that $a, b \in X$.

Suppose that $a \notin X$, $a < b$ and a, b are in configuration $\sqcup$. Let U be a ball around a such that $p \notin U$ and x, b are in configuration $\sqcup$ whenever $x \in U \cap f$. Now we can find $a_0, a_1 \in U$ such that $a_0 < a < a_1$ and a_0, a_1 are in configuration $\sqcup$. Then $[a_0, a_1, b]$ is a decided triangle. By Claim 4, $p \in \operatorname{int}[a_0, a_1, b]$, because $p \notin U$. This contradicts our assumption, thus $a \in X$. Similarly $b \in X$. $\square$

Claim 7. *If $x, y, z \in f$ are distinct and $[x,y,z]$ has undecided configuration then $[x,y,z] \not\subseteq S$.*

Proof. Suppose that $x < y < z$, and x, z are in configuration $\sqcup$. By Claim 4, y lies below the segment $[x,z]$. Assume that x, y are in configuration $\sqcup$ while y, z are in configuration $\sqcap$. Let y', z' be such that $y \le y' < z' \le z$ and that there are no points of f between y', z' (this is possible since the order of f is the same as the order of C). By Claim 4, $y', z' \in [x,y,z]$. Furthermore, we can choose y', z' in the same configuration as y, z. Thus, without loss of generality, we can assume that $y = y'$ and $z = z'$. Let $\Delta = \{p \in [x,y,z] \colon y < p < z\}$. We show that $\Delta \not\subseteq S$.

Fix a decided triangle $[a,b,c]$ such that $a \le y$, $b \ge z$ and c is not between a, b. Observe that, by Claim 4, both y, z lie on the same side of the segment $[a,b]$.

Suppose that a, b are in configuration $\sqcap$. Then y, z are above $[a, b]$ but the triangle $[a, b, c]$ lies below the line passing through a, b, because it has decided configuration (here we use Claim 4 again). Thus $[a, b, c] \cap \Delta = \emptyset$ whenever $[a, b, c]$ is a triangle with configuration $\sqcap$.

Now consider the set F of all pairs $(a, b) \in f \times f$ such that $a \leq y$, $b \geq z$ and a, b are in configuration $\sqcup$. Suppose that $\Delta \subseteq S$ and choose a sequence $\{v_n\}_{n\in\omega} \subseteq \operatorname{int} \Delta$ converging to $v = \frac{y+z}{2}$. Then for each $n \in \omega$ we can find $(a_n, b_n) \in F$ such that v_n is above or belongs to the segment $[a_n, b_n]$. By compactness, we may assume that the sequence (a_n, b_n) converges to (a, b). Observe that $(a, b) \in F$ which yields that a, b are in configuration $\sqcup$ and hence y, z lie below (or one of them belongs to) the segment $[a, b]$. Thus $v \in [a, b]$ and also $y, z \in [a, b]$. This is a contradiction to the fact that no three points of f are collinear. Thus $\Delta \not\subseteq S$. $\square$

From the last claim it follows that f is a semi-clique in S. Thus S is uncountably convex.

Assume now that P is a k-clique in S. By Claim 6, $P \setminus f$ is countable. To show that P itself is also countable, we show that $P \cap f$ is finite, actually $|P \cap f| < R(k)$, where $R(k)$ is the Ramsey number of k. Suppose otherwise. We have $[P \cap f]^2 = E \cup O$ where E consists of all pairs with configuration $\sqcup$ and $O = [f]^2 \setminus E$. There is $T \in [P \cap f]^k$ such that either $[T]^2 \subseteq E$ or $[T]^2 \subseteq O$, which means that all triangles with vertices in T have decided configurations. It follows that $\operatorname{conv} T \subseteq S$ contrary to the assumption that P is a k-clique. $\square$

The set constructed above can be modified to obtain an additional property: every clique in this set is a union of a discrete and a finite set.

Let us look at the proof of Claim 7. Let X be the set of all pairs in f which form a jump. In fact we have proved that if $(a, b) \in X$ form a jump then there is an open semi-circle $U_{a,b}$ centered at $\frac{a+b}{2}$ and determined by the segment $[a, b]$, which is disjoint from any decided triangle with vertices in f; moreover every undecided triangle contains $U_{a,b}$ for some $(a, b) \in X$. For each $(a, b) \in X$ choose a small open triangle $\Delta_{a,b} \subseteq U_{a,b}$. We can do this in such a way that the collection $\{\Delta_{a,b}\}_{(a,b)\in X}$ is discrete outside f. Now define

$$S' = \operatorname{conv} f \setminus \bigcup \{\Delta_{a,b} \colon (a, b) \in X\}.$$

Then S' has the same properties as S, namely f is a semi-clique in S' and S' is closed. Moreover, every point of $S' \setminus f$ has a neighborhood V which intersects at most one triangle $\Delta_{a,b}$. If V is a ball then $V \cap S'$ is the union of at most two convex sets. It follows that if P is a clique in S' then $P \setminus f$ is discrete. Thus the Cantor-Bendixson degree of any clique in S' is at most 1.

3. Pair colorings of Polish spaces

Let S be the set defined in (1) in the previous Section. The convexity radical of S is $f = \{x_\sigma : \sigma \in 2^\omega\}$ and $[x_{\sigma_1}, x_{\sigma_2}, x_{\sigma_3}] \subseteq S$ if and only if $\operatorname{dif}(\sigma_i, \sigma_j)$ is even for all $0 < i < j < 4$ or $\operatorname{dif}(\sigma_i, \sigma_j)$ is odd for all $0 < i < j < 4$.

Let us define a pair coloring $c : [2^\omega]^2 \to \{0, 1\}$ as follows:

$$c(\eta, \nu) = \begin{cases} 0 & \text{if } \operatorname{dif}(\eta, \nu) \text{ is even} \\ 1 & \text{if } \operatorname{dif}(\eta, \nu) \text{ is odd} \end{cases} \tag{2}$$

Call a subset $A \subseteq 2^\omega$ *homogeneous* if $c \restriction [A]^2$ is constant (and call A "homogeneous even" or "homogeneous odd" according to the constant color).

Since for every subset of $X \subseteq K(S)$ the convex hull of X is contained in S if and only if the set of corresponding points in 2^ω is homogeneous, $\gamma(S) = \mathfrak{hm}$, where:

$$\mathfrak{hm} = \min\{|\mathcal{F}| : \bigcup \mathcal{F} = 2^\omega,\ X \in \mathcal{F} \Rightarrow X \text{ is homogeneous}\} \tag{3}$$

Since every homogeneous set is clearly nowhere dense in 2^ω with the usual product topology and of measure zero in the usual product measure, $\mathfrak{hm} > \aleph_0$. In fact, $\mathfrak{hm}$ is greater than or equal to the covering numbers of the line by either meager or measure zero sets.

The first surprising property about $\mathfrak{hm}$ is that it cannot trail far behind $\mathfrak{c}$.

Lemma 8. $\mathfrak{hm}^+ \geq \mathfrak{c}$. *In particular,* $\mathfrak{hm} = \mathfrak{c}$ *whenever* $\mathfrak{c}$ *is a limit cardinal.*

Proof. For any two sequences $\eta, \nu \in 2^\omega$ let $\eta \otimes \nu \in 2^\omega$ be defined by $(\eta \otimes \nu)(n) = \eta(n/2)$ for even n and $(\eta \otimes \nu)(n) = \nu((n-1)/2)$ for odd n, namely $\eta \otimes \nu = \langle \eta(0), \nu(0), \eta(1), \nu(1), \ldots \rangle$. The operation $\otimes$ is neither commutative nor associative.

Suppose $X \subseteq 2^\omega$ is homogeneous even. Let $\eta \in 2^\omega$ be arbitrary. There is at most one sequence ν for which $\nu \otimes \eta \in X$, since if $\nu_1 \neq \nu_2$ then $\operatorname{dif}(\eta \otimes \nu_1, \eta \otimes \nu_2)$ is odd and therefore either $\eta \otimes \nu_1 \notin X$ or $\eta \otimes \nu_2 \notin X$. Given a homogeneous even X let $f_X : 2^\omega \to 2^\omega$ be the partial function that assigns for η the unique ν such that $\eta \otimes \nu \in X$ whenever such ν exists. Similarly, if Y is homogeneous odd then for every $\eta \in 2^\omega$ there is at most one $\nu \in 2^\omega$ for which $\nu \otimes \eta \in X$, and f_Y is defined analogously.

Suppose now that $\{X_\alpha : \alpha < \kappa\}$ is a collection of homogeneous even sets, $\{Y_\alpha : \alpha < \kappa\}$ a collection of homogeneous odd sets, and $\kappa^+ < \mathfrak{c}$. Fix some $A \subseteq 2^\omega$ with $|A| = \kappa^+$. Closing A under the functions f_{X_α} for all $\alpha < \kappa$ does not increase the cardinality of A, so let us suppose that A is already closed under those functions. Since $|A| = \kappa^+ < \mathfrak{c}$ we can pick $\eta \in 2^\omega \setminus A$. Choose $\nu \in A \setminus \{f_{Y_\alpha}(\eta) : \alpha < \kappa\}$. Now for every $\alpha < \kappa$ it holds that $\eta \neq f_{X_\alpha}(\nu)$ and $\nu \neq f_{Y_\alpha}(\eta)$ (one could of course use Hajnal's free-set theorem [12] to obtain this, but that would be an over-kill).

Therefore $\eta \otimes \nu \notin \bigcup \{X_\alpha : \alpha < \kappa\} \cup \{Y_\alpha : \alpha < \kappa\}$, which completes the proof. □

Corollary 9. $\gamma(S)^+ \geq \mathfrak{c}$.

We have thus bounded the convexity number of S from below by the predecessor of the continuum — if a predecessor exists, of course. If the continuum is a limit cardinal the lower bound is the continuum itself. At the end of the next Section this bound will be established for all closed uncountably convex planar sets.

To get the consistency that $\gamma(S) < \mathfrak{c}$ for all closed planar sets with no perfect 3-clique we need to bound $\gamma(S)$ from above. For that end, another cardinal invariant is introduced.

Let X be a topological space and let $c : [X]^2 \to 2$ be a pair-coloring of X. c is *continuous* if, when viewed as a symmetric two-place function on $c \restriction (X^2 \setminus \mathrm{id}_X)$, it is continuous with respect to the product topology on X^2.

Suppose $c : [X]^2 \to 2$ is given. A *c-homogeneous* subset $A \subseteq X$ is one for which $c \restriction [A]^2$ is constant. The *c-covering number* $\operatorname{cov}(c)$ is the least cardinality of a collection of c-homogeneous subsets needed to cover X.

$$\mathfrak{cb} = \max\{\operatorname{cov}(c) \mid\ c : [X]^2 \to 2 \text{ is continuous and } X \text{ is Polish}\} \tag{4}$$

Theorem 10. *Let X be a Polish space and let $c : [X]^2 \to 2$ be continuous. Suppose that no open subset of X is c-homogeneous. Then* $\operatorname{cov}(c) \geq \mathfrak{hm}$.

Proof. Let $c : [X]^2 \to 2$ be given as in the theorem, and fix some complete metric on X. We find a closed copy $T \subseteq X$ of 2^ω so that every c-homogeneous subset of T is either homogeneous odd or homogeneous even in T.

Suppose that $\{B_s : s \in 2^{\leq n}\}$ are chosen so that:

1. B_s is a closed ball of positive radius $\leq 1/\operatorname{length}(s)$
2. $s \subseteq t \Rightarrow B_t \subseteq \operatorname{int} B_s$
3. If s, t are incomparable, $x \in B_s$ and $y \in B_t$ then $\operatorname{dif}(s,t) \equiv c(x,y) \mod 2$

At step $n+1$ one chooses in each $\operatorname{int} B_s$, for $s \in 2^n$, two points $x_{s^\frown 0}$ and $x_{s^\frown 1}$ so that $c(x_{s^\frown 0}, x_{s^\frown 1}) \equiv n+1 \mod 2$. This is possible since $\operatorname{int} B_s$ is not homogeneous. By continuity of c, find open balls $x_{s^\frown i} \in B_{s^\frown i}$, $i \in \{0,1\}$, with $\overline{B}_{s^\frown i} \subseteq \operatorname{int} B_s$, each of radius $< 1/(n+1)$, so that for every choice of $x_i \in B_{s^\frown i}$ it holds that $c(x_0, x_1) \equiv n+1 \mod 2$. The proof is now clear. □

Corollary 11. $\mathfrak{cb} \geq \mathfrak{hm}$

Proof. The coloring c in (2) is continuous and 2^ω is Polish. □

Consider now any continuous pair coloring of a Polish space X. Since X has a countable basis for its topology, the set of all points for which some open neighborhood is a countable union of c-homogeneous sets is a union of countably many c-homogeneous sets. If the complement of this union is not empty, it is a perfect Polish subspace on which the coloring has no open homogeneous set. Hence:

Theorem 12. *For every continuous $c : [X] \to 2$ for a Polish space X, either X is a countable union of c-homogeneous sets or* $\operatorname{cov}(c)^+ \geq \mathfrak{c}$.

This theorem does not hold for *triple* colorings (see Section 5).

4. The structure of uncountably convex closed $S \subseteq \mathbb{R}^2$

In this Section a structure theorem is proved for closed planar sets: every $S \in \mathcal{K}$ is coverable by countably many convex subsets and countably many "special subsets". The crucial property of a special subset is that $\mathcal{I}(S)$ on it is isomorphic to the ideal generated by all monochromatic sets for some continuous pair coloring of the Baire space ω^ω.

Definition 13. *Let S be a subset of a Polish linear space. Call a nonempty subset K of S* special *in S if K is homeomorphic to ω^ω and there is a coloring $c\colon [K]^2 \to 2$ with the following properties:*

(i) *a subset of K is defected in S iff it is not c-homogeneous;*
(ii) *no open subset of K is c-homogeneous.*

By $(i) + (ii)$, every special set in S is a semi-clique in S (in particular, has no isolated points) and from Theorem 10 it follows that a special subset of S cannot be covered by less than $\mathfrak{hm}$ many convex subsets of S. A special set is homeomorphic to a complete metric space and hence is G_δ in S.

Suppose that K is a special subset of S and P is a k-clique in S. Then necessarily $|P \cap K| < R(k)$, since $|P \cap K| \geq R(k)$ implies that this intersection contains a c-homogeneous subset of size k. Thus we have:

Fact 14. *A special subset of S contains only finite cliques of S.*

The purpose of this Section is to prove the following theorem:

Theorem 15 (Decomposition Theorem for $\mathbb{R}^2$)**.** *Let S be a closed set in the plane. If S does not contain a perfect 3-clique then $S = A \cup B$, where $\gamma(A) \leq \aleph_0$ and B is a countable union of special sets.*

The proof comes at the end of the Section, after a chain of lemmas about the structure of the radical in closed planar S with no perfect 3-clique.

For the rest of this section we fix a closed uncountably convex planar set S which does not contain a perfect 3-clique. We use ideas from the proof of Shelah's theorem in [15].

Let B be a G_δ subset of a S and fix a complete metric on B. A sequence $\{U_s\}_{s\in T}$ is a *perfect tree of open sets* in B, if

1. T is a perfect subtree of $2^{<\omega}$ (or, more generally, of ω^ω);
2. $\mathrm{cl}U_s \subseteq U_t$ whenever $s \supsetneq t$;
3. $\mathrm{cl}U_s \cap \mathrm{cl}U_t = \emptyset$ whenever s, t are incompatible;
4. each U_s is a nonempty open set with diameter $\leq 2^{-\operatorname{length}(s)}$.

If we have such a tree then $\bigcap_{n\in\omega} \bigcup_{s\in T\cap 2^n} \mathrm{cl}U_s$ is a compact perfect subset of S.

A triangle $[a, b, c]$ with $a, b, c \in S$ is called *bad* if $[a, b, c] \not\subseteq S$. In this case we can find neighborhoods U_a, U_b, U_c of a, b, c respectively, so that $[a', b', c']$ is bad whenever $(a', b', c') \in (U_a \times U_b \times U_c) \cap (S \times S \times S)$.

We will consider G_δ semi-cliques in S. Every semi-clique is dense-in-itself and a G_δ semi-clique is completely metrizable. We start with the following remark: if B is a G_δ semi-clique in S then $U \cap B$ is not contained in a single line unless $U \cap B = \emptyset$. Indeed, otherwise $U \cap B$ is affinely isomorphic to an uncountably convex G_δ subset of $\mathbb{R}$ which, easily seen, contains a perfect 2-clique — which contradicts our assumption on S.

Lemma 16. *Let B be a G_δ semi-clique in S. Then there exists a nonempty open set $U \subseteq B$ such that $[a, b] \subseteq S$ for each $a, b \in U$.*

Proof. Suppose not. Then in any nonempty open subset of B we can find a defected pair of points. Thus we can construct a perfect tree of open sets $\{U_s\}_{s\in 2^{<\omega}}$ in B such that $\{a, b\}$ is defected in S whenever $a \in U_s$, $b \in U_t$ and s, t are incompatible. The perfect set obtained by this construction is a 2-clique, which is also a 3-clique. $\square$

Lemma 17. *Let B be a G_δ semi-clique in S. Then there is an open nonempty set $W \subseteq B$ which is affinely isomorphic to the graph of a Lipschitz function.*

Proof. We claim that B has the following property:

(P_1) For each nonempty open $W \subseteq B$ there are nonempty open sets $V_0, V_1, V_2 \subseteq W$ such that $[x_0, x_1, x_2]$ is not a bad triangle whenever $(x_0, x_1, x_2) \in V_0 \times V_1 \times V_2$.

Suppose not and fix a complete metric in B. As we have noticed, if $p \in \mathrm{int}[x_0, x_1, x_2] \setminus S$ then there are open sets W_0, W_1, W_2 with $x_i \in W_i$, such that $p \in \mathrm{int}[y_0, y_1, y_2]$ whenever $(y_0, y_1, y_2) \in W_0 \times W_1 \times W_2$. Thus we can construct a perfect tree of open nonempty subsets of B, $\{U_s\}_{s\in T}$, $T \subseteq 2^{<\omega}$, such that each U_s is contained in W, where $W \subseteq B$ is open nonempty which witnesses $\neg(P_1)$. Our tree should have the following property:

(i) If $s_0, s_1, s_2 \in T \cap 2^n$ are distinct then $[x_0, x_1, x_2]$ is bad whenever $(x_0, x_1, x_2) \in U_{s_1} \times U_{s_2} \times U_{s_3}$.

This property implies that $\bigcap_{n<\omega} \bigcup_{s \in T \cap 2^n} \mathrm{cl} U_s$ is a perfect 3-clique in B.

Thus B has property (P_1), which means that there are disjoint nonempty open sets $U_0, U_1, U_2 \subseteq B$ such that $[y_0, y_1, y_2]$ is a not a bad triangle whenever $(y_0, y_1, y_2) \in U_0 \times U_1 \times U_2$. Observe that in this case $\mathrm{int}[y_0, y_1, y_2] \cap B = \emptyset$, because B is a semi-clique. Shrinking U_0, U_1, U_2 if necessary, we may assume that no triple $(a, b, c) \in U_0 \times U_1 \times U_2$ is collinear; otherwise S would contain a perfect 2-clique.

Fix $a \in U_0$, $b \in U_1$. By using, if necessary, an affine transformation, we can assume that $a = (-1, 0)$, $b = (2, 0)$ and U_0 is contained in $[0, 1] \times [0, 1]$. Thus $U_0 \cap (\{t\} \times [0, 1])$ contains at most one point and if $p, q \in U_0$ then the line passing through p, q must intersect the interval (a, b). It follows that U_0 is a graph of a function. This function has the Lipschitz constant not greater than 1, because the direction between two points of U_0 is between the extremal directions of sides of triangles of the form $[a, b, x]$, where $x \in U_0$; these directions are in $[-1, 1]$. $\square$

Lemma 18. *Assume that $P \subseteq B(S)$ is a dense-in-itself G_δ set which is affinely isomorphic to the graph of a Lipschitz function f. Then f is differentiable on a dense set.*

Proof. Suppose otherwise. By using an affine transformation, we may assume that $P = b$ where $b \colon A \to \mathbb{R}$ is a nowhere differentiable function. We also assume that the Lipschitz constant of b is 1. Define

$$D_n = \{t \in A \colon \overline{b}'(t) - \underline{b}'(t) \geq \frac{1}{n}\}.$$

As A is homeomorphic to P, it is a Baire space. Let $n > 0$ and $W \subseteq A$ be such that D_n is dense in W and $W \neq \emptyset$ is open in A. Let $[-1, 1] = I_0 \cup \cdots \cup I_{k-1}$ where each I_i is open in $[-1, 1]$, $I_i \cap I_j \neq \emptyset$ iff $|i - j| \leq 1$, and if $|s - t| \geq 1/n$ then $s \in I_i$, $t \in I_j$ and $|i - j| > 1$. Thus if $t \in D_n$, then $\overline{b}'(t) \in I_i$, $\underline{b}'(t) \in I_j$ and $i - j > 1$. By shrinking W, we can fix $i, j < k$ with $i - j > 1$ and with the property that $\overline{b}'(t) \in I_i$, $\underline{b}'(t) \in I_j$ whenever $t \in D_n \cap W$.

We construct a perfect tree of open sets in $B \cap (W \times [0, 1])$. Our induction hypothesis is

($*$) If $x \in U_s$, $y \in U_t$, $s <_{\mathrm{lx}} t$ and $s, t \in T \cap 2^k$ then $\mathrm{dir}(x, y) \in I_j$,

($**$) If $s, t, r \in T \cap 2^k$ are distinct and $(x, y, z) \in U_s \times U_t \times U_r$ then $[x, y, z]$ is bad.

Suppose we want to split U_r, $r \in T \cap 2^k$. Find $x, y \in U_r$ with $\mathrm{dir}(x, y) \in I_j$ and $x < y$. This is possible if we take x or y from D_n. Then enlarge x, y to small open sets V_x, V_y, preserving this property. Let $z \in U_s$ where $s \in T \cap 2^k$, $s \neq r$. Assume $s <_{\mathrm{lx}} r$. Then the vertex x of the triangle $[z, x, y]$ has an obtuse angle. In V_x we can find a pair x', x'' with $\mathrm{dir}(x', x'') \in I_i$. Replacing, if necessary, x with x' or x'', we obtain a bad triangle. Thus we can shrink V_x, V_y and U_r to get only bad triangles spanned by all triples $(p, p', p'') \in V_x \times V_y \times V_r$. If we do this for each $s \in T \cap 2^k$ then we get open sets $U_{r^\frown 0}, U_{r^\frown 1}$ and $U_{s^\frown 0}$ for $s \in (T \cap 2^k) \setminus \{r\}$, preserving ($*$) and ($**$). By ($**$) the perfect set obtained from $\{U_s\}_{s \in T}$ is a 3-clique. $\square$

Lemma 19. *Let B be a G_δ semi-clique in S which is affinely isomorphic to a real function f. Then $f \cap V$ is not a convex function for any open $V \subseteq \mathbb{R}^2$ with $f \cap V \neq \emptyset$.*

Proof. We may assume that $B = f$ and that V is an open ball in $\mathbb{R}^2$. Suppose that $f \cap V$ is a convex function. Let $U = f \cap V$. Fix a bad triangle $[a, b, c]$, where

$a, b, c \in U$ and $a < b < c$. Let $p \in \operatorname{int}[a, b, c] \setminus S$. If for some $q < a$ the triangle $[q, a, b]$ contains p then $[q', a, b]$ is bad whenever $q' < q$. Define

$$L(a, b, c) = \{q \in U : \forall\, q' \leq q\ [q', a, b] \text{ is a bad triangle }\}.$$

Then $L(a, b, c)$ is an order interval in $(U, <)$. We claim that there exists a perfect tree of open sets $\{U_s\}_{s \in T}$ in U with the following property:

($*$) If $s_0, s_1, s_2 \in T \cap 2^n$, $s_0 <_{\mathrm{lx}} s_1 <_{\mathrm{lx}} s_2$ and $\operatorname{dif}(s_0, s_1) < \operatorname{dif}(s_1, s_2)$ then each triangle $[x_0, x_1, x_2]$ with $(x_0, x_1, x_2) \in U_{s_0} \times U_{s_1} \times U_{s_2}$ is bad.

Consider now the following condition:

(L) There is a nonempty open $W \subseteq U$ such that $L(a, b, c) \cap W = \emptyset$ whenever $[a, b, c]$ is a bad triangle with $a, b, c \in W$.

Suppose first that (L) does not hold. Then the induction step can be done as follows. Assume that we want to split U_r. Our induction hypothesis is that $U_s < U_t$ (i.e. $x < y$ whenever $x \in U_s$, $y \in U_t$) whenever $s <_{\mathrm{lx}} t$. According to $\neg(L)$, we can choose a bad triangle $[a, b, c]$ where $a, b, c \in U_r$, $a < b < c$ and $L(a, b, c) \cap U_r \neq \emptyset$. Then for each $s <_{\mathrm{lx}} r$ with $\operatorname{length}(s) = \operatorname{length}(r)$, each triangle $[q, a, b]$, where $q \in U_s$, is bad. Now we can enlarge q, a, b to small open sets $U_{s^\frown 0}, W_s^a, W_s^b$, preserving this property. Finally, we can set $U_{r^\frown 0} = \bigcap_{s <_{\mathrm{lx}} r} W_s^a$, $U_{r^\frown 1} = \bigcap_{s <_{\mathrm{lx}} r} W_s^b$, where all s's are taken from $2^{\operatorname{length}(r)}$. Clearly, ($*$) holds.

Now if (L) holds and W witnesses (L) then we can do the same construction inside W but we should take b, c instead of a, b. Let P be the perfect set obtained from $\{U_s\}_{s \in T}$. Then in each nonempty open subset of P we can find a bad triangle. Define analogously a set $R(a, b, c)$ for each bad triangle $[a, b, c]$ with $a, b, c \in P$. Similarly as above, we can construct a perfect subtree $T' \subseteq T$ such that the set P' obtained from $\{U_s\}_{s \in T'}$ is a 3-clique. This contradiction completes the proof. $\square$

Now we assume that $f : A \to \mathbb{R}$ is a Lipschitz function contained in S; assume also that f is a G_δ semi-clique in S, so A is a G_δ dense-in-itself subset of $\mathbb{R}$. Let D denote the set of differentiability points of f, which by Lemma 18, is dense in A. Recall that we write $f'(a)$ instead of $f'(a_0)$ whenever $a = (a_0, f(a_0))$.

Let $(x, y) \in D \times D$. We say that (x, y) is in *configuration* $\sim$ if $x \neq y$ and either $f'(x), f'(y) < \operatorname{dir}(x, y)$ or $f'(x), f'(y) > \operatorname{dir}(x, y)$.

Lemma 20. *There is an open nonempty set $W \subseteq f$ such that there are no pairs with configuration $\sim$ in W.*

Proof. Suppose not. Then we can construct a perfect tree of open sets $\{U_s\}_{s \in T}$ in f, such that for each distinct $s, t, r \in T \cap 2^n$ each pair $(x, y) \in U_s \times U_t$ has configuration $\sim$ and each triple $(x, y, z) \in U_s \times U_t \times U_r$ form a bad triangle. The inductive step is easy: if we want to split U_r then we find a pair $x, y \in U_r$ with configuration $\sim$; next we observe that if x, y, z are distinct and such that each two of them are in configuration $\sim$ then, as close as we wish, we can find x', y', z' which form a bad triangle. The perfect set obtained from our tree is a 3-clique. $\square$

Suppose now that f does not contain pairs with configuration $\sim$. Let $K(f)$ denote the function $f \restriction (A \setminus J)$, where J is the set of all points in A for which from at least one side are not limits of A. As A is dense-in-itself, $A \setminus J$ is a dense-in-itself G_δ set so that between any two elements from the set there is a third element from the set. Moreover, J is countable. Fix $\{a, b\} \in [K(f)]^2$. We say that a, b are in *configuration* $\sqcup$ ($\sqcap$) if there are no points of f which are between a, b and which lie above (below) the segment $[a, b]$.

Lemma 21. *Suppose f does not contain pairs with configuration $\sim$. Then:*

(a) *Each pair in $K(f)$ is either in configuration $\sqcup$ or $\sqcap$.*
(b) *If $a, b \in K(f) \cap D$ and $a < b$ then a, b are in configuration $\sqcup$ iff $f'(a) \leq \operatorname{dir}(a, b) \leq f'(b)$.*
(c) *For each $\{a, b\} \in [K(f)]^2$ there are neighborhoods $U_a, U_b \subseteq K(f)$ of a and b respectively such that a', b' are in the same configuration for each $(a', b') \in U_a \times U_b$.*

Proof. Fix $a, b \in K(f)$ with $a < b$. Let $f_0 = \{p \in f \colon a < p < b\}$. Suppose that there are $c, d \in f_0$ such that c is above the segment $[a, b]$ while d is below the segment $[a, b]$. Find $a', b', c', d' \in D$ which are close to a, b, c, d respectively and which have the same properties. Suppose that $f'(a') \leq \operatorname{dir}(a', b') \leq f'(b')$. As c' is above $[a', b']$, we have $\operatorname{dir}(a', c') > \operatorname{dir}(a', b') > \operatorname{dir}(c', b')$. On the other hand, $\operatorname{dir}(a', c') \leq f'(c')$ because otherwise a', c' would be in configuration $\sim$. Similarly, $f'(c') \leq \operatorname{dir}(c', b')$. Thus $\operatorname{dir}(a', c') \leq \operatorname{dir}(c', b')$, a contradiction.

It follows that either no point of f_0 is above $[a, b]$ or no point of f_0 is below $[a, b]$. This shows that a configuration of a, b can be defined. Observe that a, b cannot be in both configurations, because then $f_0 \subseteq [a, b]$ and consequently f_0 contains a perfect 2-clique (f_0 is a semi-clique so $S \cap [a, b]$ is uncountably convex in case $f_0 \subseteq [a, b]$). Thus configurations are defined uniquely. This shows (a). The arguments above also show (b).

For the proof of (c), let us assume that (a_n, b_n) converges to (a, b), a_n, b_n are in configuration $\sqcup$ for every $n \in \omega$ but a, b are in configuration $\sqcap$. Then there is $c \in f$ such that c is between a, b and above the segment $[a, b]$. Now, if a_n, b_n are close enough to a, b then c is also between a_n, b_n and lies above the segment $[a_n, b_n]$, a contradiction. $\square$

A triangle $[a, b, c]$ with vertices in $K(f)$ will be called *decided* if all pairs in $\{a, b, c\}$ are in the same configuration; otherwise it will be called *undecided*.

Lemma 22. *Suppose that f does not contain pairs with configuration $\sim$. Then:*

(a) *In each nonempty open subset of f there are pairs both in configuration $\sqcup$ and in $\sqcap$.*
(b) *Every undecided triangle is defected in S.*
(c) *There is a nonempty open set $U \subseteq f$ such that no decided triangle in U is defected.*

Proof. The first statement is clear: otherwise, $f \cap U$ is a convex or concave function for some nonempty open U and, by Lemma 19, f contains a perfect 3-clique.

Let $[a, b, c]$ be an undecided triangle. Assume that $a < c < b$ and a, b are in configuration $\sqcup$. Suppose that c, b are in configuration $\sqcap$. Then there is $p \in f$ which is between c, b and above the segment $[c, b]$. Then p is not above the segment $[a, b]$. Now, if $p \in [a, b]$ then we can find $p' \in f$ close to p, which is below $[a, b]$ (otherwise some neighborhood of p would be contained in a line). Thus $p' \in \operatorname{int}[a, b, c]$ which means that $[a, b, c]$ is defected in S, because no neighborhood of p' in S is countably convex. This shows (b).

For the proof of (c), suppose that in every nonempty open subset of f we can find a defected decided triangle. We may assume that we can densely find defected triangles with configuration $\sqcup$. Then we can construct a perfect tree of open sets $\{U_s\}_{s\in 3^{<\omega}}$ in $K(f)$ so that for each distinct $r,s,t \in 3^n$ each triple $(a,b,c) \in U_r \times U_s \times U_t$ forms a bad triangle in S and each pair $(a,b) \in U_r \times U_s$ is in configuration $\sqcup$. The inductive step is possible: to split U_s we can find a bad triangle $[a,b,c]$ with $a,b,c \in U_s$ and such that all pairs from a,b,c are in configuration $\sqcup$. Finally, we can enlarge these points to some open sets preserving this property. The perfect set P obtained by this construction, is a G_δ semi-clique in S and it is a graph of a convex function. This contradicts Lemma 19. $\square$

Lemma 23. *Let B be a G_δ semi-clique in S. Then there is a nonempty open set $U \subseteq B$ such that $U = K \cup L$, where K is a special subset of S and L is countable.*

Proof. By Lemma 17, there is open nonempty $W \subseteq B$ such that W is affinely isomorphic to a Lipschitz function f. By Lemma 16, we may assume that $[a,b] \subseteq S$ whenever $a,b \in W$. By Lemma 18, we may also assume that f is differentiable on a dense set. Now, using Lemma 20, we can shrink W in such a way that f does not contain pairs in configuration $\sim$. Let $U \subseteq W$ be open nonempty and such that decided triangles in U are not defected in S; such a set exists by Lemma 22. Now let $K = K(f) \cap U$, $L = U \setminus K$. Then L is countable (by the definition of $K(f)$). By Lemma 21, the coloring $c\colon [K]^2 \to 2$ defined by $c(\{a,b\}) = 0$ iff a,b are in configuration $\sqcup$, is continuous, no open subset of K is c-homogeneous, and c-homogeneous subsets of K are precisely nondefected subsets of S (by Lemma 22). Since K is homeomorphic to a subset of $\mathbb{R}$, by removal of a countable set we may assume that K is homeomorphic to ω^ω. This shows that K is a special subset of S. $\square$

Now we can prove the main theorem of the Section:

Proof of Theorem 15. Call a point $p \in S$ *special* in S if p has a neighborhood V such that $V \cap S = A \cup B$, where $\gamma(A) \le \aleph_0$ and B is a countable union of special subsets of S. Denote by X the subset of all special points in S. If $X = S$, then the proof is complete, because S can be covered with countably many open sets V with the above property. Suppose that $X \neq S$. Then $B = S \setminus X$ is closed in S and it is a semi-clique in S. Applying Lemma 23, we can find open $U \subseteq B$ such that $U = K \cup L$, where L is countable and K is a special subset of S. If $U = V \cap B$, where V is open in $\mathbb{R}^2$ then $V \cap S$ can be represented as the union of countably many convex sets and countably many special sets. Thus every point of U is special in S, a contradiction. $\square$

Corollary 24. *A closed subset of $\mathbb{R}^2$ contains an uncountable k-clique for some $k \ge 3$ iff it contains a perfect 3-clique.*

Proof. Suppose $S \subseteq \mathbb{R}^2$ is closed and does not contain a perfect 3-clique. By Theorem 15, $S = \bigcup_n C_n \cup \bigcup K_n$ where each C_n is convex and each K_n is special in S. Every clique in S has finite intersection with every C_n and also with every K_n by Fact 14. $\square$

4.1. **Uncountably convex plane G_δ sets.** All lemmas in this Section, except Lemma 19 and Lemma 22(c), apply also to G_δ subsets of the plane, because in these lemmas we use the fact that a defected triangle in S is in fact defected in its interior. Fix some G_δ set $S \subseteq \mathbb{R}^2$. Call a triangle $[a,b,c]$ in S *bad* if $\operatorname{int}[a,b,c] \not\subseteq S$. Call a semi-clique $P \subseteq S$ *strong* if any nonempty open subset of P contains a bad triangle. Then Lemma 19 holds if we replace the word "semi-clique" with "strong semi-clique". In the proof of Lemma 22(c) we construct a semi-clique which is a graph of a convex function. Now we should modify this to obtain a strong semi-clique. We do this below.

Also, the notion of a special set should be weakened: A subset K of S is *almost special* in S if K is G_δ and there exists a continuous coloring $c\colon [K]^2 \to 2$ such that no open subset of K is c-homogeneous, non-c-homogeneous subsets are defected and also no c-homogeneous triple is defected in S. The difference between special and almost special sets is that we do not require anything about defectedness of pairs: namely if a pair of points in K can be extended to a homogeneous triple then this pair is not defected, otherwise it may be defected or not.

Lemma 25. *Let f be a G_δ semi-clique in S which is a graph of a continuous convex function. Then either f is a strong semi-clique or some nonempty open subset of f can be covered by countably many convex subsets of S.*

Proof. Suppose that f is convex and $\operatorname{int}[a,b,c] \subseteq S$ whenever $a,b,c \in W$, where W is nonempty and open in f. We may assume that $W = f$. We show that f can be covered by countably many convex subsets of S. Fix $a,b \in f$ with $a < b$. Suppose that there is $c \in f$ with $b < c$ and that b is an accumulation point of $\{x \in f\colon x < b\}$. Observe that

$$[a,b] \setminus \{a,b\} \subseteq \bigcup \{\operatorname{int}[a,p,c] \subseteq S : p \in f \wedge p < b\},$$

and thus $[a,b] \subseteq S$. It follows that, if we remove the first and the last element of f (if they exist) and if we remove all pairs $\{a,b\}$ such that both a,b are not accumulation points of $\{x \in f\colon a < x < b\}$, then we get a subset $P \subseteq f$ with $\operatorname{conv}P \subseteq S$ (by Carathéodory's theorem). It follows that

$$f \subseteq \operatorname{conv}P \cup (\bigcup_{p \in f\setminus P} \{p\}),$$

and all those sets are convex subsets of S. □

Here is an analogue of Lemma 23 for G_δ sets.

Lemma 26. *Let B be a nonempty G_δ-semi-clique in S. Then there exists a nonempty open set $U \subseteq B$ such that either U can be covered by countably many convex subsets of S or $U = K \cup L$, where K is an almost special subset of S and L is countable.*

Proof. Suppose that the first possibility does not hold. Then, using the previous lemma we can apply Lemmas 17,18,19,20 and 21 to obtain a nonempty open subset of B which is affinely isomorphic to a continuous Lipschitz function f which is differentiable on a dense set, which does not contain pairs with configuration $\sim$ and which is nowhere a convex function nor a concave one. As before, denote by $K(f)$ the subset of K obtained by removing all points which are isolated in f from at least one side.

We want to know that, in some neighborhood, no decided triangle with vertices in $K(f)$ is defected in S. Then we can prove a version of Lemma 22(c). We have the following:

Claim 27. *If $[a,b,c]$ is a defected decided triangle with vertices in $K(f)$ then, as close as we wish, we can find a bad decided triangle with the same configuration as $[a,b,c]$.*

Proof. Suppose that $p \in [a,b,c] \setminus S$. Find small neighborhoods of a,b,c preserving the configurations. Now, as a,b,c are not isolated from left or right, we can move one of them in such a way that p is in the interior of the triangle. Specifically, if e.g. $a < b < c$ and $p \in [a,c]$ then we can find $c' > c$ close enough to c so that $[a,b,c']$ has the same configuration and c' lies strictly above the line passing through a,c. Then $p \in \operatorname{int}[a,b,c']$. $\square$

Now we can repeat all the arguments from the proof of Lemma 22(c): if in any neighborhood in $K(f)$ we can find a decided non-defected triangle, then using the above claim, we can construct a strong G_δ semi-clique in $K(f)$ which is the graph of a convex (or concave) function; then we can apply Lemma 19, to get a contradiction. Thus, in some nonempty open set $U \subseteq f$, the set $K = U \cap K(f)$ is almost special, of course $L = U \setminus K$ is countable. $\square$

Now, using almost the same arguments as in the proof of Theorem 15, we can prove the following.

Theorem 28. *Let S be an uncountably convex G_δ subset of the plane. If S does not contain a perfect 3-clique then $S = A \cup B$, where $\gamma(A) \leq \aleph_0$ and B is a countable union of almost special subsets of S.*

Proof. Call a point $p \in S$ *almost special* if it has a neighborhood V such that $S \cap V$ can be represented in the desired form. Let B be the set of all points in S which are not almost special. Then B is a G_δ semi-clique and no nonempty open subset of B can be covered by countably many convex subsets of S. Now, if B is nonempty then use Lemma 26 to get a contradiction. $\square$

5. Consistency Results

In this Section and the next we prove

1. Consistency of $\mathfrak{cb} < \mathfrak{c}$ and
2. There is an uncountably convex closed set $S \subseteq \mathbb{R}^3$ such that consistently $\gamma(S) < \mathfrak{hm}$.

We will use countable support iteration and countable support products of Sacks forcing. We use the representation of Sacks forcing $\mathbb{S}$ as the set of perfect subtrees of $2^{<\omega}$ ordered by inclusion.

The Sacks model, i.e., the model obtained by adding $\aleph_2$ Sacks reals to a model of CH, is the canonical model model for small cardinal invariants. Assuming the existence of certain large cardinals, Zapletal proved the following [25]: If I is a projective ideal (on the reals) and $\operatorname{cov}(I) < \mathfrak{c}$ can be forced, then $\operatorname{cov}(I) < \mathfrak{c}$ holds after adding $\mathfrak{c}^+$ Sacks reals.

There is another reason to look at the Sacks model to show the consistency of $\mathfrak{cb} < \mathfrak{c}$. Recall the proof of Lemma 8. There we assigned a function f_X to every homogeneous subset X of 2^ω such that whenever $\mathcal{H}$ is a family of homogeneous sets covering 2^ω and two sets $A, B \subseteq 2^\omega$ are closed under the functions f_X, $X \in \mathcal{H}$, then $A \subseteq B$ or $B \subseteq A$. It follows that if the universe V is a forcing extension of L and 2^ω is covered by homogeneous sets coded in L, then the constructible degrees in V are linearly ordered.

It was shown by M. Groszek in [11] that in the Sacks model over L the constructible degrees are wellordered of ordertype ω_2. In Section 6 we will construct a model of set theory in which $\mathfrak{hm} = \mathfrak{c}$ because for every family $\mathcal{H}$ of homogeneous subsets of 2^ω of size $< \mathfrak{c}$ there are incomparable sets $A, B \subseteq 2^\omega$ which are closed under the functions f_X, $X \in \mathcal{H}$.

We first show that in the Sacks model, for all continuous colorings c of ω^ω only $\aleph_1$ c-homogeneous sets are needed to cover ω^ω. In Subsection 4 we show how to generalize this to colorings of arbitrary Polish spaces.

Let us start with some preparation. Let $c : [\omega^\omega]^2 \to 2$ be continuous. Then c is coded by a real $\overline{c}$. We present one way to define $\overline{c}$. For $s \in \omega^{<\omega}$ let U_s denote the set of all $x \in \omega^\omega$ with $s \subseteq x$. For $s, t \in \omega^{<\omega}$ we write $s \perp t$ if $U_s \cap U_t = \emptyset$.

Definition 29. *Let $\overline{c} : (\omega^{<\omega})^2 \to \{0, 1, \text{undecided}\}$ be such that for $s, t \in \omega^{<\omega}$, $\overline{c}(s, t) = i$ if $s \perp t$ and for all $x \in U_s$ and all $y \in U_t$, $c(\{x, y\}) = i$, and $\overline{c}(s, t) =$ undecided otherwise.*

By the continuity of c, if $x, y \in \omega^\omega$ are different, then there are $s, t \in \omega^{<\omega}$ such that $s \subseteq x$, $t \subseteq y$, and $\overline{c}(s, t) = c(x, y)$. This shows that c is determined by $\overline{c}$. Call a tree $T \subseteq \omega^{<\omega}$ c-homogeneous of color i if the set $[T]$ of all infinite branches of T is c-homogeneous of color i. Then a tree T without leaves is c-homogeneous of color i if $\overline{c}[T^2] \subseteq \{i, \text{undecided}\}$.

If we want to show that after adding $\aleph_2$ Sacks reals to a model of CH, for every continuous coloring $c : [\omega^\omega] \to 2$ we have $\mathrm{cov}(c) \leq \aleph_1$, it is sufficient to consider colorings in the ground model since every continuous coloring is coded by a real (which is added at some initial stage of the iteration, where CH is still satisfied). We show that for every coloring c in the ground model, every new real is contained in a c-homogeneous set coded in the ground model.

5.1. **Some forcing notation.** As usual with Sacks forcing, we make heavy use of fusion arguments. (See [1] for a general treatment of fusion in forcing iterations and [2] for more on Sacks forcing.) Some of the notation in this section is inspired by or directly taken from [21].

For $n \in \omega$ and $p \in \mathbb{S}$ let p^n consist of those $t \in p$ such that t has exactly 2 immediate successors in p and t has exactly n proper initial segments with the same property. For $p, q \in \mathbb{S}$ we write $p \leq_n q$ if $p \leq q$ and $p^n = q^n$.

A sequence $(p_n)_{n\in\omega}$ in $\mathbb{S}$ is a *fusion sequence* if there is a nondecreasing unbounded function $f : \omega \to \omega$ such that for all $n \in \omega$, $p_{n+1} \leq_{f(n)} p_n$. If $(p_n)_{n\in\omega}$ is a fusion sequence, then $p_\omega = \bigcap_{n\in\omega} p_n$ is a condition in $\mathbb{S}$, the *fusion* of the sequence. In this definition, the function f is only added for technical convenience. If we only talk about the identity function instead of arbitrary f, we arrive at an essentially equivalent notion.

The idea behind fusion is that in $\mathbb{S}$, even though it is not countably closed, lower bounds exist for suitably chosen countable sequences. All we have to do while inductively thinning out a condition, is to leave more and more splittings of the tree (the condition) untouched. This method can be extended to countable support iterations.

Let α be an ordinal and let $\mathbb{S}_\alpha$ be the countable support iteration of Sacks forcing of length α. For $F \in [\alpha]^{<\aleph_0}$, $\eta : F \to \omega$, and $p, q \in \mathbb{S}_\alpha$ let $p \leq_{F,\eta} q$ if $p \leq q$ and for all $\beta \in F$, $p \restriction \beta \Vdash p(\beta) \leq_{\eta(\beta)} q(\beta)$. Roughly speaking, $p \leq_{F,\eta} q$ means that on each coordinate from F, p is $\leq_n$-below q where n is given by η.

A sequence $(p_n)_{n\in\omega}$ of conditions in $\mathbb{S}_\alpha$ is a *fusion sequence* if there is an increasing sequence $(F_n)_{n\in\omega}$ of finite subsets of α and a sequence $(\eta_n)_{n\in\omega}$ such that for all $n \in \omega$, $\eta_n : F_n \to \omega$, $p_{n+1} \leq_{F_n,\eta_n} p_n$, for all $\gamma \in F_n$ we have $\eta_n(\gamma) \leq \eta_{n+1}(\gamma)$, and for all $\gamma \in \operatorname{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \geq n$.

This notion is precisely what is needed in countable support iterations to get suitable fusions. It essentially means that once we have touched (i.e., decreased) a coordinate of p_0, we have to build a fusion sequence in that coordinate.

If $(p_n)_{n\in\omega}$ is a fusion sequence in $\mathbb{S}_\alpha$, its fusion p_ω is defined inductively. Let $F_\omega := \bigcup F_n$.

Suppose $p_\omega(\gamma)$ has been defined for all $\gamma < \beta$ for some $\beta < \alpha$. If $\beta \notin F_\omega$, let $p_\omega(\beta)$ be a name for $1_{\mathbb{S}}$. If $\beta \in F_\omega$, then $p_\omega \restriction \beta$ forces $p_n(\beta)$ to be a fusion sequence in $\mathbb{S}$. Let $p_\omega(\beta)$ be a name for the fusion of the $p_n(\beta)$'s.

5.2. **A preliminary lemma.** Our strategy is the following: Let $c : [\omega^\omega]^2 \to 2$ be continuous and $\dot{x}$ an $\mathbb{S}_{\omega_2}$-name for a new element of ω^ω. We may assume that there is $\alpha < \omega_2$ such that $\dot{x}$ is an $\mathbb{S}_\alpha$-name for a real not added at any proper initial stage of the iteration $\mathbb{S}_\alpha$. Let q be a condition in $\mathbb{S}_\alpha$. We define a tree $T_q \subseteq \omega^{<\omega}$, the *tree of q-possibilities for $\dot{x}$*, by

$$T_q := \{s \in \omega^{<\omega} : \exists q' \leq q(q' \Vdash s \subseteq \dot{x})\}.$$

It should be clear that q forces $\dot{x}$ to be a branch through T_q.

For each $p \in \mathbb{S}_\alpha$ we will construct a condition $q \leq p$ such that T_q is c-homogeneous. The next lemma tells us how to choose the color of T_q. That is, we can decrease p such that p becomes an element of one of the sets E_i, $i \in 2$, defined below. If $p \in E_i$, we can build q such that T_q is c-homogeneous of color i.

Let us fix some more notation. If $\mathbb{P}$ is any forcing notion and $\dot{y}$ is a $\mathbb{P}$-name for a new element of ω^ω let $y[p]$ be the maximal element of $\omega^{<\omega}$ such that $p \Vdash y[p] \subseteq \dot{y}$. $y[p]$ exists since $\dot{y}$ is a name for a new real.

For $i \in 2$ let

$$E_i := \{p \in \mathbb{S}_\alpha : \forall \beta < \alpha \forall q \leq p \exists q' \leq q \exists q_0, q_1 \\ (q' \restriction \beta \Vdash q_0, q_1 \leq q' \restriction [\beta, \alpha) \wedge \overline{c}(x[q_0], x[q_1]) = i)\}.$$

Recall that $\overline{c}(s,t) \in 2$ implies $s \perp t$.

Lemma 30. *E_0 and E_1 are open and $E_0 \cup E_1$ is dense in $\mathbb{S}_\alpha$.*

This lemma is true for all forcing iterations, not only of Sacks forcing. We do not even use countable supports.

Proof of Lemma 30. Let us start with two claims.

Claim 31. *Let $\mathbb{P}$ be any notion of forcing, $\dot{y}$ a $\mathbb{P}$-name for a new element of ω^ω, and c as before. Then for every condition $p \in \mathbb{P}$ there are $p_0, p_1 \leq p$ such that $\overline{c}(y[p_0], y[p_1]) \in 2$.*

For the proof of this claim let $p_0^0, p_1^0 \leq p$ be such that $y[p_0^0] \perp y[p_1^0]$. p_0^0 and p_1^0 exist since $\dot{y}$ is not decided by a single condition. For each $j \in 2$ pick a sequence $p_j^1 \geq p_j^2 \geq \dots$ below p_j^0 such that p_j^n decides $\dot{y} \restriction n$. Let $y_j := \bigcup_{n\in\omega} y[p_j^n]$. Since c is continuous, there is $n \in \omega$ such that $\overline{c}(y[p_0^n], y[p_1^n]) = c(y_0, y_1)$. Now $p_0 := p_0^n$ and $p_1 := p_1^n$ work for the claim.

Claim 32. *Let $\beta < \alpha$ and let $q \in \mathbb{S}_\alpha$ be such that for some $i \in 2$ there are q_0 and q_1 such that*

$$q \restriction \beta \Vdash q_0, q_1 \leq q \restriction [\beta, \alpha) \wedge \overline{c}(x[q_0], x[q_1]) = i.$$

Let $\gamma<\beta$. Then there are $q'\leq q$ and q'_0 and q'_1 such that

$$q'\restriction\gamma\Vdash q'_0,q'_1\leq q'\restriction[\gamma,\alpha)\wedge\overline{c}(x[q'_0],x[q'_1])=i.$$

To see this, let $q'\leq q$ be such that $q'\restriction[\beta,\alpha)=q\restriction[\beta,\alpha)$ and $q'\restriction\beta$ decides $x[q_0]$ and $x[q_1]$. For $j\in 2$ let $q'_j:=q'\restriction[\gamma,\beta)^\frown q_j$. Now q', q'_0, and q'_1 work for the claim.

For the proof of the lemma let $p\in\mathbb{S}_\alpha$. Suppose $p\notin E_0$. We show that p has an extension in E_1. Since $p\notin E_0$, there are $\gamma<\alpha$ and $q\leq p$ such that for all $q'\leq q$ and any two sequences q_0 and q_1 for names of conditions, if $q'\restriction\gamma\Vdash q_0,q_1\leq q'\restriction[\gamma,\alpha)$, then $q'\restriction\gamma\nVdash\overline{c}(x[q_0],x[q_1])=0$. We are done if we can show

Claim 33. *$q\in E_1$.*

Let $r\leq q$ and $\beta<\alpha$. Note that by Claim 32, the sets E_i are not changed if in the definition we replace "$\forall\beta<\alpha$" by "for cofinally many $\beta<\alpha$". Thus we may assume $\beta\geq\gamma$.

Let q_0 and q_1 be such that

$$r\restriction\beta\Vdash q_0,q_1\leq r\restriction[\beta,\alpha)\wedge\overline{c}(x[q_0],x[q_1])\in 2.$$

The existence of q_0 and q_1 follows from Claim 31 together with the assumption that $\dot{x}$ is not added at any proper initial stage of the iteration. Decreasing $r\restriction\beta$ if necessary, we may assume that $r\restriction\beta$ decides $\overline{c}(x[q_0],x[q_1])$ to be $i\in 2$.

By Claim 32, there are $r'\leq r$ and r_0 and r_1 such that

$$r'\restriction\gamma\Vdash r_0,r_1\leq r'\restriction[\gamma,\alpha)\wedge\overline{c}(x[r_0],x[r_1])=i.$$

By the choice of q, $i\neq 0$. Thus $i=1$. This shows $q\in E_1$. $\square$

5.3. $\operatorname{cov}(c)$ **is small in the Sacks model.** Let c, $\dot{x}$, and α be as before. The way to build a condition q for which T_q is c-homogeneous is the following: q will be the fusion of a fusion sequence $(p_n)_{n\in\omega}$ with witness $(F_n,\eta_n)_{n\in\omega}$. For each n, (p_n,F_n,η_n) will determine a finite initial segment T_n of T_q. We have to make sure that T_q is the union of the T_n and that the T_n are good enough to guarantee the c-homogeneity of T_q. The latter will be ensured by the (F_n,η_n)-faithfulness of each p_n, which is defined below.

First we introduce some tools that help us to carry out the necessary fusion arguments.

For $p\in\mathbb{S}$ and $n\in\omega$, each $\varrho\in 2^n$ determines an element s_ϱ of p^n. Let $p_\varrho:=\{t\in p:s_\varrho\subseteq t\vee t\subseteq s_\varrho\}$. Let $\operatorname{stem}(p)$ be the maximal element of p which is comparable with all the other elements of p.

Let α be an ordinal. For $F\in[\alpha]^{<\aleph_0}$, $\eta:F\to\omega$, $\sigma\in\prod_{\gamma\in F}2^{\eta(\gamma)}$, and $q\in\mathbb{S}_\alpha$ let $q*\sigma$ be defined as follows:

For $\gamma\in F$ let $(q*\sigma)(\gamma)$ be a name for a condition in $\mathbb{S}$ such that $\Vdash_{\mathbb{S}_\gamma}(q*\sigma)(\gamma)=q(\gamma)_{\sigma(\gamma)}$. For $\gamma\in\alpha\setminus F$ let $(q*\sigma)(\gamma):=q(\gamma)$.

The $q*\sigma$ form a finite maximal antichain below q. Consider the tree T generated by $\{x[q*\sigma]:\sigma\in\prod_{\gamma\in F}2^{\eta(\gamma)}\}$. If $q'\leq_{F,\eta}q$, then $T_{q'}$ is an end-extension of T.

Definition 34. *Let $i\in 2$ and $\dot{x}$ be fixed. For F and η as before, a condition $q\in\mathbb{S}_\alpha$ is* (F,η)-faithful *if for all $\sigma,\tau\in\prod_{\gamma\in F}2^{\eta(\gamma)}$ with $\sigma\neq\tau$, $\overline{c}(x[q*\sigma],x[q*\tau])=i$.*

Now we are ready to formulate the lemma that will serve for the induction steps in our construction.

Lemma 35. *Let $c : [\omega^\omega]^2 \to 2$ be continuous and let $\dot{x}$ be a $\mathbb{S}_\alpha$-name for an element of ω^ω which is not added by an initial stage of the iteration. Let F, η, and i be as in Definition 34 and suppose that $q \in \mathbb{S}_\alpha$ is (F, η)-faithful.*

a) Let $\beta \in \alpha \setminus F$ and let $F' := F \cup \{\beta\}$ and $\eta' := \eta \cup \{(\beta, 0)\}$. Then q is (F', η')-faithful.

b) Suppose $q \in E_i$. Let $\beta \in F$ and let $\eta' := \eta \restriction F \setminus \{\beta\} \cup \{(\beta, \eta(\beta) + 1)\}$. Then there is $r \leq_{F,\eta} q$ such that r is (F, η')-faithful.

Proof. a) follows immediately from the definitions.

For b) let $\delta := \max(F)$.

Claim 36. *There is a condition $q' \leq_{F,\eta} q$ such that for each $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$ there are sequences $q_{\sigma,0}$ and $q_{\sigma,1}$ of names for conditions such that*

$$q' * \sigma \restriction \delta \Vdash q_{\sigma,0}, q_{\sigma,1} \leq q' * \sigma \restriction [\delta, \alpha) \wedge \overline{c}(x[q_{\sigma,0}], x[q_{\sigma,1}]) = i$$

and $q' * \sigma \restriction \delta$ decides $x[q_{\sigma,0}]$ and $x[q_{\sigma,1}]$.

For the proof of the claim, let $\{\sigma_1, \ldots, \sigma_n\}$ be an enumeration of $\prod_{\gamma\in F} 2^{\eta(\gamma)}$. We build a $\leq_{f,\eta}$-decreasing sequence $(q_m)_{m<n}$ such that $q_0 := q$ and $q' := q_n$ works for the claim. As we construct q_m, we find suitable $q_{\sigma_m,0}$ and $q_{\sigma_m,1}$.

Let $m \in \{1, \ldots, n\}$ and assume that q_{m-1} has already been constructed. Since $q \in E_i$ and E_i is open, there are $q'_m \leq q_{m-1} * \sigma_m$ and sequences $q_{\sigma_m,0}$ and $q_{\sigma_m,1}$ of names of conditions such that

$$q'_m \restriction \delta \Vdash q_{\sigma_m,0}, q_{\sigma_m,1} \leq q' \restriction [\delta, \alpha) \wedge \overline{c}(x[q_{\sigma_m,0}], x[q_{\sigma_m,1}]) = i.$$

We may assume that $q'_m \restriction \delta$ decides $x[q_{\sigma_m,0}]$ and $x[q_{\sigma_m,1}]$. Let $q_m \leq_{F,\eta} q_{m-1}$ be such that $q_m * \sigma_m \restriction \delta = q'_m \restriction \delta$ and $q_m \restriction [\delta, \alpha) = q \restriction [\delta, \alpha)$. This finishes the construction, and it is easy to check that it works.

Continuing the proof of the lemma, let $q_{\sigma,j}$ and q' be as in the claim. Decreasing the $q_{\sigma,j}$ if necessary, we may assume that for all $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$,

$$q' * \sigma \restriction \delta \Vdash \operatorname{stem}(q_{\sigma,0}(\delta)) \perp \operatorname{stem}(q_{\sigma,1}(\delta)).$$

The assumption about the incompatibility of the stems of the $q_{\sigma,j}(\delta)$ will only be needed in the case $\beta = \delta$.

For $\varrho \in 2^{\eta(\delta)}$ let $r^{\varrho^\frown 0}$ and $r^{\varrho^\frown 1}$ be sequences of names for conditions such that for all $j \in 2$ and all $\sigma \in \prod_{\gamma\in F} 2^{\eta(\gamma)}$ with $\sigma(\delta) = \varrho$,

$$q' * \sigma \restriction \delta \Vdash r^{\varrho^\frown j} = q_{\sigma,j}.$$

If $\beta = \delta$, let r be a sequence of names for conditions such that $r \restriction \delta = q' \restriction \delta$ and for all $\sigma \in \prod_{\gamma\in F} 2^{\eta'(\gamma)}$,

$$q' * \sigma \restriction \delta \Vdash r * \sigma \restriction [\delta, \alpha) = r^{\sigma(\delta)}.$$

If β is strictly smaller than δ, let r be a sequence of names for conditions such that $r \restriction \delta = q' \restriction \delta$ and for all $\sigma \in \prod_{\gamma\in F} 2^{\eta'(\gamma)}$,

$$q' * \sigma \restriction \delta \Vdash r \restriction [\delta, \alpha) = r^{\sigma(\delta)^\frown j}$$

where $j = \sigma(\beta)(\eta(\beta))$.

Note that in any case, $r \leq_{F,\eta} q'$ and thus $r \leq_{F,\eta} q$. It follows from the construction that r is (F, η')-faithful. $\square$

Recall that every continuous $c : [\omega^\omega]^2 \to 2$ is coded by $\overline{c}$. If $M[G]$ is a generic extension of M and $c \in M$, then if we talk about c in the context of $M[G]$, we refer to the mapping that has the same definition in $M[G]$ as the original c has in M with respect to $\overline{c}$.

Lemma 37. *Let G be $\mathbb{S}_{\omega_2}$-generic over the ground model M. Let $c : [\omega^\omega]^2 \to 2$ be continuous with $c \in M$. Then in $M[G]$, ω^ω is covered by c-homogeneous sets coded in the ground model.*

Proof. We work in M. Let $\dot{x}$ be a name for an element of ω^ω. We show that $\dot{x}$ is forced to be a branch through a c-homogeneous tree in M. We may assume that for some $\alpha < \omega_2$, $\dot{x}$ is an $\mathbb{S}_\alpha$-name for a real not added by an initial stage of the iteration $\mathbb{S}_\alpha$. Clearly, $\operatorname{cf}(\alpha) \le \aleph_0$. Let $p \in \mathbb{S}_\alpha$. Using Lemma 30, we can decrease p such that for some $i \in 2$, $p \in E_i$. By induction, we define a sequence $(p_n, F_n, \eta_n)_{n\in\omega}$ such that

1. for all $n \in \omega$, $p_n \in \mathbb{S}_\beta$, $p_n \le p$, $F_n \in [\alpha]^{<\aleph_0}$, $\eta_n : F_n \to \omega$, and p_n is (F_n, η_n)-faithful,
2. for all $n \in \omega$, $F_n \subseteq F_{n+1}$, $p_{n+1} \le_{F_n,\eta_n} p_n$, and for all $\gamma \in F_n$ we have $\eta_n(\gamma) \le \eta_{n+1}(\gamma)$, and
3. for all $n \in \omega$ and all $\gamma \in \operatorname{supt}(p_n)$ there is $m \in \omega$ such that $\gamma \in F_m$ and $\eta_m(\gamma) \ge n$.

This construction can be done using parts a) and b) of Lemma 35 to extend F_n or to make η_n bigger, together with some bookkeeping to ensure 3. Now $(p_n)_{n\in\omega}$ is a fusion sequence. Let q be the fusion of this sequence. For each $n \in \omega$ let T_n be the tree generated by $\{x[p_n * \sigma] : \sigma \in \prod_{\gamma\in F_n} 2^{\eta_n(\gamma)}\}$. It is easily seen that $T_q = \bigcup_{n\in\omega} T_n$.

It now follows from the faithfulness of the p_n that T_q is c-homogeneous of color i. Moreover, q forces $\dot{x}$ to be a branch through T_q. It follows that the set of conditions in $\mathbb{S}_\alpha$ forcing $\dot{x}$ to be an element of a c-homogeneous set coded in M is dense in $\mathbb{S}_\alpha$. Since $\mathbb{S}_\alpha$ is completely embedded into $\mathbb{S}_{\omega_2}$, this finishes the proof of the lemma. □

Theorem 38. *In the Sacks model, for all continuous $c : [\omega^\omega]^2 \to 2$, not more than $\aleph_1$ c-homogeneous sets are needed to cover ω^ω.*

Proof. Let M be the ground model satisfying CH and let G be $\mathbb{S}_{\omega_2}$-generic over M. We argue in $M[G]$. Let c be a continuous coloring of the two-element subsets of ω^ω. Since every real is added by an initial stage of the iteration, there is $\alpha < \omega_2$ such that $\overline{c} \in M[G \cap \mathbb{S}_\alpha]$. By Lemma 37, ω^ω is covered by the c-homogeneous sets coded in $M[G \cap \mathbb{S}_\alpha]$. But $|\mathbb{R} \cap M[G \cap \mathbb{S}_\alpha]| = \aleph_1$. It follows that there are $\aleph_1$ c-homogeneous sets covering ω^ω. □

5.4. **$\mathfrak{cb}$ is small in the Sacks model.** We extend Theorem 40 to continuous colorings of arbitrary Polish spaces. Let X be a Polish space and $c : [X]^2 \to 2$ continuous. Since every Polish space is a continuous image of ω^ω, there is a continuous surjection $f : \omega^\omega \to X$. We define $\overline{c} : (\omega^{<\omega})^2 \to \{0, 1, \text{undecided}\}$ coding this situation.

Definition 39. *For $s, t \in \omega^{<\omega}$ let $\overline{c}(s,t) := i \in 2$ if $f[U_s] \cap f[U_t] = \emptyset$ and for all $x \in U_s$ and all $y \in U_t$, $c(\{f(x), f(y)\}) = i$. Otherwise let $\overline{c}(s,t) :=$ undecided.*

By the continuity of c and f, if $x, y \in \omega^\omega$ are such that $f(x) \neq f(y)$, then there are $s, t \in \omega^{<\omega}$ such that $s \subseteq x$, $t \subseteq y$, and $\overline{c}(s,t) = c(\{f(x), f(y)\})$.

Call a set $A \subseteq \omega^\omega$ c-homogeneous of color i if for any two different points $x, y \in A$, $f(x) \neq f(y)$ and $c(\{f(x), f(y)\}) = i$. If A is c-homogeneous, then clearly $f[A]$ is a c-homogeneous subset of X.

Note that if X and f are in the ground model, $\mathbb{P}$ is any notion of forcing, p is a condition in $\mathbb{P}$, and $\dot{x}$ is a name for an element of ω^ω such that $f(\dot{x})$ is forced to be a new element of X, then there are $p_0, p_1 \leq p$ such that $f[U_{x[p_0]}] \cap f[U_{x[p_1]}] = \emptyset$.

Now it easy to see that the lemmas needed for the proof of Theorem 38 go through as before even with this more general definition of $\overline{c}$. It follows that in the Sacks model, for every Polish space X, every continuous $c : [X]^2 \to 2$, and every continuous surjection $f : \omega^\omega \to X$, ω^ω is covered by $\aleph_1$ c-homogeneous sets. Since c-homogeneous subsets of ω^ω are mapped onto c-homogeneous subsets of X and f is onto, we have proved

Theorem 40. *In the Sacks model, for every Polish space X and every continuous $c : [X]^2 \to 2$ only $\aleph_1$ c-homogeneous sets are needed to cover X.*

Let V be a universe of set theory and suppose that $S \in V$, $S \subseteq \mathbb{R}^2$ is closed and does not contain a perfect 3-clique. If necessary, collapse the continuum to $\aleph_1$ using a σ-complete forcing. Since no new reals are introduced, no closed sets are introduced either, and S does not contain a perfect clique also after the collapse. So without loss of generality $V \models CH$.

By Theorem 15, S is a union of countably many convex sets and countably many sets which are special in S. Now by Theorem 40, in the Sacks model over V every special subset of S is a union of $\aleph_1$ convex sets. Therefore $\gamma(S) = \aleph_1$. This implies

Theorem 41. *There exists a forcing notion Q so that every closed planar set either contains a perfect 3-clique or is a union of $\aleph_1 < \mathfrak{c}$ convex sets in V^Q.*

6. Distinguishing $\mathbb{R}^2$ from $\mathbb{R}^3$

In this Section we show that the geometry of $\mathbb{R}^2$ imposes stricter restrictions on convexity numbers of closed sets than the geometry of $\mathbb{R}^3$ does. We provide a model in which all convexity numbers of uncountably convex closed planar sets are equal to the continuum — namely, a closed planar set is either countably convex or not coverable by fewer than $\mathfrak{c}$ convex sets — but in which there are closed subsets $S \subseteq \mathbb{R}^3$ with $\aleph_0 < \gamma(S) < \mathfrak{c}$.

We use the fact that countable support products of Sacks forcing have the so-called 2-localization property. This was shown by Newelski and Roslanowski in [21].

A subtree T of $\omega^{<\omega}$ is *binary* if every $t \in T$ has at most 2 immediate successors in T. A forcing notion $\mathbb{P}$ has the 2*-localization property* if every new element of ω^ω added by $\mathbb{P}$ is forced to be a branch through a binary tree from the ground model.

Theorem 42. *It is consistent that $\mathfrak{hm} = \mathfrak{c} = \aleph_{100}$ and there are $\aleph_1$ binary subtrees of $\omega^{<\omega}$ such that every $x \in \omega^\omega$ is a branch of one of these trees.*

The biggest cardinal invariant in Cichoń's diagram is $\mathrm{cof}(\mathcal{N})$, the cofinality of the ideal of measure zero subsets of the real line. Note that by Bartoszyński's characterization of $\mathrm{cof}(\mathcal{N})$ in terms of slaloms found in [3], this cardinal invariant is $\aleph_1$ in the model used for Theorem 42. In particular, it is consistent that $\mathfrak{hm}$ is strictly larger than all the cardinal invariants in Cichoń's diagram.

Theorem 42 will follow from

Lemma 43. *Let M be a model of set theory satisfying GCH. Let κ be an infinite cardinal of cofinality $> \aleph_0$ in M and let $\mathbb{P}$ be the countable support product of κ copies of Sacks forcing in M. Let G be $\mathbb{P}$-generic over M. Then in $M[G]$, $\mathfrak{hm} = \mathfrak{c}$.*

Proof. Note that forcing with a countable support product of Sacks forcing over a model of CH does not collapse cardinals (see [14]). It is clear that $\mathfrak{c} = \kappa$ in $M[G]$. Suppose that $\mathcal{F}$ is an uncountable family of homogeneous subsets of 2^ω in $M[G]$ such that $|\mathcal{F}| < \kappa$. We may assume that $\mathcal{F}$ consists of perfect sets. For $X \in M$ with $X \subseteq \kappa$ let $\mathbb{P}_X$ be the subordering of $\mathbb{P}$ consisting of those conditions the support of which is a subset of X. In M there is a set $X \subseteq \kappa$ such that $\mathcal{F} \in M[G \cap \mathbb{P}_X]$ and $|X| < \kappa$. Let $\alpha, \beta \in \kappa \setminus X$ and let $x, y \in 2^\omega$ be the generic reals added by the α-th, respectively β-th Sacks forcing in the product $\mathbb{P}$. x and y are typically not Sacks reals over $M[G \cap \mathbb{P}_X]$, however, we have $y \notin (2^\omega)^{M[G\cap\mathbb{P}_X][x]}$ and $x \notin (2^\omega)^{M[G\cap\mathbb{P}_X][y]}$. It follows from the proof of Lemma 8 that $x \otimes y$ is not an element of $\bigcup \mathcal{F}$. $\square$

Proof of the theorem. Let M and $\mathbb{P}$ be as in the lemma with $\kappa = (\aleph_{100})^M$. Let G be $\mathbb{P}$-generic over M. By the 2-localization property of $\mathbb{P}$, every $x \in (\omega^\omega)^{M[G]}$ is a branch of a binary tree in M. But $M[G]$ thinks that there are only $\aleph_1$ binary trees in M. By the lemma, $\mathfrak{hm} = \mathfrak{c}$ in $M[G]$. $\square$

In [15] a closed subset S of $\mathbb{R}^3$ was constructed such that $\gamma(S)$ is precisely the minimal size of a family of binary subtrees of $3^{<\omega}$ such that every element of 3^ω is a branch of one of these trees. (More properties of this set were established in [17].) This together with Theorem 42 clearly implies

Corollary 44. *There is a closed set $S \subseteq \mathbb{R}^3$ such that it is consistent that $\gamma(S) = \aleph_1$ while $\mathfrak{hm} = \mathfrak{c} = \aleph_{100}$.*

7. Open problems and conjectures

The results above do not contradict the following statement: the convexity number of every closed planar S is in the set $\{\aleph_0, \aleph_1, \mathfrak{c}\}$. So let us phrase:

Problem 1. *Is there a model of ZFC in which some closed $S \subseteq \mathbb{R}^2$ satisfies $\aleph_1 < \gamma(S) < \mathfrak{c}$?*

Such a model has to satisfy, of course, $\mathfrak{c} = \gamma(S)^+$.

Closed sets in $\mathbb{R}^2$ can possess at most 2 different uncountable convexity numbers. Let us phrase a wild conjecture:

Conjecture 2. *Convexity numbers of Borel subsets of $\mathbb{R}^n$ can assume at most n different uncountable values.*

From the main theorem it follows that if a closed planar set has a clique of size $\aleph_1$ then it also has a clique of size $\mathfrak{c}$.

Problem 3. *Is there a definable cardinal κ so that whenever a Borel subset of $\mathbb{R}^n$ contains a clique of cardinality $\geq \kappa$ it also contains a clique of size $\mathfrak{c}$?*

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