THE NONEXISTENCE OF UNIVERSAL METRIC FLOWS

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Abstract. We consider dynamical systems of the from (X, f) where X is a compact metric space and $f : X \to X$ is either a continuous map or a homeomorphism. Answering a question by Will Brian we show that there is no universal metric abstract ω-limit set. The same is true for metric minimal dynamical systems and for metric dynamical systems in general.

1. Introduction

We call a compact space X with a continuous map $f : X \to X$ an $\mathbb{N}$-*flow*. If f is a homeomorphism, then the pair (X, f) is a $\mathbb{Z}$-*flow*. X is the *phase space* of the flow (X, f). If (X, f) is a G-flow for $G = \mathbb{N}$ or $G = \mathbb{Z}$, then the action of G on X is given by the map

$$G \times X \to X; (n, x) \mapsto f^n(x).$$

The G-*orbit* of $x \in X$ is the set $\{f^n(x) : n \in G\}$.

Given two G-flows (X, f) and (Y, g) for $G \in \{\mathbb{N}, \mathbb{Z}\}$, a map $h : X \to Y$ is *equivariant* if $h \circ f = g \circ h$. Two G-flows are *isomorphic* if there is an equivariant homeomorphism of their phase spaces. A G-flow (Y, g) is a *factor* of a G-flow (X, f) if there is a continuous equivariant surjection $p : X \to Y$.

If $\mathcal{C}$ is a class of G-flows, then a G-flow $(X, f) \in \mathcal{C}$ is *universal* (in $\mathcal{C}$) if every $(Y, g) \in \mathcal{C}$ is a factor of (X, f).

It is clear that there are no universal objects in the class of all G-flows, simply because there are arbitrarily large phase spaces of G-flows. We investigate what happens if we restrict our attention to metric G-flows.

A G-flow (X, f) is *minimal* if X has no proper closed subsets that are G-flows with respect to the restriction of f. It is well known that there are universal minimal G-flows [3]. However, the phase space of a universal minimal G-flow for $G = \mathbb{Z}$ or $G = \mathbb{N}$ is homeomorphic to an infinite subspace of the Čech-Stone compactification of the integers and hence not metrizable.

A third class of flows that we look at is the class of *abstract ω-limit sets.*

Definition 1.1. Let $G \in \{\mathbb{N}, \mathbb{Z}\}$. For a G-flow (X, f) and $x \in X$ let

$$\omega(x) = \bigcap_{n \geq 0} \mathrm{cl}\{f^m(x) : m \geq n\}$$

be the *ω-limit set* of x.

A G-flow is an *abstract ω-limit set* if it is isomorphic to the ω-limit set of a point in some G-flow.

The ω-limit set of a point in a G-flow (X, f) is a nonempty closed subset of the phase space X that is also a G-flow. It follows that every minimal flow is the ω-limit set of each of its points.

We show that for $G \in \{\mathbb{N}, \mathbb{Z}\}$ the classes of metric minimal G-flows, metric abstract ω-limit sets and metric G-flows do not have universal elements.

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2. Algebraic flows

In [1], Anderson showed that for $G \in \{\mathbb{N}, \mathbb{Z}\}$ every metric G-flow is a factor of a G-flow whose phase space is the Cantor space $\{0,1\}^{\mathbb{N}}$. Anderson also observed that every minimal G-flow with a metric phase space is a factor of a minimal G-flow on $\{0,1\}^{\mathbb{N}}$.

An analog of this is true for abstract ω-limit sets.

Lemma 2.1. *Every metric abstract ω-limit set is a factor of a metric ω-limit set whose phase space is zero-dimensional.*

Proof. Let (X, f) be an abstract ω-limit set. Bowen's proof of his characterization of abstract ω-limit sets in [2] actually shows that (X, f) is isomorphic to the ω-limit set of a point y in a G-flow whose phase space is a subset of $X \times [0,1]$. In particular, (X, f) is isomorphic to the ω-limit set of a point y in a metric G-flow (Y, g).

By Anderson's result mentioned above, the G-flow (Y, g) is a factor of a G-flow $(\{0,1\}^{\mathbb{N}}, h)$. Let $p : \{0,1\}^{\mathbb{N}} \to Y$ be a continuous surjection witnessing this fact and let $z \in p^{-1}(y)$. It is easily checked that p is a continuous equivariant map from $\omega(z)$ onto $\omega(y)$. Hence (X, f) is a factor of the ω-limit set of z. $\square$

This shows that if there are universal elements in the class of metric G-flows, minimal metric G-flows, or metric abstract ω-limit sets, then there are zero-dimensional ones.

Via Stone duality we can investigate G-flows with a zero-dimensional phase space by studying Boolean algebras and their endomorphisms, respectively automorphisms.

If (X, f) is a G-flow and X is zero-dimensional, then its *dual* is the Boolean algebra $\mathrm{Clop}(X)$ of clopen subsets of X together with the endomorphism

$$f^* : \mathrm{Clop}(X) \to \mathrm{Clop}(X); a \mapsto f^{-1}[a].$$

The endomorphism f^* is an automorphism of $\mathrm{Clop}(X)$ iff f is a homeomorphism.

Definition 2.2. Let A be a Boolean algebra and let f be an endomorphism of A. The pair (A, f) is a *Boolean algebraic* $\mathbb{N}$*-flow* (*Ba* $\mathbb{N}$*-flow*). If f is an automorphism of A, then (A, f) is a *Boolean algebraic* $\mathbb{Z}$*-flow* (*Ba* $\mathbb{Z}$*-flow*).

The structure preserving maps between Ba G-flows are equivariant Boolean homomorphisms and we call two Ba G-flows *isomorphic* if there is an equivariant isomorphism between them.

If A is a Boolean algebra with an endomorphism f, then the space $\mathrm{Ult}(A)$ of ultrafilters of A is a compact zero-dimensional space and the Stone dual

$$f^* : \mathrm{Ult}(A) \to \mathrm{Ult}(A); p \mapsto f^{-1}(p)$$

is a continuous map. f^* is a homeomorphism iff f is an automorphism of A. The G-flow $(\mathrm{Ult}(A), f^*)$ is the *dual* of the Ba G-flow (A, f).

Taking the double dual of a zero-dimensional G-flow (X, f) yields an isomorphic G-flow.

Definition 2.3. Let $G \in \{\mathbb{N}, \mathbb{Z}\}$. If (A, f) is a Ba G-flow and $a \in A$, then by $\langle a \rangle_G$ we denote the smallest subalgebra B of A such that $a \in B$ and $(B, f \restriction B)$ is a Ba G-flow. The Boolean algebra $\langle a \rangle_G$ is the subalgebra of A generated by the G-orbit of a.

Given two Ba G-flows (A, f) and (B, g) and elements $a \in A$ and $b \in B$, we call the triples (A, f, a) and (B, g, b) *isomorphic* if there is an isomorphism between (A, f) and (B, g) that maps a to b.

Given a Ba G-flow (A, f) and $a \in A$, the *type* of a is the isomorphism type of the triple $(\langle a \rangle_G, f \restriction \langle a \rangle_G, a)$.

If (A, f) is a Ba $\mathbb{N}$-flow and $I \subseteq A$ is an ideal that is closed under f, then f induces an endomorphism f/I of the quotient A/I. If (A, f) is a Ba $\mathbb{Z}$-flow and $I \subseteq A$ is an ideal that is closed under f and f^{-1}, then f induces an automorphism f/I of the quotient A/I. On the other hand, the kernel of an G-equivariant homomorphism from a Ba G-flow (A, f) to a Ba G-flow (B, g) is an ideal that is closed under f if $G = \mathbb{N}$ and closed under f and f^{-1} if $G = \mathbb{Z}$.

Definition 2.4. For $G \in \{\mathbb{N}, \mathbb{Z}\}$ let $\mathrm{Fr}(G)$ be the free Boolean algebra over the set $\{g_n : n \in G\}$ of generators. We assume that the g_n are pairwise distinct. Let $s_G : \mathrm{Fr}(G) \to \mathrm{Fr}(G)$ be the Boolean homomorphism extending the map $g_n \mapsto g_{n+1}$.

Clearly, $s_{\mathbb{Z}}$ is an automorphism of $\mathrm{Fr}(\mathbb{Z})$ and hence $(\mathrm{Fr}(\mathbb{Z}), s_{\mathbb{Z}})$ is a Ba $\mathbb{Z}$-flow. Also, $(\mathrm{Fr}(\mathbb{N}), s_{\mathbb{N}})$ is a Ba $\mathbb{N}$-flow.

Lemma 2.5. *Let (A, f) be a BA G-flow for $G = \mathbb{N}$ or $G = \mathbb{Z}$ and let $a \in A$. Then there is a unique Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that $\pi(g_0) = a$ and $\pi(s_G(b)) = f(\pi(b))$ for all $b \in \mathrm{Fr}(G)$.*

Proof. There is a unique Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ such that for all $n \in G$, $\pi(g_n) = f^n(a)$. It is clear that π is as desired.

On the other hand, every Boolean homomorphism $\pi : \mathrm{Fr}(G) \to A$ with $\pi(g_0) = a$ and $\pi(s_G(b)) = f(\pi(b))$ for all $b \in \mathrm{Fr}(G)$ satisfies $\pi(g_n) = f^n(a)$ for all $n \in G$. □

Lemma 2.6. *Let (A, f) and (B, g) be Ba G-flows for $G = \mathbb{N}$ or $G = \mathbb{Z}$, $a \in A$, and $b \in B$. Suppose that $A = \langle a \rangle_G$ and $B = \langle b \rangle_G$. Let $\pi_A : \mathrm{Fr}(G) \to A$ and $\pi_B : \mathrm{Fr}(G) \to B$ be the unique equivariant homomorphims with $\pi_A(g_0) = a$ and $\pi_B(g_0) = b$. Then a and b have the same type iff the ideals $\pi_A^{-1}(0)$ and $\pi_B^{-1}(0)$ are identical.*

Proof. If $\pi_A^{-1}(0) = \pi_B^{-1}(0)$, then (A, f, a) and (B, f, b) are isomorphic since both triples are isomorphic to the quotient $\mathrm{Fr}(G)/\pi_A^{-1}(0)$ with the endomorphism induced by s_G and the distinguished element $g_0/\pi_A^{-1}(0)$. Note that if $G = \mathbb{Z}$, then the endomorphism induced by s_G on $\mathrm{Fr}(G)/\pi_A^{-1}(0)$ is actually an automorphism.

If a and b are of the same type, then there is an equivariant isomorphism $\iota : A \to B$ such that $\iota(a) = b$. Now $\iota \circ \pi_A = \pi_B$. Since ι is an isomorphism, $\pi_B^{-1}(0) = \pi_A^{-1}(\iota^{-1}(0)) = \pi_A^{-1}(0)$. □

3. Symbolic dynamics

Definition 3.1. Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. On the space $\{0, 1\}^G$ we consider the shift $S_G : \{0, 1\}^G \to \{0, 1\}^G$ which is defined by letting $S_G(x) : G \to \{0, 1\}$ be the map satisfying $S(x)(n) = x(n + 1)$ for all $n \in G$. Clearly, $S_{\mathbb{Z}} : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$ is a homeomorphism and $S_{\mathbb{N}} : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$ is a continuous map.

Note that the shift S_G on $\{0, 1\}^G$ is (isomorphic to) the Stone dual of the shift s_G on $\mathrm{Fr}(G)$.

Our theorem on the nonexistence of universal metric flows will follow from the fact that $(\{0, 1\}^G, S_G)$ has many minimal *subshifts*, i.e., closed subsets that are minimal G-flows with respect to the restriction of S_G. One way of constructing continuum many minimal subshifts is to consider *Sturmian subshifts.* All the facts about Sturmian subshifts that we use can be found in [4].

Definition 3.2. Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. A *Sturmian word* is a bi-infinite word $x \in \{0, 1\}^G$ such that there are two real numbers, the *slope* α and the *intercept* ρ, with $\alpha \in [0, 1)$ irrational, such that for all $i \in G$ we have

$$x(i) = 1 \quad \Leftrightarrow \quad (\rho + i \cdot \alpha) \mod 1 \in [0, \alpha).$$

In the context of $\mathbb{N}$-flows, we consider Sturmian words in $\{0,1\}^{\mathbb{N}}$ and when we talk about $\mathbb{Z}$-flows, we consider Sturmian words in $\{0,1\}^{\mathbb{Z}}$.

It is well known that the orbit closure $C_x = \mathrm{cl}\{s_G^n(x) : n \in G\}$ of a Sturmian word with the restriction of the shift is a minimal G-flow. If $x \in \{0,1\}^{\mathbb{Z}}$ is a Sturmian word of slope α, then for all y in the orbit closure of x the limit

$$\lim_{n\to\infty} \frac{|x^{-1}(1) \cap \{-n,\dots,n\}|}{2n+1}$$

exists and equals α. Similarly, if $x \in \{0,1\}^{\mathbb{N}}$ is a Sturmian word of slope α, then for all y in the orbit closure of x the limit

$$\lim_{n\to\infty} \frac{|y^{-1}(1) \cap \{0,\dots,n-1\}|}{n}$$

exists and equals α.

It follows that for different irrational numbers $\alpha, \beta \in [0,1)$, Sturmian words of slope α and β have different (even disjoint) orbit closures. We call the orbit closure of a Sturmian word together with the restriction of S_G a *Sturmian subshift.* A Sturmian subshift is a G-flow.

Given a Sturmian subshift $(X, S_G \restriction X)$, we denote the common slope of all Sturmian words that generate X by $\alpha(X)$.

Lemma 3.3. *Let $(X, S_G \restriction X)$ be a Sturmian subshift and let $p : \mathrm{Fr}(G) \to \mathrm{Clop}(X)$ be the homomorphism dual to the embedding of X into $\{0,1\}^G$. Then $\langle p(g_0)\rangle_G = \mathrm{Clop}(X)$ and the type of $p(g_0)$ determines $\alpha(X)$.*

Proof. Since X is a subspace of $\{0,1\}^G$, p is onto. Since $\mathrm{Fr}(G) = \langle g_0\rangle_G$, $\langle p(g_0)\rangle_G = \mathrm{Clop}(X)$. By Lemma 2.6, the type of $p(g_0)$ determines the kernel $p^{-1}(0)$. But by standard Stone duality, the ideals of $\mathrm{Fr}(G)$ are in 1-1 correspondence to the subspaces of $\{0,1\}^G$. It follows that the type of $p(g_0)$ determines the subspace X of $\{0,1\}^G$ and hence the slope $\alpha(X)$. □

Definition 3.4. In the context of Lemma 3.3 we call $p(g_0)$ the *generator* of $\mathrm{Clop}(X)$ and denote it by g_X.

Theorem 3.5. *Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. Then there is no metric G-flow that has all Sturmian subshifts as factors.*

Proof. Suppose there is a metric G-flow (X, f) such that every Sturmian subshift is a factor of (X, f). By Anderson's result mentioned above, we may assume that X is zero-dimensional. Let (A, f^*) be the Stone dual of (X, f). Then A is a countable Boolean algebra.

If a Sturmian subshift $(Y, S_G \restriction Y)$ is a factor of (X, f), then there is an equivariant embedding of $\mathrm{Clop}(Y)$ into A. In particular, A has an element whose type is the same as the type of the generator g_Y of $\mathrm{Clop}(Y)$.

Since there are uncountably many slopes of Sturmian words, by Lemma 3.3 there are uncountably many different types of generators of algebras of the form $\mathrm{Clop}(Y)$ where $(Y, S_G \restriction Y)$ is a Sturmian subshift. But since A is countable, its elements realize only countably many different types. A contradiction. □

Corollary 3.6. *Let $G = \mathbb{N}$ or $G = \mathbb{Z}$. The following classes of G-flows contain no universal elements:*

(1) *Metric G-flows*
(2) *Metric minimal G-flows*
(3) *Metric abstract ω-limit sets*

Proof. The corollary follows from the previous theorem together with the fact that all Sturmian subshifts are contained in each of the three classes of G-flows. □

The proof of Theorem 3.5 shows that no G-flow of weight less that $2^{\aleph_0}$ has all Sturmian subshifts as factors. A universal minimal G-flow has a phase space that is a subspace of the Čech-Stone compactification of G. It follows that the weight of a universal minimal G-flow is at most $2^{\aleph_0}$. Since every Sturmian subshift is minimal, every Sturmian subshift is a factor of the universal minimal G-flow. Hence the weight of a universal minimal G-flow is exactly $2^{\aleph_0}$.

References

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