FUNCTIONS WITH MANY LOCAL EXTREMA

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Abstract. Answering a question addressed by Dirk Werner we show that the set of local extrema of a nowhere constant continuous function $f : [0,1] \to \mathbb{R}$ is always meager but possibly of full measure. The set of local extrema of a nowhere constant C^∞-function from $[0,1]$ to $\mathbb{R}$ can be of arbitrarily large measure below 1.

1. Introduction

In [1], Behrends, Natkaniec and the author studied the question whether a continuous function f from a topological space X into the real line can have a local extremum at every point of X without being constant. Among other things it was observed that if X is a connected space of weight $< |\mathbb{R}|$, then every continuous function $f : X \to \mathbb{R}$ that has a local extremum at every point of X is constant. Also, if X is a connected linear order in which every family of pairwise disjoint open intervals is of size $< |\mathbb{R}|$ and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of X, then f is constant.

The proof of the latter fact given in [1] shows that if X is a connected linear order and $f : X \to \mathbb{R}$ is continuous and has a local extremum at every point of X, then f is constant on a nonempty open interval. In fact, the collection of open intervals on which f is constant has a dense union.

Recently, the results mentioned above have been improved by Fedeli and Le Donne (see [2]), who showed that if X is a connected space in which every family of pairwise disjoint open sets is of size $< |\mathbb{R}|$, then every continuous function $f : X \to \mathbb{R}$ that has a local extremum at every point is constant.

In this note we answer a question addressed by Dirk Werner, namely how many local extrema a non-constant continuous function, say from the unit interval, into the reals can actually have.

It is relatively easy to construct a continuous function $f : [0,1] \to \mathbb{R}$ that is not constant and whose set of local minima is open and dense. Just choose a closed nowhere dense set $A \subseteq [0,1]$ of positive measure (see Lemma 1) and let $f(x)$ be the measure of $A \cap [0,x]$. Then clearly, f is continuous, not constant and constant

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on every open interval disjoint from A. In particular, f has a local minimum and maximum at every point of $X \setminus A$.

This example shows that we should consider functions that are not constant on any nonempty open interval.

2. Measure

The following lemma is well known.

Lemma 1. *Let $\varepsilon > 0$. Then there is a closed nowhere dense set $A \subseteq [0,1]$ of measure at least $1-\varepsilon$.*

Proof. Let $\{(a_n, b_n) : n \in \mathbb{N}\}$ be the collection of all open subintervals of $[0,1]$ with rational endpoints. For each $n \in \mathbb{N}$ let $(c_n, d_n) \subseteq (a_n, b_n)$ be an open interval of length at most $2^{-n} \cdot \varepsilon$. Now $B = \bigcup_{n\in\mathbb{N}}(c_n, d_n)$ is a dense open set of measure at most ε. Hence, the set $A = [0,1] \setminus B$ is closed, nowhere dense and of measure at least $1-\varepsilon$. $\square$

By removing a suitable open interval from A we can actually assume that A is exactly of measure $1-\varepsilon$.

Lemma 2. *Let $a, b \in \mathbb{R}$ be such that $a < b$. Let $A \subseteq [a,b]$ be closed and nowhere dense. Then the function $f^A_{a,b} : [a,b] \to \mathbb{R}$ that assigns to every point x its distance from A is continuous and has local minima exactly at the points of A. Moreover, whenever $I \subseteq [a,b]$ is a maximal open interval disjoint from A, then $f^A_{a,b} \restriction \mathrm{cl}(I)$ is piecewise linear and in fact consists of two linear (in the sense of affine linear) pieces, one of slope 1 and one of slope -1.*

Theorem 3. *There is a continuous function $g : [0,1] \to \mathbb{R}$ such that g is not constant on any non-empty open interval and the set of local minima of g is of measure 1. In particular, the set of local minima of g is dense in $[0,1]$.*

Proof. Let $a, b \in [0,1]$ be such that $a < b$. Suppose that $f : [a,b] \to \mathbb{R}$ is linear (in the sense of affine linear) with $f(a) = c$ and $f(b) = d$. Let $c = a + \frac{1}{8}(b-a)$ and $d = b - \frac{1}{8}(b-a)$. Let A be a closed nowhere dense subset of $[c,d]$ of measure $\frac{1}{2}(b-a)$. We may assume $c, d \in A$.

Now let $f^* : [a,b] \to \mathbb{R}$ be defined as follows. For each $x \in [a,b]$ let

$$f(x) = \begin{cases} 4\frac{f(b)-f(a)}{b-a}(x-a) + f(a), & x \le c \\ f^A_{c,d}(x) + \frac{1}{2}(f(a)+f(b)), & c \le x \le d \\ 4\frac{f(b)-f(a)}{b-a}(x-b) + f(b), & x \ge d \end{cases}$$

In other words, f^* is a continuous function whose graph starts and ends at the same points as the graph of f, but f^* has local minima at every point of A, except possibly the first and last points of A, i.e., c and d. In particular, the set of local minima of f^* is of measure at least $\frac{1}{2}(b-a)$. We **observe** that

$$\sup\{|f^*(x) - f(x)| : x \in [a,b]\} \leq \max(b-a, |f(b)-f(a)|).$$

Given a function $f : [0,1] \to \mathbb{R}$, we define $f^* : [0,1] \to \mathbb{R}$ as follows. If $I \subseteq [0,1]$ is a maximal open interval such that f is linear in I, we let $f^* \restriction \operatorname{cl} I = (f \restriction \operatorname{cl} I)^*$. If $x \in [0,1]$ is not contained in a maximal open interval on which f is linear, we let $f^*(x) = f(x)$. From our construction it follows that f^* is continuous if f is.

Now choose $A \subseteq [0,1]$ closed, nowhere dense, and of measure $\frac{1}{2}$. Let $f_0 = f_{0,1}^A$. For every $n > 0$ let $f_n = f_{n-1}^*$. The sequence $(f_n)_{n\in\mathbb{N}}$ is a sequence of continuous functions. By our observation above, the sequence converges uniformly. It follows that the limit g of this sequence is a continuous function from $[0,1]$ to $\mathbb{R}$.

It is easily checked that g is nowhere constant. Also, the set of local minima of g is the union of the sets of local minima of the f_n. By induction it follows that the measure of the set of local minima of f_n is at least $\sum_{k=1}^{n+1} \frac{1}{2^k}$. Hence the measure of the set of local minima of g is 1. $\square$

Clearly, if $f : [0,1] \to \mathbb{R}$ is continuously differentiable and has a dense set of local extrema, then f has to be constant. In particular, a nowhere constant, continuously differentiable function on the unit interval cannot have a set of local extrema of full measure. However, nowhere constant C^∞-functions can have sets of local extrema of large measure.

Theorem 4. *For every $\varepsilon > 0$ there is an infinitely often differentiable function $f : [0,1] \to \mathbb{R}$ such that f is not constant on any non-empty open interval and the set of local minima of f is of measure at least $1 - \varepsilon$.*

Proof. We start the proof with a preliminary remark.

Claim 5. For all $a, b \in [0,1]$ with $a < b$ there is a C^∞-function $h : [0,1] \to \mathbb{R}$ such that h vanishes outside (a,b) and is positive and nowhere constant on (a,b).

For the proof of the claim we define $g : \mathbb{R} \to \mathbb{R}$ as follows: For all $x \in \mathbb{R}$ let

$$g(x) = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1) \\ 0 & x \notin (-1,1). \end{cases}$$

It is well known that g is infinitely often differentiable. Clearly, g is nowhere constant and positive on the set $(-1,1)$. The claim is witnessed by translations of scaled versions of g.

Now let $A \subseteq [0,1]$ be as in Lemma 1 and choose a maximal family $\mathcal{G}$ of non-negative C^∞-functions on $[0,1]$ with the following properties:

(1) For every $g \in \mathcal{G}$ the set $g^{-1}[(0,\infty)]$ is a non-empty open interval $I_g \subseteq [0,1] \setminus A$.
(2) For $g, h \in \mathcal{G}$ with $g \neq h$ the intervals I_g and I_h are disjoint.

Such a family $\mathcal{G}$ exists by Zorn's Lemma. Since $\{I_g : g \in \mathcal{G}\}$ is a disjoint family of non-empty open intervals, it is countable. It follows that $\mathcal{G}$ is countable.

By the claim, $\bigcup\{I_g : g \in \mathcal{G}\}$ is a dense subset of $[0,1] \setminus A$. Since A is nowhere dense and yet of positive measure, $[0,1] \setminus A$ is not the union of finitely many open intervals and hence $\mathcal{G}$ is infinite. Let $(g_n)_{n\in\omega}$ be an enumeration of $\mathcal{G}$ without repetition.

For every $n \in \omega$ choose $\varepsilon_n > 0$ such that for all $m \leq n$ we have

$$\varepsilon_n \cdot \sup\left\{\left|g_n^{(m)}(x)\right| : x \in [0,1]\right\} < 2^{-n}.$$

Here $g_n^{(m)}$ denotes the m-th derivative of g_n.

For every $n \in \omega$ let $f_n = \sum_{m=0}^{n} \varepsilon_m g_m$. Since the I_{g_m}, $m \in \omega$, are pairwise disjoint and by the choice of the ε_m, the sequence $(f_n)_{n\in\omega}$ converges uniformly in every derivative and hence converges to a C^∞-function $f : [0,1] \to \mathbb{R}$.

Clearly, $B = f^{-1}(0) = [0,1] \setminus \bigcup\{I_g : g \in \mathcal{G}\}$ and B is a closed nowhere dense superset of A. Moreover, f is not constant on any open interval disjoint from B. Since B is nowhere dense, this implies that f is nowhere constant. Clearly, every point of B, and hence of A, is a local minimum of f. □

Let us point out that the use of Zorn's Lemma in the proof of Lemma 4 can be easily avoided and that for any given ε a suitable function f can be defined explicitly using a closed, but lengthy, formula.

3. Category

We point out that the analog of Theorem 3 for category fails badly.

Theorem 6. *If $f : [0,1] \to \mathbb{R}$ is continuous and not constant on any non-empty open interval, then the set of local minima of f is meager.*

The proof of this theorem uses the following lemma.

Lemma 7. *The set of local minima of a continuous function $f : [0,1] \to \mathbb{R}$ is F_σ.*

Proof. For $a, b, c, d \in [0,1] \cap \mathbb{Q}$ with $a < b < c < d$ consider the set

$$M_{a,b,c,d} = \{x \in [b,c] : f(x) = \min(f[(a,d)])\}.$$

Clearly, $M_{a,b,c,d}$ is closed and every element of $M_{a,b,c,d}$ is a local minimum of f. On the other hand, if x is a local minimum of f, then there are $a, b, c, d \in [0,1] \cap \mathbb{Q}$ such that $a < b < c < d$ and $x \in M_{a,b,c,d}$. It follows that the set of local minima of

f is equal to

$$\bigcup\{M_{a,b,c,d} : a, b, c, d \in [0, 1] \cap \mathbb{Q} \wedge a < b < c < d\},$$

which is clearly F_σ. □

Proof of Theorem 6. By Lemma 7, the set M of local minima of f can be written as $\bigcup_{n\in\mathbb{N}} M_n$ where each M_n is closed. Assume that M is not meager. Then for some $n \in \mathbb{N}$, M_n is somewhere dense. Since M_n is closed, M_n actually contains a non-empty open interval (a, b). But a continuous function that has a local minimum at each point of a nonempty interval is constant on that interval. A contradiction. □

Corollary 8. *If $f : [0, 1] \to \mathbb{R}$ is not constant on any non-empty open interval, then the set of local extrema of f is meager. However, even the set of local minima can be of measure 1.*

References

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