AN INDUCED RAMSEY THEOREM FOR MODULAR PROFINITE GRAPHS

STEFANIE HUBER, STEFAN GESCHKE, AND MENACHEM KOJMAN

ABSTRACT. It is proved that the class of modular profinite ordered graphs of countable weight satisfies with high probability the A-partition property for a finite ordered graphs A with respect to Baire measurable partitions: for every B in the class there exists a G in the class

$$G \rightarrowtail_{\text{Baire}} (B)^A.$$

More generally, for every finite ordered graph A there is a number $k(A) \in \mathbb{N}$ such that for every universal modular profinite graph G and for every partition of $\binom{G}{A}$ to finitely many Baire measurable parts, there is a closed ordered copy $G' \subseteq G$ of G such that $\binom{G'}{A}$ meets at most $k(A)$ parts. The probability that a random graph A on $\{0, 1, \ldots, n-1\}$ satisfies $k(A) = 1$ tends to 1 as n grows to infinity.

1. INTRODUCTION

A class of structures $\mathcal{A}$ satisfies the *A-partition property* for a structure A if for every $B \in \mathcal{A}$ there is $C \in \mathcal{A}$ such that for every partition of the set $\binom{C}{A}$ of all isomorphic copies of A in C to two parts there exists an isomorphic copy B' of B in C such that $\binom{B'}{A}$ is contained in one of the parts. A class $\mathcal{A}$ which satisfies the A-partition property for every $A \in \mathcal{A}$ is called a *Ramsey Class*.

The class of all complete finite graphs is Ramsey by Ramsey's theorem:

$$\forall r \forall k \exists n \, n \to (k)^r.$$

The A-partition property holds for all finite A also in the class of countable complete graphs, by the infinite Ramsey theorem:

$$\forall r \, \mathbb{N} \to (\mathbb{N})^r. \tag{1}$$

Galvin and Prikry proved that with only Borel partitions, also

$$\mathbb{N} \to (\mathbb{N})^{\mathbb{N}} \tag{2}$$

holds. Thus, the class of countable complete graphs is Ramsey with respect to to Borel partitions. A restriction of the partitions is necessary, as with the Axiom of Choice (AC) counterexamples to (2) are easily constructed.

Partially supported by a German-Israel Foundation for Scientific Research & development grant.

With the class of all finite graphs the picture is different. One has to impose order to turn this class into a Ramsey class:

Theorem 1.1 (Nešetřil and Rödl). *For every finite ordered graphs A and B there exists a finite ordered graph C such that*

$$C \rightarrowtail (B)^A.$$

The ω-saturated countable ordered graph G^* is a universal element in the class of countable ordered graphs, hence the A-partition property in this class for some finite element A is equivalent to the relation

$$G^* \rightarrowtail (G^*)^A. \tag{3}$$

But this relation holds only for A of size 1.

We consider in the present paper the induced Ramsey properties of the class of modular profinite graphs of countable weight. This is the class of all inverse limits of a countable system of finite ordered graphs. This class has universal elements for order preserving and continuous embeddings. We call such graphs Universal Modular Profinite graphs, or UMP graphs, for short. There is no unique UMP graph, but all UMP graphs form a single bi-embeddability class with respect to these embeddings, hence for the purpose of the partition theorem below it does not matter which one is chosen.

UMP graphs appeared first in the investigation [6] of co-chromatic numbers of continuous graphs on Polish spaces as attaining the maximal possible co-chromatic number among all continuous graphs and were called $G_{\max}$ there. The two element set $\{G_{\max}, G_{\min}\}$ attains all possible uncountable co-chromatic numbers of separable continuous graphs, where $G_{\min}$ from [5] is also a profinite graph (see [6]).

UMP graphs can be economically characterised up to isomorphism as follows. The vertex set of a UMP graph is a subset of the ordered irrationals which has no isolated points and which is compact with respect to the induced topology from the space of the irrationals. Any two vertices are separated by some finite modular ordered partition and every nonempty open interval of the vertex set contains induced ordered copies of all finite ordered graphs. See below for more details.

Universality of UMP graphs implies that for every finite ordered A, the A-partition property in the class we are discussing is equivalent to the relation

$$G \rightarrowtail (G)^A \tag{4}$$

with any fixed UMP graph G.

A few words of caution are in place here. First, since the number of closed subgraphs of an UMP graph G is equal to the graph's cardinality, trivial diagonalization using AC forbids this relation altogether (even for partitions of points). So we consider from now on only Baire measurable partitions, which include all Borel partitions. Let $G \rightarrowtail_{\text{Baire}} (G)^A$ denote the weakening of (4) to only Baire measurable partitions.

Before we state the results we need to recall the following standard graph-theoretic notation. Let $\mathcal{G}_n$, for $n > 0$, denote the probability space of all finite graphs on the ordered set $n = \{0, 1, \ldots, n-1\}$ with uniform probability. A property of finite ordered graphs holds *with high probability* if its probability to hold in $\mathcal{G}_n$ tends to 1 as n grows to infinity.

1.1. **The Results.** Let P_n denote the probability that a graph A in P_n satisfies (4).

Theorem. $\lim_{n\to\infty} P_n = 1$.

That is, for a finite ordered graph A the relation

$$G \rightarrowtail_{\text{Baire}} (G)^A$$

holds with high probability for every UMP graph G.

Most of the work below is required, though, to include the finite ordered graph violating this relation — which occur in probability tending to 0.

Theorem. *For every finite ordered A there exists a natural number $k(A) \geq 1$ such that for every UMP G and every finite Baire measurable partition of the set $\binom{G}{A}$ all copies of A in G, there exists a closed copy G' of G in G such that $\binom{G'}{A}$ is contained in the union or at most $k(A)$ cells of the partition.*

In fact, we identify a clopen partition of all copies of A in G to $k(A)$ many parts and prove that (i) the partition is *persistent*, that is, $\binom{G'}{A}$ meets every cell of the partition for every closed copy G' in G; and (ii) the partition is *basic*, that is, for every finite Baire measureable partition there is a copy of G in itself on which the basic cell determines the cell of the given partition.

The first theorem can be re-stated now as saying that with probability tending to 1 the basic partition given by the second theorem has one cell.

In a forthcoming article [4] we will discuss Borel partitions of infinite closed subgraphs. A version of the theorem above is true in this situation as well, but its proof requires forcing techniques and absoluteness arguments, which are not used in the present article.

2. Preliminaries and Notation

2.1. **Modular profinite graphs of countable weight.** Let X be a topological space. A graph G with the vertex set X is *clopen* if the edge-relation of G is a clopen subset of $X^2 \setminus \{(x,x) : x \in X\}$.

G is *modular profinite* if it is the inverse limit of a system of finite graphs whose bonding maps are modular. Here we call a map f from the vertex set $V(G)$ of a graph G to the vertex set $V(H)$ of a graph H *modular* if for all $v, w \in V(G)$, either $f(v) = f(w)$ or $\{v,w\} \in E(G)$ iff $\{f(v), f(w)\} \in E(G)$. Clearly, for each modular map $f : V(G) \to V(H)$ and each vertex $v \in V(H)$, the inverse image $f^{-1}(v)$ is a module of G, i.e., a set M of vertices of G such that for all $u \in V(G) \setminus M$ either u is adjacent to all vertices in M or to no vertex in M. If $f : G \to H$ is modular, then $\{f^{-1}(v) : v \in V(H)\}$ is a *modular partition* of G, i.e., a partition of the vertex set of G into modules, and the induced subgraph of H on the range of f is the *modular quotient* of G by this modular partition.

In [3] modular profinite graphs have been studied in detail and it was shown that a graph G on a topological space X is modular profinite if and only if X is compact and zero-dimensional, G is a clopen graph on X, and the modular partitions of G into finitely many clopen sets separate the vertices of G.

For the purpose of this article it is unnecessary to go into more details of the definition of modular profinite graphs. It is enough to use the following description of modular profinite graphs of countable weight:

For distinct points $x, y \in \omega^\omega$ let $\Delta(x, y)$ denote the minimal $n \in \omega$ with $x(n) \neq y(n)$. Let $S \subseteq \omega^\omega$ be a closed set. A coloring $c : [S]^2 \to 2$ is of *depth* 1 if for all $\{x, y\} \in [S]^2$ the color $c(x, y)$ only depends on $x \restriction (\Delta(x, y)+1)$ and $y \restriction (\Delta(x, y) + 1)$. The coloring c corresponds to the graph $G_c = (S, c^{-1}(1))$. If the coloring $c : [S]^2 \to 2$ is of depth 1 and S is compact, then the graph G_c is modular profinite. Moreover, every modular profinite graph of countable weight is isomorphic to a graph of the form G_c.

A modular profinite graph G of countable weight is described by the following data: the set V of vertices, which is a compact subset of the *Baire space* ω^ω, and for each

$$t \in T = T(V) = \{x \restriction n : x \in V \wedge n \in \omega\}$$

a graph G_t on the set $\mathrm{succ}_T(t)$ of immediate successors of t in T such that distinct successors $s_0, s_1 \in \mathrm{succ}_T(t)$ form an edge in G_t iff all $x, y \in V$ with $s_0 \subseteq x$ and $s_1 \subseteq y$ form an edge in G. Since the coloring $c : [V]^2 \to 2$ corresponding to G is of depth 1, for t, s_0, s_1 as above, either all $x, y \in V$ with $s_0 \subseteq x$ and $s_1 \subseteq y$ form an edge in G or no such pair forms an edge.

In [6] it was proved that there is a universal modular profinite graph of countable weight, but using the language of continuous colorings. Such universal graphs are not unique up to isomorphism, but are unique with respect to bi-embeddability. A detailed description of such a graph is as follows:

Let R be the Rado graph, i.e., the unique countable universal and homogeneous graph. We assume that the set of vertices of R is just the set ω of natural numbers. For each $n \in \omega$ let R_n denote the induced subgraph of R on the set $\{0, \dots, n\}$. Now consider a subtree $T_{\max} \subseteq \omega^{<\omega}$ such that each $t \in T_{\max}$ has $\mathrm{succ}_{T_{\max}}(t) = \{0, \dots, |t|\}$, where $|t|$ is the length of the finite sequence t. For each t in $T_{\max}$ we choose G_t such that the map

$$\mathrm{succ}_{T_{\max}}(t) \to \{0, \dots, |t|\}; s \mapsto s(|t|)$$

is an isomorphism between G_t and $R_{|t|}$. This induces a graph structure on the space $[T_{\max}]$ of branches of $T_{\max}$. We call this graph $G_{\max}$.

Let us introduce some more notation for trees. We will only consider subtrees T of the $\omega^{<\omega}$ of finite sequences of natural numbers ordered by set-theoretic inclusion. For such a tree T and $t \in T$ let

$$T_t = \{s \in T : t \subseteq s \vee s \subseteq t\}.$$

For a subset Z of T let

$$T_Z = \{s \in T : s \text{ is comparable with some } t \in Z\}.$$

For $n \in \omega$ let $\mathrm{Lev}_T(n) = \{t \in T : |t| = n\}$. Observe that for all $t \in \omega^{<\omega}$ the natural number $|t|$ coincides with the domain of t. Also, $|t|$ is the height of t in any subtree T of $\omega^{<\omega}$ with $t \in T$.

2.2. **Order.** For our purposes we have to consider order, mainly because we want to use the Nešetřil-Rödl theorem (Theorem 1.1 above), which only works for ordered graphs.

Note that the Nešetřil-Rödl theorem implies that for any finite number N and all finite ordered graphs B and L there is a finite ordered graph D such that for any family $\mathcal{C}$ of at most N colorings of the induced copies of L in D by two colors, there is a copy B' of B in D such that each of the colorings in $\mathcal{C}$ is constant on the induced copies of L in B'. This is because we can code the N colorings in $\mathcal{C}$ by a single coloring c with 2^N colors and then apply the Nešetřil-Rödl theorem with $r = 2^N$ many colors, which, of course, follows from the version with 2 colors.

Obtaining an order on $G_{\max}$ is easy. We simply take the *lexicographic order* on the Baire space[1]:

For $x, y \in \omega^\omega$ with $x \neq y$ let $\Delta(x, y) = \min\{n \in \omega : x(n) \neq y(n)\}$. Now let $x \leq y$ if either $x = y$ or $x(\Delta(x, y)) < y(\Delta(x, y))$. This defines a linear order on ω^ω and hence on $G_{\max}$. Even though this order does not seem to have any relation to the graph structure, we have the following fact:

Lemma 2.1. *If H is any finite ordered graph and $t \in T_{\max}$ then there is an extension s of t in $T_{\max}$ such that H embeds (via an order preserving embedding) into G_s, where we consider G_s as an ordered graph with respect to the lexicographic order on the vertices of G_s.*

Proof. We consider the random graph with the usual ordering on ω. Since the graphs G_t are isomorphic to the initial segments R_n of R, it is enough to show that each finite ordered graph H embeds into the ordered random graph R.

Let $v_1, \dots, v_k$ be the increasing enumeration of the vertices of H. We define an order preserving embedding $e : H \to R$ as follows: Choose

[1] The Baire space ω^ω is homeomorphic to the space of irrational numbers via continued fractions expansion, but this is not an order preserving homeomorphism. It is trivial to find an order-preserving homeomorphisms between the ordered irrationals and the lexicographically ordered ω^ω.

$e(v_1) \in V(R)$ arbitrarily. Now assume that we have chosen $e(v_1), \dots, e_(v_i)$ for some $i \in \{1, \dots, k-1\}$ such that $e \restriction \{v_1, \dots, v_i\}$ is an order preserving embedding of the induced subgraph on $\{v_1, \dots, v_i\}$ into R. By the extension property of the random graph, there is a vertex w of R such that $(e \restriction \{v_1, \dots, v_i\}) \cup \{(v_{i+1}, w)\}$ is an embedding of the induced subgraph on the set $\{v_1, \dots, v_i, v_{i+1}\}$ into R as unordered graphs. But it is easy to see that in fact, there are infinitely many such vertices w. Hence we can find one that is larger than all the vertices $e(v_1), \dots, e(v_i)$ and call it $e(v_{i+1})$. This finishes the recursive construction of e. □

We are interested in induced subgraphs of $G_{\max}$ that contain copies of $G_{\max}$ itself. One way of getting such subgraphs is by constructing sufficiently large subtrees of $T_{\max}$. Given a subtree T of $T_{\max}$, for each $t \in T$ let G_t^T denote the induced subgraph of G_t on the set of immediate successors of t in T.

We call a tree $T \subseteq T_{\max}$ a $G_{\max}$*-tree* if for every finite ordered graph H and all $t \in T$ there is $s \in T$ such that $t \subseteq s$ and H embeds into G_s^T.

A $G_{\max}$-tree T is *normal* if for all $t, s \in T$ with $t \subseteq t'$ either G_t^T embeds into G_s^T or G_s^T has only a single vertex. A subtree T of $\omega^{<\omega}$ is *skew* if T has at most one splitting node at level n.

Clearly, every $G_{\max}$-tree has a $G_{\max}$-subtree that is both skew and normal. For every subtree T of $T_{\max}$ let $G(T)$ denote the induced subgraph of $G_{\max}$ on the set $[T]$ of infinite branches of T.

The main techniques of building a $G_{\max}$-subtree S of a $G_{\max}$-tree T is *fusion*: A sequence $(T_k)_{k\in\omega}$ is a *fusion sequence with witness* $(m_k)_{k\in\omega}$ if the following hold:

(1) $(m_k)_{k\in\omega}$ is a strictly increasing sequence of natural numbers.
(2) For all $k, \ell \in \omega$, if $k < \ell$, then T_ℓ is a $G_{\max}$-subtree of T_k such that $\mathrm{Lev}_{T_k}(m_k) = \mathrm{Lev}_{T_\ell}(m_k)$.
(3) For every finite ordered graph H, every $k \in \omega$, and every $t \in \mathrm{Lev}_{T_k}(m_k)$ there is $\ell > k$ such that t has an extension s in T_ℓ such that $|s| < m_\ell$ and H embeds into $G_s^{T_\ell}$.

It is easily checked that if $(T_k)_{k\in\omega}$ is a fusion sequence witnessed by $(m_k)_{k<\omega}$, then the *fusion* $\bigcap_{k\in\omega} T_k = \bigcup_{k\in\omega}(T_k \cap \omega^{\leq m_k})$ is a $G_{\max}$-tree. In practice, whenever we construct a fusion sequence $(T_k)_{k\in\omega}$ witnessed by $(m_k)_{k\in\omega}$, we will use some book-keeping that tells us that when we have already chosen T_k and m_k, we now have to find a splitting node s above a certain $t \in \mathrm{Lev}_{T_k}(m_k)$ such that a certain finite ordered graph H embeds into $G_s^{T_{k+1}}$. With the right book-keeping, which we will not specify precisely, this guarantees that $(T_k)_{k\in\omega}$ is a fusion sequence witnessed by $(m_k)_{k\in\omega}$.

2.3. **Types.** Let T be a $G_{\max}$-tree and let H and H' be finite induced subgraphs of $G(T)$. We say that H and H' are *strongly isomorphic* if there is an isomorphism $\varphi : H \to H'$ of ordered graphs such that for all 2-element sets $\{x, y\}$ and $\{x', y'\}$ of vertices of H we have

$$\Delta(x, y) \leq \Delta(x', y') \Leftrightarrow \Delta(\varphi(x), \varphi(y)) \leq \Delta(\varphi(x'), \varphi(y')).$$

The *type* of a finite induced subgraph H of $G(T)$ is its strong isomorphism type. Given a type τ of a finite induced subgraph of $G(T)$, we denote by $\binom{G(T)}{\tau}$ the set of all induced subgraphs of $G(T)$ of type τ.

2.4. **The Halpern-Läuchli theorem.**

Definition 2.2. *Let T be a subtree of $\omega^{<\omega}$. For $n \in \omega$ let $\mathrm{Lev}_T(n) = T \cap \omega^n$. For $t \in T$ let $T_t = \{s \in T : s \subseteq t \vee t \subseteq s\}$. For subtrees $T_1, \ldots, T_\ell$ of $\omega^{<\omega}$ let*

$$\bigotimes_{i=1}^{\ell} T_i = \bigcup_{n \in \omega} \prod_{i=1}^{\ell} \mathrm{Lev}_{T_i}(n).$$

For $D_1 \subseteq T_1, \ldots, D_\ell \subseteq T_\ell$ let

$$\bigotimes_{i=1}^{\ell} D_i = \prod_{i=1}^{\ell} D_i \cap \bigotimes_{i=1}^{\ell} T_i.$$

For $n, m \in \omega$ with $n \leq m$, a sequence $(D_1, \ldots, D_\ell)$ with $D_1 \subseteq T_1, \ldots, D_\ell \subseteq T_\ell$ is (n, m)-dense *in $\bigotimes_{i=1}^{\ell} T_i$ if for all $(t_1, \ldots, t_\ell) \in \prod_{i=1}^{\ell} \mathrm{Lev}_{T_i}(n)$ there is $(s_1, \ldots, s_\ell) \in \prod_{i=1}^{\ell} \mathrm{Lev}_{D_i}(m)$ with $t_1 \subseteq d_1, \ldots, t_\ell \subseteq d_\ell$.*

Theorem 2.3 (Halpern-Läuchli [8]). *Let $\ell, k > 0$ be natural numbers and let $T_1, \ldots, T_\ell$ be finitely splitting subtrees of $\omega^{<\omega}$. For every coloring $c : \bigotimes_{i=1}^{\ell} T_i \to k$ there are $t_1 \in T_1, \ldots, t_\ell \in T_\ell$ and $D_1 \subseteq (T_1)_{t_1}, \ldots, D_\ell \subseteq (T_\ell)_{t_\ell}$ such that c is constant on $\bigotimes_{i=1}^{\ell} D_i$ and for every $n \in \omega$ there is $m \geq n$ such that $(D_1, \ldots, D_\ell)$ is (n, m)-dense in $\bigotimes_{i=1}^{\ell} (T_i)_{t_i}$.*

3. The main theorem

Definition 3.1. *Let τ be the type of a finite induced subgraph of $G_{\max}$. We define a topology on $\binom{G_{\max}}{\tau}$ as follows: A set $O \subseteq \binom{G_{\max}}{\tau}$ is* open *if for all $H \in O$ there are open neighborhoods $U_1, \ldots, U_n$ of the vertices of H such that all induced subgraphs of $G_{\max}$ that have exactly one vertex in each U_i and no other vertices are also in O. This topology on $\binom{G_{\max}}{\tau}$ is separable and induced by a complete metric. A coloring $c : \binom{G_{\max}}{\tau} \to 2$ is* continuous *if it is continuous with respect to this topology.*

Theorem 3.2. *For every type τ of a finite induced subgraph of $G_{\max}$, every $G_{\max}$-tree T and every continuous coloring $c : \binom{G(T)}{\tau} \to 2$ there is a $G_{\max}$-subtree S of T such that c is constant on $\binom{G(S)}{\tau}$.*

We prove this theorem in a series of lemmas. First we fix a type τ of a finite induced subgraph of $G_{\max}$ and a continuous coloring $c : \binom{G_{\max}}{\tau} \to 2$.

We may assume that the type τ is skew in the sense that whenever H is a finite induced subgraph of $G_{\max}$, then the tree $T(V(H))$ of initial segments of vertices of H is skew. Otherwise, given a $G_{\max}$-tree T, we choose a skew

$G_{\max}$-subtree S of T and then $\binom{G(S)}{\tau}$ is empty. In particular, c is constant on $\binom{G(S)}{\tau}$. This proves Theorem 3.2 in the case that τ is not skew.

Now fix a $G_{\max}$-tree T. If H is a finite induced subgraph of $G(T)$, let $\Delta(H)$ denote the maximal $\Delta(x,y)$ of two distinct vertices of H. For $n \in \omega$ let $H \restriction n = \{x \restriction n : x \text{ is a vertex of } H\}$

Lemma 3.3. *There is a $G_{\max}$-subtree S of T such that for the induced subgraphs H of $G(S)$ of type τ the color $c(H)$ only depends on $H \restriction (\Delta(H)+1)$.*

Proof. First consider a single finite induced subgraph H of $G(T)$ of type τ. By our definition of $\Delta(H)$, the map $x \mapsto x \restriction (\Delta(H)+1)$ is a bijection from the set $V(H)$ of vertices of H onto $H \restriction (\Delta(H)+1)$. Let $t_1,\dots,t_\ell$ denote the elements of $H \restriction (\Delta(H)+1)$. For all $\overline{x} = (x_1,\dots,x_\ell) \in [T_{t_1}] \times \dots \times [T_{t_\ell}]$ the induced subgraph of $G(T)$ on the set $\{x_1,\dots,x_\ell\}$ is isomorphic to H. By the continuity of c, for all such $\overline{x}$ there are open neighborhoods $U_1^{\overline{x}} \ni x_1,\dots,U_\ell^{\overline{x}} \ni x_\ell$ such that for all $(y_1,\dots,y_\ell) \in U_1^{\overline{x}} \times \dots \times U_\ell^{\overline{x}}$ for the induced subgraph H' of $G(T)$ on the vertices $y_1,\dots,y_\ell$ we have $c(H) = c(H')$.

We may assume that the $U_i^{\overline{x}}$ are basic open sets, i.e., sets of the form $T_r \cap [T]$ for some $r \in T$. Since the space $[T_{t_1}] \times \dots \times [T_{t_\ell}]$ is compact, there is a finite set $F \subseteq [T_{t_1}] \times \dots \times [T_{t_\ell}]$ such that

$$[T_{t_1}] \times \dots \times [T_{t_\ell}] = \bigcup_{\overline{x} \in F} \prod_{i=1}^{\ell} U_i^{\overline{x}}.$$

Hence there is some $m \in \omega$, namely the maximal length of the r's with $[T_r] = U_i^{\overline{x}}$ for some $\overline{x} \in F$ and $i \in \{1,\dots,\ell\}$, such that for all induced subgraphs H' of $G(T)$ with $H' \restriction (\Delta(H)+1) = H \restriction (\Delta(H)+1)$ the color $c(H')$ only depends on $H' \restriction m$.

Since for each $n \in \omega$ there are only finitely many sets of the form $H \restriction m$ where H is a subgraph of $G(T)$, there is a function $f : \omega \to \omega$ such that for every finite induced subgraph H of $G(T)$ with $\Delta(H)+1 = n$, the color $c(H)$ only depends on $H \restriction f(n)$. Now let S be a $G_{\max}$-subtree of T such that whenever $s \in S$ is a splitting node of S of length n, then S has no splitting node t whose length is in the interval $(n, f(n)]$. Now for subgraphs H of $G(S)$ of type τ the color $c(H)$ only depends on $H \restriction (\Delta(H)+1)$. $\square$

Lemma 3.4. *Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \restriction (\Delta(H)+1)$. Then there is a $G_{\max}$-subtree S of T such that for the induced subgraphs H of $G(S)$ of type τ the color $c(H)$ only depends on $H \restriction \Delta(H)$.*

Proof. Let H be an induced subgraph of $G(T)$ of type τ. We call the unique node $t_0 \in \mathrm{Lev}_T(\Delta(H))$ that has at least to incomparable extensions in $H \restriction (\Delta(H)+1)$ the *highest splitting node* of H. Let L be the induced subgraph of G_{t_0} whose vertices are the extensions of t_0 in $H \restriction (\Delta(H)+1)$.

Let $t_1, \dots, t_\ell$ be the elements of $H \restriction (\Delta(H)+1)$ that are not extensions of t_0. Let $\bar{t} = (t_1, \dots, t_\ell)$. The ℓ-tuple $\bar{t}$ determines a coloring $c^{\bar{t}}_{t_0}$ of the induced copies of L in G_{t_0} by two colors:

Given an induced copy L' of L in G_{t_0} let $s_1, \dots, s_k$ be the vertices of L'. Choose

$$(z_1, \dots, z_k, y_1, \dots, y_\ell) \in [T_{s_1}] \times \dots \times [T_{s_k}] \times [T_{t_1}] \times \dots \times [T_{t_\ell}].$$

Now $\{z_1, \dots, z_k, y_1, \dots, y_\ell\}$ is the set of vertices of a copy H' of H in $G(T)$ of type τ. Let $c^{\bar{t}}_{t_0}(L') = c(H')$. Since $c(H')$ depends only on $H' \restriction (\Delta(H)+1)$, $c(H')$ does not depend on the choices of the z_i and y_j.

We construct the required subtree S of T. We do that by choosing a fusion sequence $(T_k)_{k\in\omega}$ along with a strictly increasing sequence $(m_k)_{k\in\omega}$ of natural numbers witnessing that the T_k form a fusion sequence. First let $T_0 = T$ and $m_0 = 0$. Suppose T_k and m_k have been chosen already. Some book-keeping device tells us that for are certain node $t \in \mathrm{Lev}_{T_k}(m_k)$ and a certain finite ordered graph B there has to be an extension $t_0 \in T_k$ of length $< m_{k+1}$ such that F embeds into $G^{T_{k+1}}_{t_0}$.

We will choose T_{k+1} and m_{k+1} such that t is the only element of $\mathrm{Lev}_{T_k}(m_k)$ that has an extension of length $< m_{k+1}$ in T_{k_1} that is a splitting node. Also, t will only have a single extension of length $< m_{k+1}$ in T_{k+1} that is a splitting node. In particular, we know in advance the size of $\mathrm{Lev}_{T_{k+1}}(|t_0|+1)$. This gives us a finte upper bound on the number of colorings of the form $c^{\bar{t}}_{t_0}$ of the induced copies of L in $G^{T_k}_{t_0}$. Let N denote this upper bound.

By the Nešetřil-Rödl theorem there is a finite ordered graph D such that for every collection of at most N colorings by two colors of the induced copies of L in D, there is an induced copy B' of B in D such that all induced copies of L in B' have the same colors with respect to all of the N colorings.

Since T_k is a $G_{\max}$-tree, there is an extension t_0 of t in T_k such that D embeds into $G^{T_k}_{t_0}$. Let $m_{k+1} = |t_0|+1$. Choose a set $Z \subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ such that each element of $\mathrm{Lev}_{T_k}(m_{k+1})$ other than t has exactly one extension in Z. Now for all $\bar{t} = (t_1, \dots, t_\ell) \in Z^\ell$ such that there is an induced subgraph H of $G(T_k)$ with

$$H \restriction |t_0| = \{t_0, t_1 \restriction |t_0|, \dots, t_\ell \restriction |t_0|\}$$

we consider the coloring $c^{\bar{t}}_{t_0}$. By the choice of D, $G_{t_0}(T_k)$ contains an induced copy B' of B such that all the relevant colorings $c^{\bar{t}}_{t_0}$ are constant on the set of induced copies of L in B'.

Let Y be the set of vertices of B' and let $T_{k+1} = (T_k)_{Y\cup Z}$. This finishes the recursive construction of the trees T_k and of the natural numbers m_k. Let S be the fusion $\bigcap_{k\in\omega} T_k$ of the sequence $(T_k)_{k\in\omega}$. Then S is a $G_{\max}$-tree by our book-keeping.

Let H be an induced subgraph of $G(S)$ of type τ. Let t_0 be the highest splitting node of H. Choose $k \in \omega$ such that $m_k \le |t_0| < m_{k+1}$. Let $t_1, \dots, t_\ell$ denote the elements of $H \restriction (|t_0|+1)$ and let $\bar{t} = (t_1, \dots, t_\ell)$. Since

S is skew by construction, the set $\{t_1, \dots, t_\ell\}$ is uniquely determined by $H \upharpoonright |t_0|$. But since T_{k+1} was chosen so that all induced copies of L in $G_{t_0}^{T_{k+1}}$ have the same color with respect to $c_{t_0}^{\bar{t}}$, $c(H)$ actually does not depend on the copy of L inside $G_{t_0}^{T_{k+1}}$ that lives on the vertices $\mathrm{succ}_{T_{k+1}}(t_0)$. Any other copy of L in $G_{t_0}^{T_{k+1}}$ would yield the same color. Now $G_{t_0}^{T_{k+1}} = G_{t_0}^S$. It follows that $c(H)$ only depends on $H \upharpoonright |t_0| = H \upharpoonright \Delta(H)$. $\square$

For a finite induced subgraph H of $G(T)$ of type τ let $\Delta'(H)$ denote the minimal $n \in \omega$ such that $|H \upharpoonright n| = |H \upharpoonright \Delta(H)|$.

Lemma 3.5. *Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \upharpoonright \Delta(H)$. Let H_0 be an induced subgraph of $G(T)$ of type τ. Let $n = \Delta'(H_0)$. Call an induced copy H of H_0 in $G(T)$ of type τ* compatible *with H_0 if $H \upharpoonright n = H_0 \upharpoonright n$. Then there is a $G_{\max}$-subtree S of T such that $\mathrm{Lev}_S(n) = \mathrm{Lev}_T(n)$ and c is constant on the set of graphs compatible with H_0.*

Proof. Let $t_1, \dots, t_\ell$ be an enumeration of $H_0 \upharpoonright n$ without repetition such that the highest splitting node of $T(H_0)$ is an extension of t_1. Note that for every graph H of type τ that is compatible with H_0, the highest splitting node of H is an extension of t_1. We may assume that for every splitting node t of T_{t_1} there is a graph H whose highest splitting node is t. This can be achieved by thinning out the tree T above level n in order to make sure that the graphs G_t are sufficiently large for all splitting t nodes of T that extend t_1.

We define an auxiliary coloring

$$\bar{c} : \bigotimes_{i=1}^{\ell} T_{t_i} \to 2$$

as follows:

Given $(s_1, \dots, s_\ell) \in \bigotimes_{i=1}^{\ell} T_{t_i}$, for every $i \in \{1, \dots, \ell\}$ let x_i be the minimal vertex of $G(T)$ such that $s_i \subset x_i$. Let $m \geq n$ be minimal such that $x_1 \upharpoonright m$ is a splitting node of T. Let $s' = x_1 \upharpoonright m$ and choose a graph H that is compatible with H_0 such that s' is the highest splitting node of H. The graph H exists by our assumptions on T. We can choose H in such a way that $x_2, \dots, x_\ell$ are vertices of H. Now let $\bar{c}(s_1, \dots, s_\ell) = c(H)$.

By our assumptions on T, the color $c(H)$ only depends on $H \upharpoonright m$. This means that $\bar{c}(s_1, \dots, s_\ell)$ depends on our choice of m and the sequences $x_1 \upharpoonright m, x_2 \upharpoonright m, \dots, x_\ell \upharpoonright m$ and on the fact that $x_1 \upharpoonright m$ is the highest splitting node of H, but it does not depend on the choice of H above the m-th level.

By the Halpern-Läuchli theorem there are $r_1 \in T_{t_1}, \dots, r_\ell \in T_{t_\ell}$ and sets $D_1 \subseteq T_{r_1}, \dots, D_\ell \subseteq T_{r_\ell}$ such that $\bar{c}$ is constant on $\bigotimes_{i=1}^{\ell} D_i$ and for all $m \geq n$ there is $k \geq m$ such that $(D_1, \dots, D_\ell)$ is (m, k)-dense in $\bigoplus_{i=1}^{\ell} (T_i)_{t_i}$.

We now construct a fusion sequence $(T_j)_{j \in \omega}$ and a strictly increasing sequence $(m_j)_{j \in \omega}$ of natural numbers as follows:

Let

$$T_0 = \bigcup_{i=1}^{\ell} T_{r_i} \cup \{t \in T : t \text{ is not comparable with any } t_i,\ i \in \{1, \dots, \ell\}\}$$

and $m_0 = n$. Suppose we have chosen T_j and m_j for some $j \in \omega$. Choose $m > m_j$ such that all $t \in \mathrm{Lev}_T(m_j)$ have an extension of length $< m$ that is a splitting node in T_j. Let $k \geq m$ be such that $(D_1, \dots, D_\ell)$ is (m, k)-dense in $\bigoplus_{i=1}^{\ell} (T_i)_{t_i}$.

Now choose a set $Z \subseteq \mathrm{Lev}_{T_j}(k)$ such that the following hold:

(1) For all $t \in \mathrm{Lev}_{T_j}(m)$, if t and t_1 are incomparable, then t has exactly one extension in Z.
(2) For all $t \in \mathrm{Lev}_{T_j}(m_j)$, if $t_1 \subseteq t$, then t has exactly one extension in Z.
(3) For each $i \in \{1, \dots, \ell\}$, $T_{t_i} \cap Z \subseteq D_i$.

Finally, we choose m_{j+1} such that each element of $Z \cap (T_j)_{t_1}$ has an extension of length $< m_{j+1}$ that is a splitting node of T_j. We choose $Z' \subseteq \mathrm{Lev}_{T_j}(m_{j+1})$ such that the following hold:

(1′) Let $t \in Z \cap (T_j)_{t_1}$ and let s be the minimal splitting node of T_j that extends t. Then every immediate successor of s has exactly one extension in Z'.
(2′) For every $r \in Z' \cap (T_j)_{t_1}$ there are a splitting node s of T_j and $t \in Z$ such that $t \subseteq s \subseteq r$.
(3′) For all $t \in Z$ that are incomparable with t_1, $Z' \cap (T_j)_t$ consists of the lexicographically minimal extensions of elements of $Z \cap (T_j)_t$ in $\mathrm{Lev}_{(T_j)_{t_i}}(m_{j+1})$.

Now let

$$T_{j+1} = \{t \in T_j : t \text{ is comparable with an element of } Z'\}.$$

This finishes the construction of the sequences $(T_j)_{j\in\omega}$ and $(m_j)_{j\in\omega}$.

Now let $S = \bigcap_{j\in\omega} T_j$. Observe that S is generated by the set $\bigcup_{j\in\omega} T_j \restriction m_j$.

We now show that c is constant on the set of all induced subgraphs H of $G(S)$ that are compatible with H_0. Let H be an induced subgraph of $G(S)$ that is compatible with H_0. Let s be the highest splitting node of H and chose $j \in \omega$ such that $m_j < |s| < m_{j+1}$. The construction of S guarantees that no restriction of s to some number in the interval $[m_j, |s|)$ is a splitting node of S.

In the construction of m_{j+1} and T_{j+1} we chose integers m and k such that $m_j < m \leq k < m_{j+1}$. T_{j+1} has no splitting node whose length is in the interval $[m_j, k)$ and thus $k \leq |s|$. Now for each vertex x of H with $t_1 \not\subseteq x$, $x \restriction |s|$ is the lexicographically minimal extension of $x \restriction k$ in T. But $\mathrm{Lev}_{T_{j+1}}(k) \cap \bigcup_{i=1}^{\ell} T_{t_i} \subseteq \bigcup_{i=1}^{\ell} D_i$. It follows that $c(H)$ is equal to the constant color that $\bar{c}$ assumes on the set $\bigoplus_{i=1}^{\ell} D_i$. This shows that c is constant on the set of induced subgraphs of $G(S)$ compatible with H_0. $\square$

Lemma 3.6. *Assume that for all finite induced subgraphs H of $G(T)$ of type τ the color $c(H)$ only depends on $H \restriction \Delta(H)$. Then there is a $G_{\max}$-subgraph S of T such that for all finite induced subgraphs H of type τ, $c(H)$ only depends on $H \restriction \Delta'(H)$.*

Proof. We may assume that for every induced subgraph H of $G(T)$ of type τ and every splitting node $t \in T$, H embeds into G_t^T. Also, if for every subgraph H of $G(T)$ of type τ the tree of initial segments of vertices of H only has a single splitting node, Lemma 3.5 gives us a $G_{\max}$-subtree S of T such that c is constant on $G(S)$. Hence we can assume that the tree of initial segments of a graph H of type τ has at least two different splitting nodes.

Again we construct a fusion sequence $(T_k)_{k\in\omega}$ and a sequence $(m_k)_{k\in\omega}$ witnessing this. In our construction we make sure that for all $k \in \omega$, T_{k+1} has exactly one splitting node whose length is in the interval $[m_k, m_{k+1})$ and the length of this splitting node is exactly $m_{k+1} - 1$.

Let t be the minimal splitting node of T. Let $T_0 = T$ and $m_0 = |t| + 1$. Suppose T_k and m_k have been chosen. By Lemma 3.5, for every finite induced subgraph H_0 of $G(T)$ of type τ such that $\Delta'(H_0) = m_k$ there is a $G_{\max}$-subtree T_k' of T_k such all copies H of H_0 of type τ in $G(T_k)$ that are compatible with H_0 have the same color $c(H)$. Iterating this argument finitely often, we find a G_{max}-subtree T_k'' of T_k such that for all induced subgraphs H of $G(T)$ of type τ with $\Delta'(H) = m_k$ the color $c(H)$ only depends on $H \restriction m_k$.

Now some book-keeping device tells us that a certain $t \in \mathrm{Lev}_{T_k}(m_k)$ should have an extension t_0 of length $< m_{k+1}$ such that a certain finite ordered graph F embeds into $G_{t_0}^{T_{k+1}}$. We choose an extension t_0 of t such that F embeds into $G_{t_0}^{T_{k+1}}$ and let $m_{k+1} = |t_0| + 1$. Let $Z \subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ be such that $\mathrm{succ}_{T_k}(t_0) \subseteq Z$ and each $s \in \mathrm{Lev}_{T_k}(m_k) \setminus \{t\}$ has exactly one extension in Z. Now let $T_{k+1} = (T_k)_Z$. This finishes the definition of the fusion sequence $(T_k)_{k\in\omega}$ and the sequence $(m_k)_{k\in\omega}$.

Finally let $S = \bigcap_{k\in\omega} T_k$. By our book-keeping, S is a $G_{\max}$-tree. Whenever H is an induced subgraph of $G(S)$ of type τ there is a unique $k \in \omega$ such that $\Delta'(H) = m_k$. Since S is a $G_{\max}$-subtree of T_{k+1}, by the choice of T_{k+1}, the color $c(H)$ only depends on $H \restriction \Delta'(H)$. □

Proof of Theorem 3.2. Let ℓ denote the number of vertices of graphs of type τ. We prove the theorem by induction on ℓ. If $\ell = 1$, then we just observe that every continuous coloring c of the subgraphs of $G(T)$ of type τ is constant on an open set $U \subseteq [T]$.

Now assume that $\ell > 1$ and for all types τ' of subgraphs of $G(T)$ with less than ℓ vertices the theorem holds. By Lemma 3.6 there is a $G_{\max}$-subtree T' of T such that on T' the color $c(H)$ of a graph of type τ only depends on $H \restriction \Delta'(H)$. Given such a graph H, let $t_1, \dots, t_k$ denote the distinct elements of $H \restriction \Delta'(H)$ and choose $(x_1, \dots, x_k) \in [T'_{t_1}] \times \dots \times [T'_{t_k}]$. Now the type τ' of the induced subgraph of $G(T)$ on the vertices $x_1, \dots, x_k$ only depends on τ.

We define a coloring c' on the set of all subgraph H' of $G(T')$ of type τ'. Given such a subgraph, let H be a graph of τ such that $H \restriction \Delta'(H) = H' \restriction (\Delta(H') + 1)$. Such a graph H exists since T' is a $G_{\max}$-tree. Now let $c'(H') = c(H)$. By our assumption on T', $c(H)$ only depends on $H \restriction \Delta'(H)$ and hence $c'(H')$ is independent of the choice of H. Clearly, graph of type τ' have less than ℓ vertices and hence, by our inductive hypothesis, there is a $G_{\max}$-subtree S of T' such that c' is constant on subgraphs of S of type τ'. But now c is constant on subgraphs of S of type τ. This finishes the proof of the theorem. □

3.1. **The Baire measurable case.**

Definition 3.7. *Let τ be the type of a finite induced subgraph of $G_{\max}$ and let T be a $G_{\max}$-tree. A coloring $c : \binom{G(T)}{\tau} \to 2$ is* Baire measurable *if the sets $c^{-1}(0)$ and $c^{-1}(1)$ have the Baire property in the Polish space $\binom{G(T)}{\tau}$.*

Our main Theorem 3.2 can be extended to Baire measurable colorings using standard methods from descriptive set theory.

We need the following lemma.

Lemma 3.8. *Let τ be the type of a nonempty finite induced subgraph of $G_{\max}$ and let $c : \binom{G(T)}{\tau} \to 2$ be a Baire measurable measurable coloring. Then there is a $G_{\max}$-subtree S of T such that c is continuous on $\binom{G(S)}{\tau}$.*

Proof. We choose open sets $U, V \subseteq \binom{G(T)}{\tau}$ such that the symmetric differences $c^{-1}(0) \triangle U$ and $c^{-1}(1) \triangle V$ are meager. Let $(N_n)_{n\in\omega}$ be a sequence of closed nowhere dense subsets of $\binom{G(T)}{\tau}$ such that

$$(c^{-1}(0) \triangle U) \cup (c^{-1}(1) \triangle V) \subseteq \bigcup_{n\in\omega} N_n.$$

Our goal is to construct a $G_{\max}$-subtree S of T such that $\binom{G(S)}{\tau}$ is disjoint from $\bigcup_{n\in\omega} N_n$. In this case, the preimages of 0 and 1 of the restriction of c to the set $\binom{G(S)}{\tau}$ are the open subsets $U \cap \binom{G(S)}{\tau}$ and $V \cap \binom{G(S)}{\tau}$ of $\binom{G(S)}{\tau}$. It follows that c is continuous on $\binom{G(S)}{\tau}$.

It remains to find the $G_{\max}$-subtree S that avoids the set $\bigcup_{n\in\omega} N_n$. We construct a fusion sequence $(T_k)_{k\in\omega}$ of $G_{\max}$-subtrees of T and a strictly increasing sequence $(m_k)_{k\in\omega}$ of natural numbers and then put $S = \bigcap_{k\in\omega} T_k$.

Suppose T_k and m_k have already been chosen. We assume that for all $t \in \mathrm{Lev}_{T_k}(m_k)$ and all $s \in T$ with $t \subseteq s$ we have $s \in T_k$. Some bookkeeping will tell us that we have to find a splitting node s above a certain $t \in \mathrm{Lev}_{T_k}(m_k)$ such that a certain finite ordered graph H embeds into $G_s^{T_{k+1}}$. Since T_k is a $G_{\max}$-tree, there is $m > m_k$ such that t has an extension $s \in T_k$ of length $< m$ such that H embeds into $G_s^{T_k}$.

Now suppose H is a subgraph of $G(T_k)$ of type τ such that $\Delta(H) < m$. Let ℓ be the number of vertices of H. The set $H \restriction m$ determines an open subset O of $\binom{G(T)}{\tau}$. The set $\bigcup_{n\leq k} N_n$ is closed and nowhere dense in $\binom{G(T)}{\tau}$. Hence there is a nonempty open subset of O that is disjoint from $\bigcup_{n\leq k} N_n$. It follows that the ℓ elements of $H \restriction m$ have extensions $s_1, \dots, s_\ell \in T_k$ such that the open subset of $\binom{G(T)}{\tau}$ determined by $s_1, \dots, s_\ell$ is disjoint from $\bigcup_{n\leq k} N_n$. We may assume that $s_1, \dots, s_\ell$ are all of the same length $m' > m$.

Let $Z \subseteq \mathrm{Lev}_{T_k}(m')$ be a set that contains exactly one extension of every element of $\mathrm{Lev}_{T_k}(m)$ and in particular the elements $s_1, \dots, s_\ell$. Now consider the $G_{\max}$-tree T' consisting of all elements of T_k that are comparable to one of the elements of Z. Whenever H' is a subgraph of $G(T')$ of type τ with $H' \restriction m = H \restriction m$, then $H' \restriction m' = \{s_1, \dots, s_\ell\}$. In particular, H' is not an element of $\bigcup_{n\leq k} N_n$.

We can iterate this argument and obtain $m_{k+1} > m'$ and a set $X \subseteq \mathrm{Lev}_{T_k}(m_{k+1})$ with the following property: If H' is a subgraph of $G(T_k)$ with $\Delta(H') < m$ such that $H' \restriction m_{k+1} \subseteq X$, then H' is not an element of $\bigcup_{n\leq k} N_n$.

Let

$$T_{k+1} = \{t \in T_k : \exists s \in X (s \subseteq t \vee t \subseteq s)\}.$$

Now for every subgraph H' of $G(T_{k+1})$ of type τ with $\Delta(H') < m$ we have $H' \notin \bigcup_{n\leq k} N_n$. This finishes the recursive definition of the sequences $(T_k)_{k\in\omega}$ und $(m_k)_{k\in\omega}$.

Finally let $S = \bigcap_{k\in\omega} T_k$. We use the book-keeping in the construction of the T_k to make sure that S is a $G_{\max}$-tree. Let $n \in \omega$ and suppose H is a subgraph of $G(S)$ of type τ. Then there is $k \in \omega$ such that $\Delta(H) < m_k$. We can choose $k \geq n$. Note that $\mathrm{Lev}_S(m_k) = \mathrm{Lev}_{T_k}(m_k)$. By the choice of T_{k+1} and since $S \subseteq T_{k+1}$, $H \notin \bigcup_{i\leq k} N_i$. In particular, $H \notin N_n$. This shows that $\binom{G(S)}{\tau}$ is disjoint from $\bigcup_{n\leq\omega} N_n$. It follows that c is continuous on the set $\binom{G(S)}{\tau}$. □

The generalization of Theorem 3.2 to Baire measurable colorings now follows easily from Lemma 3.8.

Theorem 3.9. *For every type τ of a finite induced subgraph of $G_{\max}$, every $G_{\max}$-tree T and every Baire measurable coloring $c : \binom{G(T)}{\tau} \to 2$ there is a $G_{\max}$-subtree S of T such that c is constant on $\binom{G(S)}{\tau}$.*

Proof. By Lemma 3.8, there is a $G_{\max}$-subtree T' of T such that c is continuous on $\binom{G(T')}{\tau}$. Now by Theorem 3.2 there is a $G_{\max}$-subtree S of T' such that c is constant on $\binom{G(S)}{\tau}$. □

Let A be a finite ordered graph. Let $k(A)$ denote the number of different (skew) types of A.

Theorem 3.10. *For every finite graph A and UMP G, every finite Baire measurable partition of $\binom{G}{A}$, there is a closed ordered copy G' of G in G such that the type of each $B \in \binom{G}{A}$ determines its cell in the partition.*

This theorem follows by iterating Theorem 3.9.

As proved in [3], a 4-saturated finite graph has no modular partitions except the trivial ones, so in particular a finite ordered graph which without the ordering is 4-saturated has exactly one possible type in any $T_{\max}$ tree. This gives:

Theorem 3.11. *The probability that an ordered graph A on $\{0, 1, \ldots, n-1\}$ satisfies*

$$G \rightarrowtail_{Baire} (G)^A$$

converges to 1 as n grows to infinity.

Proof. It is a standard fact in random graphs that the probability that a random graph on n is 4-saturated converges to 1. □

References

[1] A. Blass, *A partition theorem for perfect sets*, Proc. Amer. Math. Soc. **82**(2), 271–277 (1981)

[2] S. Frick, *Continuous Ramsey theory in higher dimensions*, Dissertation, Freie Universität Berlin (2008)

[3] S. Geschke, *Clopen graphs*, Fundamenta Mathematicae 220, No. 2 (2013), 155–189

[4] S. Geschke. S. Huber, *Partitions of closed subgraphs*, in preparation

[5] S. Geschke, M. Kojman, W. Kubis and R. Schipperus. *Convex decompositions in the plane and continuous pair colorings of the irrationals.* Israel J. Math. 131 (2002), 285–317

[6] S. Geschke, M. Goldstern, M. Kojman, *Continuous Ramsey theory on Polish spaces and covering the plane by functions*, Journal of Mathematical Logic, Vol. 4, No. 2 (2004), 109–145

[7] A. Hajnal and P. Komjáth. *Embedding graphs into colored graphs.* Trans. Amer. Math. Soc. 307 (1988), no. 1, 395–409.

[8] J. Halpern, H. Läuchli, *A partition theorem*, Trans. Amer. Math. Soc. 307 (1988), 395–409

[9] A. Kechris, **Classical descriptive set theory,** Graduate Texts in Mathematics 156, Springer-Verlag (1995)

[10] J. Nešetřil, V. Rödl, *A structural generalization of the Ramsey theorem*, Bull. Amer. Math. Soc. Volume 83, Number 1 (1977), 127–128

[11] J. Nešetřil. *For Graphs There Are Only Four Types of Hereditary Ramsey Classesand.* J. Comb. Theory. Series B 46 (1989), 127-132

[12] M. Pouzet, N. Sauer, *Edge partitions of the Rado graph*, Combinatorica 16 (4) (1996), 1–16

[13] N. Sauer, *Coloring subgraphs of the Rado graph*, Combinatorica 26.2 (2006), 231–253

[14] Y. Sheu, *Partition properties and Halpern-Läuchli theorem on the $C_{\min}$ forcing*, Dissertation, University of Florida (2005)

E-mail address: frick.stefanie@googlemail.com

Department of Mathematics, University of Hamburg, Bundesstr. 55, 20249 Hamburg, Germany
E-mail address: stefan.geschke@uni-hamburg.de

Department of Mathematics, Ben-Gurion University of the Negev, P.O.B. 653, Be'er Sheva, 84105 Israel
E-mail address: kojman@math.bgu.ac.il