Note

A Short Proof of Fleischner's Theorem

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Abstract

We give a short proof of the fact that the square of a finite graph is Hamiltonian.

1 Introduction

The *square* G^2 of a graph G is the graph on $V(G)$ in which two vertices are adjacent if and only if they have distance at most 2 in G. In 1974, Fleischner [3, 4] proved that the square of every 2-connected finite graph has a Hamilton cycle. Thomassen [7] extended this fact to locally finite 1-ended graphs, where a Hamilton cycle is taken to be an infinite path containing all vertices. Using Thomassen's method, Říha (see [8] or [2]) produced a shorter proof of Fleischner's Theorem. History repeated itself, and once again the study of infinite graphs led to a new proof of Fleischner's Theorem: a proof is presented here that is shorter than Říha's, and uses techniques developed for the recent extension of Fleischner's Theorem to locally finite graphs with any number of ends[1].

In [6] the present proof is adapted to give a short proof of another theorem of Fleischner [3], stating that the total graph of every finite 2-edge-connected graph has a Hamilton cycle.

2 Definitions

We will be using the terminology of [2]. Let G be a multigraph, and J a walk in G. A *pass* of J through a vertex x is a subwalk of J of the form $uexfv$, where e and f are edges. By *lifting* this pass we mean replacing it in J by the walk ugv, where g is a u-v edge, if $u \neq v$, or by the trivial walk u if $u = v$ (in fact, the latter case will never occur).

A *double edge* is a pair of parallel edges, and a *multipath* is a multigraph obtained from a path by replacing some of its edges by double edges. If $C \subseteq G$ are multigraphs, then a C*-trail* in G is either a path having precisely its endvertices (but no edge) in common with C, or a cycle having precisely one vertex in common with C. A vertex y on some cycle C is called C*-bound* if all neighbours of y lie on C.

[1]Settling a problem of Diestel [1], it is shown in [5] that the square of every locally finite 2-connected graph contains a *Hamilton circle*, a homeomorphic image of the complex unit circle S^1 in the topological space $|G|$ formed by G and all its ends.

3 The proof

We will use the following lemma of Říha [8]. For the convenience of the reader the proof is repeated here.

Lemma 1. *If G is a 2-connected finite graph and $x \in V(G)$, then there is a cycle $C \subseteq G$ that contains x as well as a C-bound vertex $y \neq x$.*

Proof. As G is 2-connected, it contains a cycle C' that contains x. If C' is a Hamilton cycle there is nothing more to show, so let D be a component of $G - C'$. Assume that C' and D are chosen so that $|D|$ is minimal. Easily, C' contains a path P' between two distinct neighbours u, v of D whose interior $\mathring{P}'$ does not contain x and has no neighbour in D. Replacing P' in C' by a u-v–path through D, we obtain a cycle C that contains x and a vertex $y \in D$. By the minimality of $|D|$ and the choice of P', y has no neighbour in $G - C$, so C satisfies the assertion of the lemma. □

We will prove Fleischner's Theorem in the following stronger form, which is similar to the assertion proved by Říha [8].

Theorem 1. *If G is a 2-connected finite graph and $x \in V(G)$, then G^2 has a Hamilton cycle whose edges at x lie in $E(G)$.*

Proof. We perform induction on $|G|$. For $|G| = 3$ the assertion is trivial. For $|G| > 3$, let C be a cycle as provided by Lemma 1. Our first aim is to define, for every component D of $G - C$, a set of C-trails in $G^2 + E'$, where E' will be a set of additional edges parallel to edges of G. Every vertex of D will lie in exactly one such trail, and every edge of an element of such a trail that is incident with a vertex of C will lie in $E(G)$ or in E'.

If D consists of a single vertex u, we pick any C-trail in G containing u, and let E_D be the set of its two edges. If $|D| > 1$, let $\tilde{D}$ be the (2-connected) graph obtained from G by contracting $G - D$ to a vertex $\tilde{x}$. Applying the induction hypothesis to $\tilde{D}$, we obtain a Hamilton cycle $\tilde{H}$ of $\tilde{D}^2$ whose edges at $\tilde{x}$ lie in $E(\tilde{D})$. Write $\tilde{E}$ for the set of those edges of $\tilde{H}$ that are not edges of G^2. Replacing these by edges of G or new edges $e' \in E'$, we shall turn $E(\tilde{H})$ into the edge set of a union of C-trails. Consider an edge $uv \in \tilde{E}$, with $u \in D$. Then either $v = \tilde{x}$, or u, v have distance at most 2 in $\tilde{D}$ but not in G, and are hence neighbours of $\tilde{x}$ in $\tilde{D}$. In either case, G contains a u–C edge. Let E_D be obtained from $E(\tilde{H}) \setminus \tilde{E}$ by adding at every vertex $u \in D$ as many u–C edges as u has incident edges in $\tilde{E}$; if u has two incident edges in $\tilde{E}$ but sends only one edge e to C, we add both e and a new edge e' parallel to e. Then every vertex of D has the same degree (two) in $(V(G), E_D)$ as in $\tilde{H}$, so E_D is the edge set of a union of C-trails. Let $G' := (V(G), E(C) \cup \bigcup_D E_D)$ be the union of C and all those trails, for all components D of $G - C$ together.

Let y be a C-bound vertex of C and pick a vertex z and edges d_1, d_2, g_1, g_2 of C, so that $C = xg_1z \dots d_1yd_2 \dots g_2x$ (the vertices and edges named here need not be distinct). We will add parallel edges to some edges of $C - g_1$, to turn G' into an eulerian multigraph $G_{\emptyset}$ — i.e. a connected multigraph in which every vertex has even degree (and which therefore has an Euler tour [2]). Every vertex in $G' - C$ already has degree 2. In order to obtain even degrees at the vertices in C we consider these vertices in reverse order, starting with x and ending with z. Let u be the vertex currently considered, and let v be the vertex to be considered next. Add a new edge parallel to uv if and only if u has odd degree in the multigraph obtained from G' so far. When finally $u = z$ is considered, every other vertex has even degree, so by the "hand-shaking lemma" z must have even degree too and no edge parallel to g_1 will be added. Let $G_{\between}$ be the resulting multigraph, and let $C_{\between} = G_{\between}[V(C)]$.

If g_2 has a parallel edge g_2' in $G_{\between}$, then delete both g_2, g_2'. If g_2 has no parallel edge, and d_2 has a parallel edge d_2', then delete both d_2 and d_2'. Let $G_{\not\between}$ be the resulting (eulerian) multigraph. If g_2 has been deleted, then let P_3 be the multipath $C_{\between} - \{g_2, g_2'\}$. If not, let P_1 be the maximal multipath in $C_{\between}$ with endvertices x, y containing g_1, and let P_2 be the multipath containing all edges in $E(C_{\between} \cap G_{\not\between}) - E(P_1)$ (Figure 1).

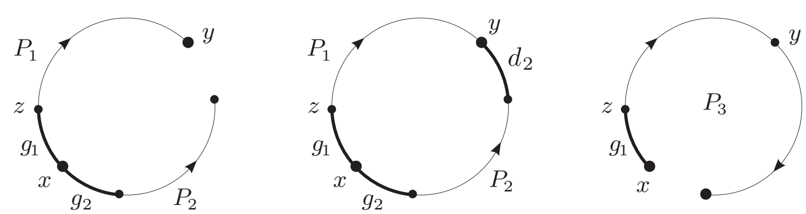

Figure 1: The paths P_i (three cases). The bold edges are known to be single.

Our plan is to find an Euler tour J' of $G_{\not\between}$ that can be transformed into a Hamilton cycle of G^2. In order to endow J' with the required properties we will derive it from an Euler tour of an auxiliary multigraph, which we define next.

For every i such that P_i has been defined, do the following. Write $P_i = x_0^i x_1^i \ldots x_{l_i}^i$ with $x_0^i = x$, and e_j^i or just e_j for the x_{j-1}^i–x_j^i edge of P_i in $E(C)$. Its parallel edge, if it exists, will again be denoted by e_j' (when i is fixed). Now for $j = 1, \ldots, l_i - 1$, if e_{j+1}' exists, replace e_j and e_{j+1}' by a new edge f_j joining x_{j-1} to x_{j+1}; we say that f_j *represents* the walk $x_{j-1} e_j x_j e_{j+1}' x_{j+1}$ (Figure 2). Note that every such replacement leaves the current multigraph connected, and it preserves the parity of all degrees. Hence, the multigraph $G^{\lhd}$ finally obtained by all these replacements is eulerian, so pick an Euler tour J of $G^{\lhd}$. Transform J into an Euler tour J' of $G_{\not\between}$ by replacing every edge in $E(J) - E(G_{\not\between})$ by the walk it represents.

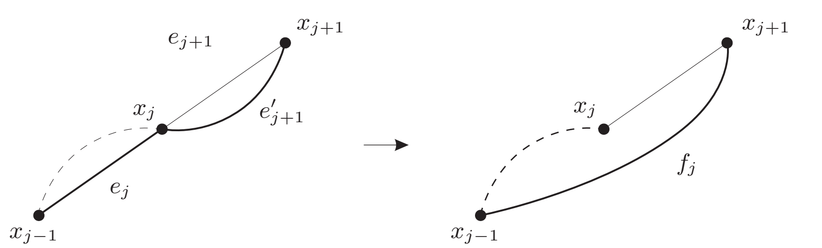

Figure 2: Replacing e_j and e_{j+1}' by a new edge f_j.

Our next aim is to perform some lifts in J' to transform it into a Hamilton cycle. To this end, we will now mark some passes for later lifting. Start by marking all passes of J' through x except for one arbitrarily chosen pass. We want to mark some more passes, so that for any vertex $v \in V(C) - x$ the following assertion holds:

for any i, j, if $v = x^i_j$ then all passes of J' through v are marked except for the pass containing e^i_j. (1)

This is easy to satisfy for $v \neq y$, as there is precisely one pair i, j so that $v = x^i_j$ in that case. A difficulty can only arise if $v = y = x^1_{l_1} = x^2_{l_2}$, in case both P_1 and P_2 contain y. By the definition of the P_i, this case only materialises if there are no edges g'_2, f'_2 in $G_{\between}$, and as y is C-bound, it has degree at most 3 and hence degree 2 in $G_{\between}$ in that case. But then, there is only one pass of J' through v, which consists of $e^1_{l_1}, e^2_{l_2}$, and leaving it unmarked satisfies (1).

So we assume that (1) holds, and now we claim that

for every edge $e = uv$ in J', at most one of the two passes of J' that contain e is marked, and moreover if $u = x$, then the pass of J' through v containing e is unmarked. (2)

This is clear for edges in $E(G_{\not\between}) - E(C_{\between})$, so pick an $e \in P_i$. If $e = e_j$ for some j, then by (1) the pass of J' through x^i_j containing e is unmarked; in particular, if e is incident with $x = x^i_0$, then $j = 1$ and the pass of J' through x^i_1 containing e is unmarked. If $e = e'_j$, then e is not incident with x by the construction of $G_{\not\between}$, and an edge f_{j-1} was defined to represent the walk $x_{j-2}e_{j-1}x_{j-1}e'_jx_j$. Since J contained f_{j-1}, this walk is a pass in J'. This pass is unmarked by (1), because it is a pass through x_{j-1} containing e_{j-1}.

So we proved our claim, which implies that no two marked passes share an edge. Thus we can now lift each marked pass of J' to an edge of G^2, to obtain a new closed walk H' in $G^2 + E'$. Every vertex of G is traversed precisely once by H', since by (1) we marked, and eventually lifted, for each vertex v of G all passes of J' through v except precisely one pass. (This is trivially true for a vertex u in $G - C$, as there is only one pass of J' through u and this pass was not marked.) In particular, H' cannot contain any pair of parallel edges, so we can replace every edge e' in H' that is parallel to an edge e of G by e to obtain a Hamilton cycle H of G^2. Since by the second part of (2) no edge incident with x was lifted at its other end, both edges of H at x lie in G as desired.

□

4 Total graphs

The *subdivision graph* $S(G)$ of a graph G is the bipartite graph with partition classes $V(G), E(G)$ where $x \in V(G)$ and $e \in E(G)$ are joined by an edge if x is incident with e in G. The *total graph* $T(G)$ of G is the square of $S(G)$; equivalently, $T(G)$ is the graph on $V(G) \cup E(G)$ where two vertices are adjacent if the respective objects are adjacent or incident in G. Fleischner [3] proved that:

Theorem 2. *If G is a finite, 2-edge-connected graph then $T(G)$ has a Hamilton cycle.*

In [6] the proof of Section 3 was adapted to give a short proof of Theorem 2, exploiting the fact that $T(G)$ is the square of a graph. We do not repeat that proof here, but we will point out the main differences to the proof in Section 3.

Instead of looking for a cycle C with a C-bound vertex, we just pick any cycle C in G; the reason is that later we will consider the subdivision graph C' of C, and then any of the vertices of degree 2 that will arise after subdividing an edge will be C'-bound. Again we use induction, and apply the induction hypothesis to all components of $S(G)-S(C')$ to obtain a set of C'-trails covering all vertices in $S(G)-S(C')$ (this step is more complicated though). After constructing the C'-trails we have a very similar situation to that in the proof of Section 3, and we can proceed in the same way; the fact that we have a big choice of C'-bound vertices only simplifies the proof.

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