Infinite Highly Connected Planar Graphs of Large Girth

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Abstract

We construct infinite planar graphs of arbitrarily large connectivity and girth, and study their separation properties. These graphs have no thick end but continuum many thin ones. Every finite cycle separates them, but they corroborate Diestel's conjecture that every k-connected locally finite graph contains a possibly infinite cycle — see [3] — whose deletion leaves it $(k-3)$-connected.

1 Introduction

For finite graphs, it is well known that large enough connectivity forces arbitrarily large complete minors [1]. For infinite graphs, this is not true: there are planar graphs of arbitrarily high finite connectivity. (Planar graphs cannot be infinitely connected, as infinite connectivity clearly forces an infinite complete topological minor.) In fact we can go further and demand that their girth, as well as their connectivity, exceed any given finite bound while retaining planarity.

A simple example of a planar k-connected graph of girth k, for $k > 4$, is the k-regular tessellation of the hyperbolic plane by k-gons. This graph clearly has only one end, which is thick. (An end is *thick* if it contains infinitely many disjoint infinite paths; otherwise it is *thin*.) A natural question to ask, therefore, is whether there are planar graphs of arbitrarily large connectivity and girth that have more than one end, or that have no thick end. We show that the answer to both of these questions is positive, by constructing such graphs with no thick end but continuum many thin ones (Section 2).

By a theorem of Thomassen [6], every finite $(k+3)$-connected graph contains a cycle whose deletion leaves the graph k-connected. Aharoni & Thomassen [2] showed that this is not true for infinite graphs, even for locally finite ones. Their counterexample is constructed by a non-trivial recursion. Our graphs are also counterexamples, but this requires no further proof: their planarity implies at once that every cycle separates them (Section 3).

Diestel [3] conjectured that Thomassen's theorem might generalise to locally finite graphs if we allow infinite cycles (as defined in [3]). We do not prove this but show in Section 3 that our counterexamples corroborate Diestel's conjecture: each of these graphs contains an infinite cycle whose deletion reduces the connectivity by at most two.

We will use the terminology of [1].

2 The Graphs Δ^k_M

In this section we define a rather different type of graph from the tesselations mentioned in the Introduction. As before, our graphs will have arbitrarily large girth and connectivity, but they will have uncountably many thin ends and no thick ends.

Given an integer k and $M \leq \omega$, let Δ^k_M be the graph constructed inductively as follows. We start with a cycle $\mathcal{C}$ of length $2k+1$ in the plane. Then at each of ω steps, we add a new vertex in every inner face of the current plane graph (we call these vertices *centers*), and join it by independent paths of length k to all the vertices on the boundary of that face that are less than M steps old (we call such a path a *radial path*). Clearly all these graphs have girth $2k+1$, and for $M < \omega$ they are locally finite. It will be easy to check the following:

Theorem 1. *Δ^k_M has no thick end, but continuum many thin ones.*

Confirming a conjecture of Diestel (personal communication), we shall then prove the following:

Theorem 2. *For every k there is an $M_0 \in \mathbb{N}$ such that Δ^k_M is $(2k-1)$-connected if $M > M_0$.*

Before we prove these theorems, we need to fix some terminology. Call the cycles that bound a face at some step *primitive*. In what follows, C denotes an arbitrary primitive cycle. Define the *father* of C, as the primitive cycle whose interior contains C, and which was constructed in the step immediately preceding the step in which C was constructed. Call C a *child* of C' if C' is the father of C. Define the *ancestor* relation between primitive cycles as the reflexive transitive closure of the father relation.

Proof of Theorem 1. Note that because any ray can cross a given primitive cycle only finitely often, it must have a subray that lies entirely inside or entirely outside it, and a ray that lies inside a primitive cycle C cannot be equivalent with one that lies outside C, since the vertex set of C separates them. Thus any infinite sequence of primitive cycles each of which is a child of the previous one, uniquely specifies an end, namely the class of rays having a subray inside all cycles of the sequence. Since there are continuum many such sequences, Δ^k_M has continuum many ends. As no infinite set of independent rays can enter any given primitive cycle, all ends are thin.

□

We now set off to prove Theorem 2. In what follows we will consider k fixed and prove Lemmas 1-4 for a sufficiently large M. Our main task is to show the following Lemma:

Lemma 1. *Any center can be connected to any other by (at least) $2k-1$ independent paths.*

Then, because any non-center is connected to a lot of centers by independent paths if M is large enough, Theorem 2 will follow easily.

In order to prove Lemma 1, we will make use of Lemmas 2 and 3 (to be proved later).

For any primitive cycle and any center c that lies in its inside, Lemma 2 allows us to draw them as shown in Figure 1, where the heavy dots represent the young vertices. In order to state it, we will need the following definitions.

The construction of any primitive cycle C other than $\mathcal{C}$ is completed with the addition of $2k-1$ vertices in one step (that belong to two radial paths). We call them the *young vertices* of C. The rest of C's vertices we call its *old* vertices. Pick any two consecutive vertices of $\mathcal{C}$ and call them its old vertices, calling the rest its young vertices. Note that the young vertices of any primitive cycle form a subpath of it.

If C is any (plane) cycle, let $\hat{C} = C \cup \dot{C}$ where $\dot{C}$ is the bounded component of $\mathbb{R}^2 \backslash C$.

Call the first center constructed inside C the *center of* C, and call C the *father* of its center. C is an *ancestor of a center* if it is an ancestor of its father.

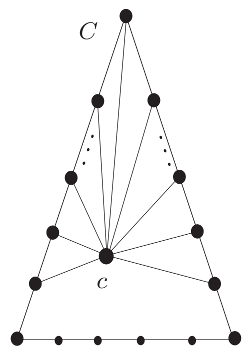

Figure 1: The paths described in Lemma 2

Lemma 2. *For every center c and any of its ancestors C, there are in $\hat{C}$ independent paths from c to all of the young vertices of C, meeting C only at their endpoints.*

Lemma 3 allows us to draw any primitive cycle as shown in Figure 2, where the heavy dots represent the young vertices.

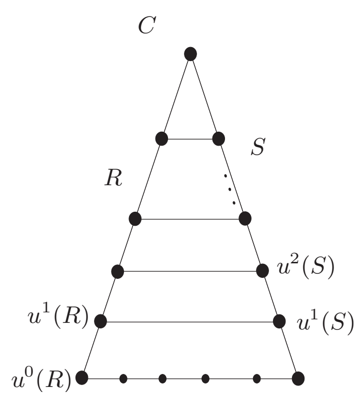

Figure 2: The paths described in Lemma 3

Again, we will need some definitions:

For any vertex a of C, we denote the radial path, if one exists, that connects the center of C to a, by $R_a = R_a(C)$. If R_a is any radial path, then name its vertices as $u^i = u^i(R_a)$, $0 \leq i \leq k$, so that $R = (a =)u^0 u^1 u^2 \ldots u^k$.

If R, S are radial paths of C, let $C(R, S)$ denote the cycle bounded by R, S and by that path on C that connects their endpoints, that contains less young vertices of C (Figure 3).

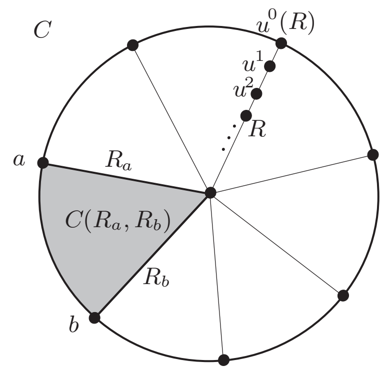

Figure 3:

Lemma 3. *For every primitive cycle $C = C(R, S)$ there are $k-1$ disjoint paths between $\{u^i(R)|1 \leq i \leq k-1\}$ and $\{u^i(S)|1 \leq i \leq k-1\}$ that lie in $\hat{C}$, meeting C only at their endpoints.*

Figures 4 and 5 show how we make use of Figure 1 (Lemma 2) and Figure 2 (Lemma 3) to prove Lemma 1. The former figures correspond to the two possible relative positions of any two centers within a primitive cycle. In both of them, x, y denote the arbitrary centers to be connected by $k-1$ independent paths.

The next Lemma will help prove both Lemmas 2 and 3, and is thus the cornerstone of the whole proof.

We will say that a primitive cycle $C = C(R, S)$ satisfies $\mathcal{A}^l$, if there are l disjoint paths in $\hat{C}$ from the first l inner vertices of R (i.e. $\{u^i(R)|1 \leq i \leq l\}$) to the first l vertices of S (i.e. $\{u^i(S)|0 \leq i \leq l-1\}$) that meet C only at their endpoints. (Note that by construction, R and S can be interchanged.)

Similarly, we will say that C satisfies $\mathcal{B}^l$, if there are l disjoint paths in $\hat{C}$ from the first l vertices of R to the first l vertices of S that meet C only at their endpoints.

We will call C *young* if it does not meet any old vertices of its father.

Lemma 4. *Every young primitive cycle satisfies $\mathcal{A}^{k-1}$ and $\mathcal{B}^{k-1}$.*

Proof. We will perform induction.

Let $C' = C(R, S)$ be an arbitrary young primitive cycle and let $a_i = u^i(R)$ and $b_i = u^i(S)$ for $0 \leq i \leq k$.

It is obvious that C' satisfies $\mathcal{A}^1$ (respectively $\mathcal{B}^1$) if $M > 1$: The desired path can be constructed by joining the paths $R_{a_1}(C')$ and $R_{b_0}(C')$ (respectively R_{a_1} and R_{b_1}).

So suppose that every young primitive cycle satisfies $\mathcal{A}^m$ and $\mathcal{B}^m$ for some $m < k-1$, and pick any young primitive cycle C. We will prove that C satisfies $\mathcal{A}^{m+1}$ and $\mathcal{B}^{m+1}$.

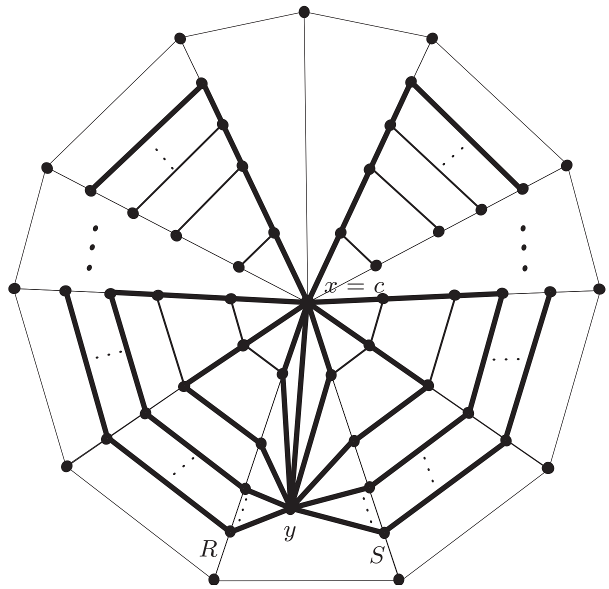

Figure 4: Combining Lemmas 2 and 3 to prove Lemma 1 (first case)

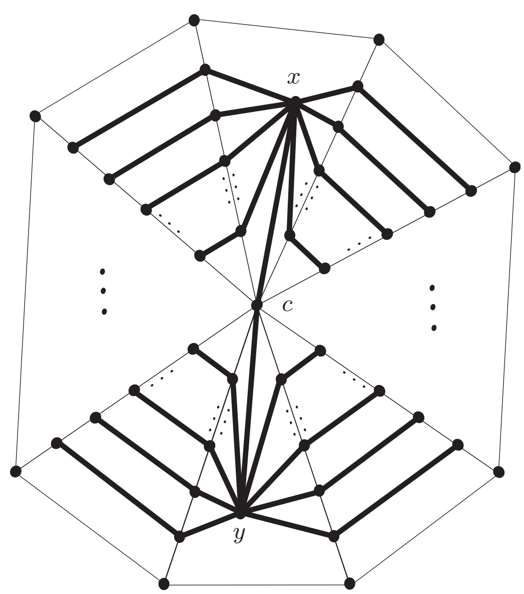

Figure 5: Combining Lemmas 2 and 3 to prove Lemma 1 (second case)

In order to prove that C satisfies $\mathcal{A}^{m+1}$ (respectively $\mathcal{B}^{m+1}$), join R_{a_1} to R_{b_0} (respectively R_{a_1} to R_{b_1}) to get one of the desired paths. The other m paths will be constructed in three steps.

For the first step, note that for $2 \leq i \leq m+1$ the primitive cycle $C_i = C(R_{a_i}, R_{a_{i+1}})$ is young, so by the induction hypothesis it satisfies $\mathcal{A}^m$, which means that there are in $\hat{C}_i$ disjoint paths from the first $i-1$ vertices of R_{a_i} to the first $i-1$ inner vertices of $R_{a_{i+1}}$ (Figure 6). The union of these paths for all i gives a set of disjoint paths between $\{a_i | 2 \leq i \leq m+1\}$ and $\{u^i(R_{a_{m+2}}) | 1 \leq i \leq m\}$.

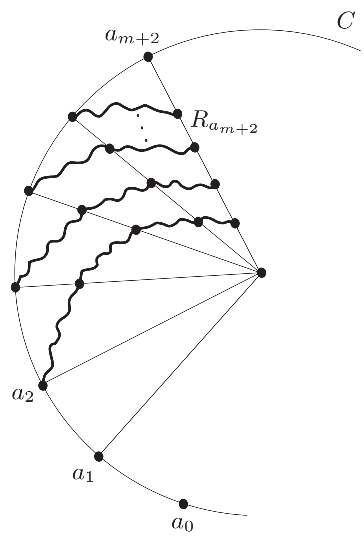

Figure 6: The wavy curves depict the paths of the first step

For the second step, repeating the argumentation of the first on the children of C between R_{b_1} and $R_{b_{m+1}}$ (respectively between R_{b_2} and $R_{b_{m+2}}$ in order to prove that C satisfies $\mathcal{B}^{m+1}$), we obtain a set of disjoint paths between $\{b_i | 1 \leq i \leq m\}$ and $\{u^i(R_{b_{m+1}}) | 1 \leq i \leq m\}$ (respectively between $\{b_i | 2 \leq i \leq m+1\}$ and $\{u^i(R_{b_{m+2}}) | 1 \leq i \leq m\}$).

For the third step, note that all children of C in $\hat{C}(R_{a_{m+2}}, R_{b_{m+1}})$ (respectively $\hat{C}(R_{a_{m+2}}, R_{b_{m+2}})$) are also young. Joining the paths provided by the satisfaction of $\mathcal{B}^m$ by each of those children of C, yields disjoint paths between the first m inner vertices of $R_{a_{m+2}}$ and the first m inner vertices of $R_{b_{m+1}}$ (respectively $R_{b_{m+2}}$) (Figure 7). Note that if $m = k-2$ then $R_{a_{m+2}} = R_{b_{m+2}}$ and no paths are constructed in this step.

By joining the paths constructed in these three steps we obtain the paths needed to prove that C satisfies $\mathcal{A}^{m+1}$ and $\mathcal{B}^{m+1}$ (Figure 8 shows the paths that prove the satisfaction of $\mathcal{A}^{m+1}$). This concludes the inductive step and thus the proof of Lemma 4.

□

Lemma 3 seems at first glance to be a special case of Lemma 4 but it is not, because the latter demands that the cycle be young, so we have to prove the former separately:

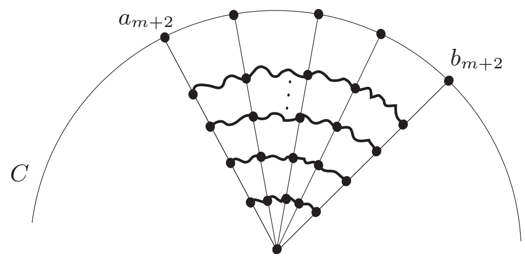

Figure 7: The paths of the third step

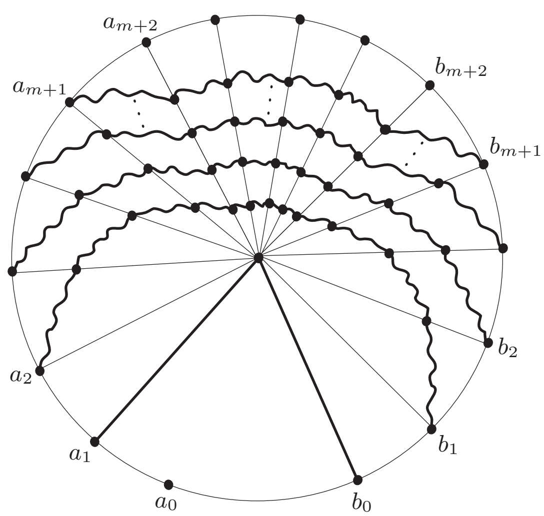

Figure 8: The paths obtained by joining the paths of steps 1-3 (thick)

Proof of Lemma 3. Let $a_i = u^i(R)$ and $b_i = u^i(S)$ for $1 \leq i \leq k-1$.

All children of C in $\hat{C}_1 = \hat{C}(R_{a_1}, R_{a_k})$ are young, so we can imitate the first step of the proof of Lemma 4 (this time using directly the fact that the primitive cycles satisfy $\mathcal{A}^{k-1}$, by Lemma 4, instead of any induction hypothesis) to construct a set of disjoint paths in $\hat{C}_1$ between $\{a_i | 1 \leq i \leq k-1\}$ and the first $k-1$ inner vertices of R_{a_k}. In the same way we can construct a set of disjoint paths in $\hat{C}_2 = \hat{C}(R_{b_1}, R_{b_k})$ between $\{b_i | 1 \leq i \leq k-1\}$ and the first $k-1$ inner vertices of R_{b_k}. Since $R_{a_k} = R_{b_k}$, joining the paths of these two sets in pairs yields the desired paths.

□

Proof of Lemma 2. If C is the father of c, then the radial paths connecting c to the young vertices of C are the desired paths. It now suffices to show that for any primitive cycle $C' \neq \mathcal{C}$, there is a set of disjoint paths that connect every young vertex of C' to a young vertex of its father C'', and lie in $\hat{C}'' \backslash \hat{C}'$. Indeed, if this is true, we can perform induction on the number of generations between c and C to prove the Lemma.

So let C' be any primitive cycle and C'' its father. Let R_a, R_b be the radial paths of C'' that bound C', where a, b are vertices of C'' (and C') (Figure 9).

Applying Lemma 3 to all children of C'' except C' and joining the resulting paths, we obtain a set of disjoint paths $\{Q_i | 1 \leq i \leq k\}$, where Q_i connects $u^i(R_a)$ to $u^i(R_b)$ (Q_k being a trivial path). We can now construct the desired paths by combining subpaths of the paths Q_i with subpaths of the radial paths of C''. Indeed, for any young vertex v of C'', let P_v be the path constructed as follows. If the minimum number of young vertices on a subpath of C'' connecting v to one of a_1, a_2 is i, then travel along $R_v(C'')$ until you meet Q_i, and then use the shortest of the two subpaths of Q_i leading to a young vertex of C' (the second part of the path will be trivial if v is one of a_1, a_2 or $i = k$). It is easy to confirm that P_v and $P_{v'}$ are disjoint if $v \neq v'$.

□

We now have all what we need to prove Lemma 1.

Proof of Lemma 1. Let x, y be any two centers, let C be their last common ancestor, and c its center. Suppose, without loss of generality, that $y \neq c$. There are two cases:

If $x = c$ (Figure 4), let $C' = C(R, S)$ be the child of C in whose inside y lies. Lemma 2 supplies for each young vertex v of C' an independent $y-v$-path P_v in $\hat{C}'$. Our plan is to find a $v - x$ path P'_v in $\hat{C} \backslash \dot{C}'$, so that P'_v and $P'_{v'}$ are independent if $v' \neq v$. Joining each P'_v to P_v, we obtain the $2k-1$ needed paths.

One of the young vertices of C' coincides with x, so we may simply set $P'_x = x$.

Applying Lemma 3 to all children of C except C', and joining the resulting paths, we obtain a set of disjoint paths $\{Q_i | 1 \leq i \leq k-1\}$, where Q_i connects $u^i(R)$ to $u^i(S)$. Now for $v = u^i(R)$ (respectively $u^i(S)$) P'_v is constructed as follows: Travel $k-1-i$ steps along Q_i, one step being a subpath between two consecutive radial paths of C, and then use the radial path of C on which you landed to reach x. (Figure 4) Note that as C has at least $2k-1$ radial paths, a path of this form that begins at a R vertex cannot meet one that begins at a S vertex.

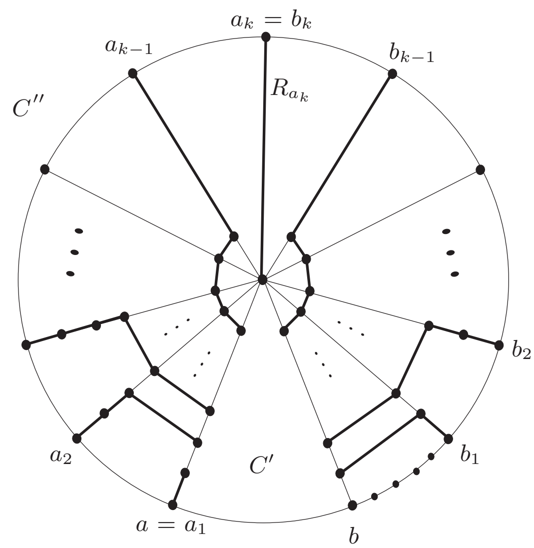

Figure 9: The paths between the young vertices of C'' and C' (thick). In this figure, a happens to coincide with a_1. The thin vertices next to b represent old vertices of C''

For the second case, if $x \neq c$ (Figure 5), let C_x (respectively C_y) be the child of C in whose inside x (respectively y) lies. C_x cannot coincide with C_y, because if this was the case we would have chosen C_x rather than C as the last common ancestor of x, y.

Again making use of Lemma 2, we obtain inside C_x a set of independent paths from x to all young vertices of C_x, and similarly for y and C_y. Then apply Lemma 3 to all children of C except C_x and C_y. Joining the resulting paths and the ones mentioned above to one another yields the desired paths.

□

Theorem 2 now follows easily:

Proof of Theorem 2. Pick M_0 so large that at least $2k-1$ radial paths begin at any non-center of $\Delta^k_{M_0}$, and so that Lemma 1 holds. Now suppose there is a set B of at most $2k-2$ vertices that disconnects Δ^k_M. By Lemma 1, all centers of $\Delta^k_M - B$ lie in the same component K. Pick any vertex v from some component $K' \neq K$. This must be a non-center, and by the choice of M at least $2k-1$ radial paths begin at it. Since these radial paths are independent paths from v to a center, at least one center is still connected to v in $\Delta^k_M - B$, contradicting the fact that $K' \neq K$.

□

3 Separation Properties

Let $\dot{\Delta}^k_M$ be the graph obtained from Δ^k_M by copying the inner face of $\mathcal{C}$ on its outer. Then, Theorem 2 transfers verbatim to $\dot{\Delta}^k_M$:

Theorem 3. *For every k there is an $M_0 \in \mathbb{N}$ such that $\dot{\Delta}^k_M$ is $2k-1$-connected for $M > M_0$.*

Proof. Set M_0 equal to the respective value of Theorem 2, and suppose there is a set B of $\leq 2k-2$ vertices that disconnects $\dot{\Delta}^k_M$. By Theorem 2, there is a component K that contains all vertices in $\hat{\mathcal{C}}$ not it B, and a component K' containing all vertices in $\mathbb{R}\backslash\dot{\mathcal{C}}$ not it B. But both K, K' contain $\mathcal{C}\backslash\mathcal{B}$, and as $|V(\mathcal{C})| > |V(B)|$, $K \cap K' \neq \emptyset$, a contradiction.

□

$\dot{\Delta}^k_M$ has yet another interesting property. By a theorem of Thomassen [6], every $(k+3)$-connected finite graph contains a cycle after whose deletion the resulting graph is still k-connected. This is not true for infinite graphs, even locally finite ones. Aharoni & Thomassen [2] constructed a locally finite counterexample. Their graph is constructed by a fairly complicated recursion, where in each step they attach a copy of some fixed graph to all the cycles in the graph of the previous step. It is easy to see that $\dot{\Delta}^k_M$ is also such a counterexample for $k \geq 3$ and M large enough but finite: since no cycle of this graph bounds a face, the deletion of every cycle separates the graph.

Diestel & Kühn [4, 5] have suggested a topological generalization of finite cycles for infinite graphs called *circles*. These are defined as the homeomorphic images of S^1 in the graph's — seen as a 1-complex — Freudenthal compactification $|G|$ (see [3] for details; we do not need more here). By replacing the concept of cycle with that of a circle, it has been possible to extend to infinite graphs some standard results about finite graphs, that otherwise fail. In this context, Diestel [3] poses the following question in an attempt to extend Thomassen's theorem:

Problem 1. *If G is $(k+3)$-connected, does $|G|$ contain a circle C such that $G - C$ is k-connected?*

We want to show that the answer to this question is positive for the graphs that we have just seen to be counterexamples to the naive extension of Thomassen's theorem.

Let $G = \dot{\Delta}^k_M$, where M is finite but large enough to guarantee that G is $(2k-1)$-connected. An instance of a circle is a double ray whose rays belong to the same end, together with this end. We will find such a circle C in G, and prove that $G - C$ is $(2k-4)$-connected.

In what follows we will be working in $\hat{\mathcal{C}}$, so we can, with a slight abuse, use the terminology introduced for Δ^k_M to refer to objects in $\dot{\Delta}^k_M$. We will define C recursively. Pick an edge w_1v_1 of $\mathcal{C}$ and let C_1 be the child of $\mathcal{C}$ that contains w_1v_1. For the recursive step, suppose that we have already defined a path $P_n = w_n \ldots w_2w_1v_1v_2 \ldots v_n$ that lies on some primitive cycle C_n, and for any $i \in \{1, \ldots, n-1\}$, v_i and w_i are at least M steps (of the construction of Δ^k_M) older than v_n and w_n. Since by construction no radial path reaches v_i or w_i for $1 < i < n$ after the construction of w_n and v_n, any primitive cycle that contains P_n has a child that also contains it. So let C_{n+1} be the most distant descendant of C_n that has young vertices that send edges to w_n and v_n, and name these vertices w_{n+1} and v_{n+1} respectively (note that this guarantees that w_{n+1}, v_{n+1} are at least M steps younger than the other w_i, v_i). Repeating ad infinitum, a double ray $D = \ldots w_2w_1v_1v_2 \ldots$ is defined. To see that both rays of D belong to the same end ω, note that $C_i \backslash \mathring{P}_i$ is a path that joins v_i to w_i and avoids D, and that these paths are disjoint for distinct i's. Let C consist of D and ω.

Theorem 4. *$G - C$ is $(2k-3)$-connected.*

Proof. Pick any two vertices v, w of $G - C$, and let $\mathcal{P}$ be a set of $2k-1$ independent $v - w$-paths in G ($\mathcal{P}$ exists by Theorem 3). We will show that C cannot meet more than two elements of $\mathcal{P}$, and thus there are at least $(2k-3)$ independent paths between any two vertices of $G - C$.

Pick $j \in \mathbb{N}$ such that C_j (as described in the definition of C) was constructed later than all vertices of D that lie on a path in $\mathcal{P}$. Then, as no vertex constructed before some primitive cycle F can lie in $\hat{F}$, no path in $\mathcal{P}$ meets $\dot{C}_j$. Since the graph is plane, the paths in $\mathcal{P}$ are curves that meet only at their endpoints, and so $\hat{C}_j$ can meet at most two of them. But D lies in $\hat{C}_j$: for every i, C_{i+1} is a child of C_i and thus $\hat{C}_{i+1}$ lies in $\hat{C}_i$, which means that D lies in $\hat{C}_i$ for all i. Thus D as well cannot meet more that two elements of $\mathcal{P}$, which completes the proof.

□

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