Bases and closures under infinite sums

Henning Bruhn   Agelos Georgakopoulos

Abstract

Motivated by work of Diestel and Kühn on the cycle spaces of infinite graphs we study the ramifications of allowing infinite sums in a module R^M. We show that every generating set in this setup contains a basis if the ground set M is countable, but not necessarily otherwise. Given a family $\mathcal{N} \subseteq R^M$, we determine when the infinite-sum span $\langle \mathcal{N} \rangle$ of $\mathcal{N}$ is closed under infinite sums, i.e. when $\langle \mathcal{N} \rangle = \langle \langle \mathcal{N} \rangle \rangle$. We prove that this is the case if R is a field or a finite ring and each element of M lies in the support of only finitely many elements of $\mathcal{N}$. This is, in a sense, best possible. We finally relate closures under infinite sums to topological closures in the product space R^M.

1 Introduction

The first homology group of a graph G is known in graph theory as the *cycle space of G*. For finite graphs, the cycle space (usually over $\mathbb{Z}_2$) is well-studied and many results are known; see [9] for instance. In infinite graphs, however, several of these results become false. To remedy this, Diestel and Kühn [10, 11, 12] proposed a new homology. This *topological cycle space* has been surprisingly successful; indeed, various authors [1, 2, 3, 4, 5, 6, 15, 16] have shown that all the standard properties of the cycle space in a finite graph generalise to infinite graphs if the topological cycle space is used. We refer to Diestel [8] for formal definitions and an introduction to the topological cycle space from the combinatorial point of view, and to Diestel and Sprüssel [13] for a study of its relationship to homology theory.

One essential property of the topological cycle space is that, in it, it is necessary to use well-defined infinite sums called *thin sums*. In this paper we study two problems about the topological cycle space in which thin sums play an important role. These problems were previously solved by ad-hoc methods appealing to the structure of the underlying graph. We show that the graph theoretic formulation is unnecessary: the problems can be rephrased, and solved, in a purely algebraic setting. Our solutions yield more general results of independent interest. Moreover, our algebraic approach sheds some light on a technique that has appeared in several proofs about the topological cycle space; indeed, in Section 4 we obtain a corollary that can be used to simplify those proofs.

Let R be a ring and let M be an arbitrary set. If $\mathcal{N} \subseteq R^M$ is an infinite family for which in each coordinate almost all entries are zero, then there is an obvious way to define its sum, namely by pointwise addition. More formally, we call $\mathcal{N}$ *thin* if for all $m \in M$ the number of members N of $\mathcal{N}$ with $N(m) \neq 0$ is finite. For a thin family $\mathcal{N}$ we define the sum $\sum_{N \in \mathcal{N}} N = \sum \mathcal{N}$ to be the element S of R^M with $S(m) = \sum_{N \in \mathcal{N},\, N(m) \neq 0} N(m)$ for all $m \in M$. We remark that thin families, also called *summable* families, occur in the context of slender modules, see Göbel and Trlifaj [17, Chapter 1].

For a (not necessarily thin) family $\mathcal{N} \subseteq R^M$ denote by $\langle \mathcal{N} \rangle$ the space consisting of all sums of thin subfamilies of $\mathcal{N}$. Our first problem, discussed in Section 2, concerns the existence of *bases*: is there a subfamily $\mathcal{B}$ of $\mathcal{N}$ with $\langle \mathcal{N} \rangle = \langle \mathcal{B} \rangle$ such that each element of $\langle \mathcal{N} \rangle$ has a unique representation in $\mathcal{B}$? We will see that even if R is a field there are generating sets that do not contain a basis (Theorem 2), although we can always find a basis if the underlying set M is countable (Theorem 1). Bases were used as a tool in [5] in order to characterise those locally finite graphs that can be drawn in the plane without any two edges crossing. There, bases in certain generating sets were needed, and their existence was proved under very restrictive additional requirements. Theorem 1 yields these bases immediately.

The second problem we consider is whether the space $\langle \mathcal{N} \rangle$ is closed under taking thin sums, i.e. whether $\langle \mathcal{N} \rangle = \langle \langle \mathcal{N} \rangle \rangle$. While this is false in general, we will show in Section 3 that it is true if $\mathcal{N}$ is thin—provided R is a field or a finite ring (Theorem 7). We will see that, in a sense, this is best possible. Closedness is relevant to the study of the topological cycle space: it is important to know that the cycle space as well as the space of all cuts is closed under taking thin sums (a *cut* is the set of all edges that meet both classes of a given bipartition of the vertices of a graph). Both of these facts follow immediately from Theorem 7.

The question whether a space $\mathcal{N}$ is closed under taking thin sums is related to the question whether $\mathcal{N}$ is topologically closed as a subspace of the product space R^M. We investigate this connection in the last section.

A line of research very much in the direction of the present work, albeit more oriented towards graph theory, has been pursued by Casteels and Richter [7].

2 Bases

Before we start we need some definitions. Let M be a set, R a ring, and let $\mathcal{N}$ be a family with its members in R^M. Often we will multiply families with coefficients before adding them: if $a = (a(N))_{N \in \mathcal{N}}$ is a family of coefficients in R, one for each $N \in \mathcal{N}$, we use the shorthand $a\mathcal{N}$ for the family $(a(N)N)_{N \in \mathcal{N}}$. For a $K \in R^M$, we call a family a of coefficients a *representation of K in $\mathcal{N}$* if $a\mathcal{N}$ is thin and if $K = \sum a\mathcal{N}$—that is, $K(m) = \sum_{N \in \mathcal{N}} a(N)N(m)$ for every $m \in M$. Denote by $\langle \mathcal{N} \rangle$ the set of elements of R^M that have a representation in $\mathcal{N}$. Intuitively, $\mathcal{N}$ is a generating set, and $\langle \mathcal{N} \rangle$ is the space it generates.

For a family $\mathcal{N} \subseteq R^M$, we call a subfamily $\mathcal{B}$ of $\mathcal{N}$ a *basis of* $\langle \mathcal{N} \rangle$, if $\langle \mathcal{B} \rangle = \langle \mathcal{N} \rangle$ and $\mathbf{0} \in R^M$ has a unique representation in $\mathcal{B}$. Note that $\mathbf{0}$ has a unique representation in some family $\mathcal{N}$ if and only if every element in $\langle \mathcal{N} \rangle$ has a unique representation in $\mathcal{N}$.

It is well known that a generating set in a module does not need to contain a basis (in the classical sense), and, clearly, this is also the case in our setting: Take for example $R = \mathbb{Z}$, $M = \{0\}$, and $\mathcal{N} = \{a_2, a_3\}$, where a_2, a_3 are defined by $a_2(0) = 2$ and $a_3(0) = 3$. Then $\mathcal{N}$ does not contain a basis of $\langle \mathcal{N} \rangle$. When R is a field, however, we can say more about the existence of a basis:

Theorem 1. *Let M be a countable set, F be a field and let $\mathcal{N}$ be a family with its members in F^M. Then $\mathcal{N}$ contains a basis of $\langle \mathcal{N} \rangle$.*

In linear algebra the analogous assertion is usually proved with Zorn's lemma as follows. Given a chain $(\mathcal{B}_\lambda)_\lambda$ of linearly independent subsets of the generating set, it is observed that $\bigcup_\lambda \mathcal{B}_\lambda$ is still linearly independent since any violation of linear independence is witnessed by finitely many elements, and these would already lie in one of the $\mathcal{B}_\lambda$. Thus, each chain has an upper bound, which implies, by Zorn's lemma, that there is a maximal linearly independent set, a basis. This approach, however, fails in our context, as dependence does not need to be witnessed by only finitely many elements, thus we cannot get the contradiction that already one of the $\mathcal{B}_\lambda$ was not independent.

As an illustration, put $F = \mathbb{Z}_2$, $M := \mathbb{Z}$ and $\mathcal{B}_i := \{\{j, j+1\} : -i \leq j < i\}$ for $i = 1, 2, \ldots$. (Here, and later we shall freely identify elements of $\mathbb{Z}_2^M$ with subsets of M.) Now, while no nonempty finite subset of $\mathcal{B}_\infty := \bigcup_{i=1}^\infty \mathcal{B}_i$ is dependent, the whole set is: $\sum_{B \in \mathcal{B}_\infty} B = \emptyset$.

The standard proof outlined above pursues a bottom-up approach. There is also a, perhaps more pedestrian, top-down proof, where successively those elements of $\mathcal{N}$ are weeded out that can be replaced by others. Slightly more precisely, a (possibly transfinite) enumeration $N_1, N_2, \ldots$ of $\mathcal{N}$ is processed step-by-step, and in each step it is checked whether N_λ can be expressed as a sum in earlier N_μ (i.e. $\mu < \lambda$) that are still left. If yes, N_λ is not needed to generate $\langle \mathcal{N} \rangle$, and therefore deleted. If no, N_λ is kept. It is not hard to check that this process yields a basis.

Our proof of Theorem 1 relies on an adaptation of this argument. Clearly, we cannot expect it to work as it is, since it fails to take infinite sums into account. However, if we restrict ourselves to those elements of $\mathcal{N}$ that share a given $m \in M$ then all sums are finite, and we can employ the argument. So, we will partition $\mathcal{N}$ into sets $\mathcal{N}_1, \mathcal{N}_2, \ldots$ so that all the elements in each of the $\mathcal{N}_j$ share an $m \in M$. Then we will use an (adapted) top-down argument on each of the $\mathcal{N}_i$.

Proof of Theorem 1. Let $m_1, m_2, \ldots$ be a (possibly finite) enumeration of M, and for $i = 1, 2, \ldots$ define $\mathcal{N}_i$ to be the set of those elements $N \in \mathcal{N} \setminus \bigcup_{j<i} \mathcal{N}_j$ for which $N(m_i) \neq 0$. Clearly, $\{\mathcal{N}_i : i \in \mathbb{N}\}$ is a partition of $\mathcal{N}$. For every $i \in \mathbb{N}$, let $N_{i1}, N_{i2}, \ldots, N_{i\lambda}, \ldots$ be a (possibly transfinite) enumeration of $\mathcal{N}_i$.

Now, for each $i = 1, 2, \ldots$ we perform a transfinite induction as follows. Start by setting $\mathcal{B}_{i0} = \emptyset$, and then for every ordinal $\lambda > 0$ define the set $\mathcal{B}_{i\lambda} \subseteq \mathcal{N}_i$ as follows ($\mathcal{B}_{i\lambda}$ is the set of those elements among the first λ $N \in \mathcal{N}_i$ that we will put in our basis): If $L := N_{i\lambda}$ has a representation a_L in $\mathcal{N}$ such that

$$a_L(N) = 0 \textit{ for } N \notin \bigcup_{\mu<\lambda} \mathcal{B}_{i\mu} \cup \bigcup_{k>i} \mathcal{N}_k, \tag{1}$$

then let $\mathcal{B}_{i\lambda} = \bigcup_{\mu<\lambda} \mathcal{B}_{i\mu}$. Otherwise, set $\mathcal{B}_{i\lambda} = \bigcup_{\mu<\lambda} \mathcal{B}_{i\mu} \cup \{N_{i\lambda}\}$. Having defined all $\mathcal{B}_{i\lambda}$, we put $\mathcal{B}_i := \bigcup_\lambda \mathcal{B}_{i\lambda}$.

For later use we note that

$$\textit{if } L \in \mathcal{N}_i \setminus \mathcal{B}_i,\ N \in \mathcal{N}_j \setminus \mathcal{B}_i \textit{ and } j \leq i \textit{ then } a_L(N) = 0. \tag{2}$$

Indeed, if $a_L(N) \neq 0$ then by (1) we obtain $N \in \bigcup_{\mu<\lambda} \mathcal{B}_{i\mu} \cup \bigcup_{k>i} \mathcal{N}_k$ for some λ, and as $j \leq i$ we have $N \in \bigcup_{\mu<\lambda} \mathcal{B}_{i\mu}$. But this means that $N \in \mathcal{B}_i = \bigcup_\mu \mathcal{B}_{i\mu}$, a contradiction.

We claim that $\mathcal{B} := \bigcup_i \mathcal{B}_i$ is a basis of $\langle \mathcal{N} \rangle$. To show that $\mathbf{0} \in R^M$ has a unique representation, suppose there are coefficients $b : \mathcal{B} \to F$, not all of which are zero, such that $b\mathcal{B}$ is thin and $\sum b\mathcal{B} = 0$. Let $i \in \mathbb{N}$ be minimal so that there is an ordinal μ with $b(N_{i\mu}) \neq 0$, and observe that since for all the elements B in $\mathcal{B}_i$ we have $B(m_i) \neq 0$, there is a maximal ordinal λ such that $b(N_{i\lambda}) \neq 0$ (because $b\mathcal{B}$ is thin). Then $N_{i\lambda} = \sum_{N \in \mathcal{B} \setminus \{N_{i\lambda}\}} b^{-1}(N_{i\lambda}) b(N) N$ is (or, more precisely, can be extended to) a representation of $N_{i\lambda}$ that satisfies (1), a contradiction to that $N_{i\lambda} \in \mathcal{B}_i$.

Next, consider a $K \in \langle \mathcal{N} \rangle$. We will show that K has a representation in $\mathcal{B}$. Starting with any representation b^0 of K in $\mathcal{N}$, we inductively define for $i = 1, 2, \ldots$ representations $b^i : \mathcal{N} \to F$ as follows. (Intuitively, b^i is a representation of K using only elements of $\mathcal{N}$ that are left after step i of the construction of $\mathcal{B}$, that is, after we have finished deleting elements of $\mathcal{N}_i$.) Set $\mathcal{E}_i := \{N \in \mathcal{N}_i \setminus \mathcal{B}_i : b^{i-1}(N) \neq 0\}$. Since $b^{i-1}\mathcal{N}$ is thin and since $N(m_i) \neq 0$ for all $N \in \mathcal{E}_i \subseteq \mathcal{N}_i$, it follows that $\mathcal{E}_i$ is a finite set. Put

$$\begin{aligned} b^i(N) &= 0 \textit{ for } N \in \mathcal{E}_i, \textit{ and} \\ b^i(N) &= b^{i-1}(N) + \sum_{L \in \mathcal{E}_i} b^{i-1}(L) a_L(N) \textit{ for } N \notin \mathcal{E}_i. \end{aligned} \tag{3}$$

(Note that a_L is defined for every $L \in \mathcal{E}_i$, since $\mathcal{E}_i \subseteq \mathcal{N}_i \setminus \mathcal{B}_i$.)

We claim that this definition yields a representation of K that uses only those elements of $\mathcal{N}_1, \ldots, \mathcal{N}_i$ that lie in $\mathcal{B}$, in other words, we claim that for every $i \in \mathbb{N}$ it is true that

$$b^i(N) = 0 \textit{ if } N \in \bigcup_{j=1}^{i} \mathcal{N}_j \setminus \mathcal{B} \tag{4}$$

and

$$K = \sum b^i \mathcal{N} \textit{ (in particular, } b^i\mathcal{N} \textit{ is thin).} \tag{5}$$

To prove the two claims we proceed by induction. For (4), consider an $N \in \mathcal{N}_j \setminus \mathcal{B}$ where $j \leq i$. If $N \in \mathcal{E}_i$ then $b^i(N) = 0$ by definition, so consider the case when $N \notin \mathcal{E}_i$. If $j = i$ this implies that $b^{i-1}(N) = 0$; if $j < i$ then we get $b^{i-1}(N) = 0$ too, this time using induction. Since, by (2), $a_L(N) = 0$ for every $L \in \mathcal{E}_i$, (3) implies $b^i(N) = 0$, as desired.

For (5), first note that $b^i\mathcal{N}$ is indeed thin as $\mathcal{E}_i$ is finite and both of $a_L\mathcal{N}$ and $b^{i-1}\mathcal{N}$ thin (the latter by induction). Furthermore:

$$\begin{aligned} &\sum b^i\mathcal{N} - \sum b^{i-1}\mathcal{N} = \\ &\quad = \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} (b^i(N) - b^{i-1}(N))N + \sum_{N \in \mathcal{E}_i} (b^i(N) - b^{i-1}(N))N \\ &\quad = \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} (\sum_{L \in \mathcal{E}_i} b^{i-1}(L) a_L(N) N) + \sum_{N \in \mathcal{E}_i} (\mathbf{0} - b^{i-1}(N)N) \\ &\quad = \sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} a_L(N)N - \sum_{N \in \mathcal{E}_i} (\mathbf{b}^{i-1}(N)N). \end{aligned}$$

As $a_L(N) = 0$ if $N, L \in \mathcal{E}_i$ by (2), we obtain for the first sum in the last line

$$\sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N} \setminus \mathcal{E}_i} a_L(N)N = \sum_{L \in \mathcal{E}_i} b^{i-1}(L) \sum_{N \in \mathcal{N}} a_L(N)N = \sum_{L \in \mathcal{E}_i} b^{i-1}(L)L.$$

Together with the previous equation this yields $\sum b^i\mathcal{N} - \sum b^{i-1}\mathcal{N} = \mathbf{0}$, which proves (5).

For every $N \in \mathcal{N}$, define $b^\infty(N) := b^j(N)$ if $N \in \mathcal{N}_j$, and note that

$$b^i(N) = b^\infty(N) \text{ for } N \in \mathcal{N}_j \text{ and } i \geq j. \tag{6}$$

Indeed, consider $i > j$ and observe that, by (2), $a_L(N) = 0$ for all $L \in \mathcal{E}_i \subseteq \mathcal{N}_i \setminus \mathcal{B}_i$. So, from (3) it follows that $b^i(N) = b^{i-1}(N)$.

We immediately get from (4) that

$$\text{if } N \in \mathcal{N} \setminus \mathcal{B} \text{ then } b^\infty(N) = 0. \tag{7}$$

Thus, the $b^\infty(N)$ can be seen as coefficients on $\mathcal{B}$, and our next claim states that b^∞ is what we are looking for, namely a representation of K in $\mathcal{B}$:

$$K = \sum_{B \in \mathcal{B}} b^\infty(B)B \text{ (in particular, } b^\infty\mathcal{N} \text{ is thin).} \tag{8}$$

Consider an $m_i \in M$. By definition of the $\mathcal{N}_j$, every N with $N(m_i) \neq 0$ lies in $\bigcup_{j=1}^i \mathcal{N}_j$. By (6), b^i and b^∞ are identical on $\bigcup_{j=1}^i \mathcal{N}_j$. Since $b^i\mathcal{N}$ is thin, there are therefore only finitely many $N \in \mathcal{N}$ so that $b^\infty(N)N(m_i) \neq \infty$. Thus $b^\infty\mathcal{N}$ is thin. Furthermore, we obtain with (5) and (6):

$$\sum_{N \in \mathcal{N}} b^\infty(N)N(m_i) = \sum_{N \in \bigcup_{j=1}^i \mathcal{N}_j} b^\infty(N)N(m_i) = \sum_{N \in \mathcal{N}} b^i(N)N(m_i) = K(m_i).$$

Claim (8) now follows from (7). This completes the proof. □

Observe that contrary to conventional linear algebra, two bases do not need to have the same cardinality—even over a field. Indeed, putting $F = \mathbb{Z}_2$ and $M = \{m_0, m_1, \ldots\}$ we see that $\mathcal{B} := \{\{m_i\} : i \geq 0\}$ is a countable basis of F^M. On the other hand, $\mathcal{N} := \{\{m_0\} \cup N : N \subseteq M\}$ clearly generates F^M too, and contains, by Theorem 1, a basis $\mathcal{B}'$. Since all thin subsets of $\mathcal{N}$ are finite, $\mathcal{B}'$ needs to be uncountable to generate the uncountable set F^M. Thus $\mathcal{B}$ and $\mathcal{B}'$ are two bases of F^M that do not have the same cardinality.

We have formulated Theorem 1 only for countable sets M. The following result shows that this is indeed best possible.

Theorem 2. *For an uncountable set M there exists a family $\mathcal{N}$ of elements of $\mathbb{Z}_2^M$, so that $\mathcal{N}$ does not contain a basis of $\langle \mathcal{N} \rangle$.*

Before proving Theorem 2, let us recall some basic graph theoretical terminology; for a much more comprehensive treatment we refer to Diestel [9]. A *graph* G consists of a set of vertices $V(G)$ and a set of edges $E(G) \subseteq [V(G)]^2$; thus, the elements of $E(G)$ are 2-element subsets of $V(G)$. A *path* (resp. *ray*) is a graph on distinct vertices $x_1, \ldots, x_k$ (resp. $x_1, x_2, \ldots$) where x_i is linked to x_{i+1} by an edge. A closed path (i.e. if, in addition, there is an edge between x_k and x_1) is called a *cycle*. A *component* of G is a maximal subgraph of G in which any two vertices are connected by a path.

Proof of Theorem 2. Let A, B be two disjoint sets with cardinalities $|A| = \aleph_0$ and $|B| = \aleph_1$. Define G to be the graph with vertex set $M := A \cup B$ and edge set $\mathcal{N} := A \times B$. As $\mathcal{N} \subseteq \mathcal{P}(M)$, we may ask whether $\mathcal{N}$ contains a basis of $\langle \mathcal{N} \rangle$. We claim that it does not.

Let us show that each countable subset N of M is an element of $\langle \mathcal{N} \rangle$. Indeed, let $n_1, n_2, \dots$ be a (possibly finite) enumeration of N, and choose for $i = 1, 2, \dots$ a ray R_i that starts at n_i, and does not meet the first $i-1$ vertices of each of $R_1, \dots, R_{i-1}$ except, possibly, at n_i. Then, the set $\bigcup_{i \in \mathbb{N}} E(R_i)$ of edges of these rays is thin, and its sum equals N since $\sum_{e \in E(R_i)} e = \{n_i\}$.

Suppose that $\langle \mathcal{N} \rangle$ has a basis $\mathcal{B} \subseteq \mathcal{N}$, and let H be the graph with vertex set M and edge set $\mathcal{B}$. Since $\mathcal{B}$ must contain for each element in B at least one edge incident with it, $\mathcal{B}$ is uncountable. Therefore, one of the vertices in the countable set A, say v, is incident with infinitely many edges in $\mathcal{B}$. Delete from H the vertex v (and its incident edges) and denote by $\mathcal{C}$ the set of components of the resulting graph that (in H) are adjacent to v.

Observe that for each $C \in \mathcal{C}$ there is exactly one edge in H between v and some vertex, u_C say, in C; indeed, if there were two edges between v and vertices u, u' in C, then the union of these edges with an u–u' path in C would be a cycle in H, contradicting that $\emptyset$ has a unique representation in $\mathcal{B}$ since the sum of the edges of a cycle equals $\emptyset$.

Next, suppose there are distinct $C, D \in \mathcal{C}$ each containing a ray; then, C (respectively D) also contains a ray R (resp. S) starting at u_C (resp. at u_D). Then $R \cup S$ together with the two edges between v and $\{u_C, u_D\}$ yields a set of edges which sums to $\emptyset$, again a contradiction.

Pick a countably infinite number of $C \in \mathcal{C}$ none of which contains a ray, and denote the set of these by $\mathcal{C}'$. As $N := \{u_C : C \in \mathcal{C}'\}$ is countable it lies in $\langle \mathcal{N} \rangle$, thus there is a $\mathcal{B}_N \subseteq \mathcal{B}$ such that $\sum_{e \in \mathcal{B}_N} e = N$.

Suppose there is a $C \in \mathcal{C}'$ such that an edge $e \in \mathcal{B}_N$ incident with u_C lies in C. As C does not contain any cycle or any ray, we can run from e along edges in $E(C) \cap \mathcal{B}_N$ to a vertex $w \neq u_C$ that is only incident with one edge in $\mathcal{B}_N$. This implies that $w \in \sum_{e \in \mathcal{B}_N} e = N$, a contradiction since $w \notin N$. However, $u_C \in N$ must be incident with an edge from $\mathcal{B}_N$. Consequently, for each $C \in \mathcal{C}'$ the edge between v and u_C lies in $\mathcal{B}_N$, contradicting that $\mathcal{B}_N$ is thin. $\square$

3 Closedness under taking thin sums

In this section we investigate the following question:

Question 3. *Let M be a set, R be a ring and $\mathcal{N} \subseteq R^M$. When is $\langle \mathcal{N} \rangle$ closed under taking thin sums, i.e. when is $\langle \mathcal{N} \rangle = \langle \langle \mathcal{N} \rangle \rangle$?*

In conventional algebra, the answer is easy: always. Once we allow infinite sums, however, the answer is not that straightforward. Consider, for instance, the case when $M = \mathbb{N}$, $R = \mathbb{Z}_2$ and $\mathcal{N} := \{\{1, i\} : i \in \mathbb{N}\}$. Clearly, we have $\{i\} \in \langle \mathcal{N} \rangle$ for all $i \in \mathbb{N}$ and thus $\mathbb{N} = \sum_{i \in \mathbb{N}} \{i\} \in \langle \langle \mathcal{N} \rangle \rangle$. On the other hand, $\mathbb{N} \notin \langle \mathcal{N} \rangle$ as all thin sums of elements in $\mathbb{N}$ are necessarily finite. Thus, $\langle \mathcal{N} \rangle$ is indeed a proper subset of $\langle \langle \mathcal{N} \rangle \rangle$, and therefore not closed under taking thin sums.

The example seems to indicate that $\mathcal{N}$ needs to be thin. Indeed, if we require $\mathcal{N}$ to be thin then we will see that $\langle \mathcal{N} \rangle$ is closed under taking thin sums—provided that R is a field or a finite ring (Theorem 7). At the end of the section we will give an example showing that this is a fairly complete answer to Question 3: If R is neither a field nor finite, then we cannot guarantee that $\langle\mathcal{N}\rangle = \langle\langle\mathcal{N}\rangle\rangle$.

We remark that there is another way to overcome the counterexample above. Vella [18] shows that a family $\mathcal{N}$ of elements in $\mathbb{Z}_2^{\mathbb{N}}$ is closed under taking thin sums if, instead of being thin, $\mathcal{N}$ has the property that every sum of finitely many members of $\mathcal{N}$ is the disjoint union of members of $\mathcal{N}$.

It turns out that Question 3 is closely related to the topological closure in the product space $R^M = \prod_{m\in M} R_m$ where each R_m is a copy of R endowed with the discrete topology. In what follows we denote by $\overline{\mathcal{N}}$ the topological closure of a subset $\mathcal{N}$ of R^M in the space $\prod_{m\in M} R_m$. Given a $K \in R^M$, note that $K \in \overline{\mathcal{N}}$ if and only if for every finite subset M' of M there is a $N \in \mathcal{N}$ with $K(m) = N(m)$ for all $m \in M'$. In the next two lemmas we will see that for a thin family $\mathcal{T}$ of elements of R^M, $\langle\mathcal{T}\rangle$ is topologically closed if R is a finite ring or a field. Moreover, we will conclude in Lemma 6 that $\langle\langle\mathcal{T}\rangle\rangle$ lies in $\overline{\langle\mathcal{T}\rangle}$, and combining these results yields an answer to Question 3. We will pursue the relationship between topological closedness and closedness under taking thin sums further in the next section.

The first proof uses a typical compactness argument.

Lemma 4. *Let M be a set, R be a finite ring, and let $\mathcal{T}$ be a thin family of elements of R^M. Then $\overline{\langle\mathcal{T}\rangle} = \langle\mathcal{T}\rangle$.*

Proof. Consider an element $K \in \overline{\langle\mathcal{T}\rangle}$. By Tychonoff's theorem, the product space $X := \prod_{T\in\mathcal{T}} R$ where R bears the discrete topology is compact. For any finite subset $M' \subseteq M$, we consider the set $A_{M'}$ of families of coefficients a, such that $a\mathcal{T}$ agrees with K on M'; formally, define

$$A_{M'} := \{a \in X : \sum_{T\in\mathcal{T}} a(T)T(m) = K(m) \textit{ for every } m \in M'\}.$$

(Note that we view the elements of X as coefficients for the family $\mathcal{T}$.) We claim that these sets are closed in X, and that their collection has the finite intersection property.

To show that each $A_{M'}$ is closed, let $\mathcal{S}_{M'}$ be the subfamily of all those $T \in \mathcal{T}$ for which there is a $m \in M'$ with $T(m) \neq 0$. As $\mathcal{T}$ is thin, $\mathcal{S}_{M'}$ is finite. Since R is finite as well, there are only finitely many $b : \mathcal{S}_{M'} \to R$ such that $\sum_{S\in\mathcal{S}_{M'}} b(S)S(m) = K(m)$ for all $m \in M'$. For each such b, $B_b := \{a \in X : a(S) = b(S) \text{ for every } S \in \mathcal{S}_{M'}\}$ is closed in X, and since $A_{M'}$ is the union of these finitely many sets B_b it is closed too.

Next, if we have finite sets $M_1, \dots, M_l \subseteq M$ then, clearly, $\bigcap_{i=1}^{l} A_{M_i} = A_{M''}$ for $M'' := \bigcup_{i=1}^{l} M_i$. As $K \in \overline{\langle\mathcal{T}\rangle}$, there is an element L of $\langle\mathcal{T}\rangle$ that agrees with K on M''. But any representation of L in $\mathcal{T}$ is an element of $A_{M''}$, thus the $A_{M'}$ have the finite intersection property as claimed.

Now, the assertion of the theorem follows easily: X is compact, therefore, the intersection of all $A_{M'}$ is non-empty. For an element a of that intersection, we have $\sum_{T\in\mathcal{T}} a(T)T(m) = K(m)$ for all $m \in M$, i.e. it is a representation of K in $\mathcal{T}$. □

We need a completely different approach to prove a similar result in the case when R is a (possibly infinite) field:

Lemma 5. *Let M be a set, F be a field, and let $\mathcal{T}$ be a thin family of elements of F^M. Then $\overline{\langle \mathcal{T} \rangle} = \langle \mathcal{T} \rangle$.*

Proof. Consider a $K \in \overline{\langle \mathcal{T} \rangle}$. We will reduce the problem of finding a representation of K in $\mathcal{T}$ to the solution of an infinite system of equations. To do this, we associate a variable x^T with every member T of $\mathcal{T}$, and for each $m \in M$ we define e_m to be the linear equation

$$\sum_{T \in \mathcal{T} : T(m) \neq 0} x^T T(m) = K(m)$$

in the variables x^T. Note that as $\mathcal{T}$ is thin, each e_m contains only finitely many variables. Let $E = \{e_m : m \in M\}$. By construction, if there is an assignment $a : \mathcal{T} \to F$ such that setting $x^T = a(T)$ for every $T \in \mathcal{T}$ yields a solution to every equation in E, then a is a representation of K in $\mathcal{T}$. So in what follows, our task is to find such a solution.

For every $T \in \mathcal{T}$, define d_T to be the linear equation $x^T = 1$, put $D = \{d_T : T \in \mathcal{T}\}$, and denote by $\mathcal{E}$ the set of all those sets E' with $E \subseteq E' \subseteq E \cup D$ such that every finite subset $E'' \subseteq E'$ has a solution. We claim that $\mathcal{E}$ contains a $\subseteq$-maximal element E^*. First, note that $\mathcal{E} \neq \emptyset$ as $E \in \mathcal{E}$; indeed, for a finite subset $E'' \subseteq E$, the set M' of all $m \in M$ for which $e_m \in E''$ is finite. As $K \in \overline{\langle \mathcal{T} \rangle}$, there is an element L of $\langle \mathcal{T} \rangle$ that agrees with K on M', and any representation a of L in $\mathcal{T}$ yields a solution of E''. Second, if $(E_i)_{i \in I}$ is a chain in $\mathcal{E}$ then, clearly, the union $\bigcup_{i \in I} E_i$ lies in $\mathcal{E}$, too. Thus, Zorn's lemma ensures the existence of E^*.

Next, we show that for every $T \in \mathcal{T}$ there is a finite $E_T \subseteq E^*$ and an element f_T of the field F such that $x^T = f_T$ in every solution of E_T. Suppose not, and observe that then, clearly, $d_T \notin E^*$. Consider a finite subset E' of E^*, and note that, by assumption, there are two solutions in which x^T takes two distinct values, g_1 and g_2, say. Then, for every $f \in F$ there is a solution of E' where $x^T = fg_1 + (1 - f)g_2$. Setting $f = (1 - g_2)(g_1 - g_2)^{-1}$ yields a solution of E' with $x^T = 1$, which means that $E' \cup \{d_T\}$ has a solution. As E' was an arbitrary finite subset of E^*, it follows that every finite subset of $E^* \cup \{d_T\}$ has a solution, contradicting the maximality of E^*. Thus, E_T and f_T exist, as we have claimed.

Finally, define coefficients $a(T) := f_T$ for $T \in \mathcal{T}$. To see that a is a solution of E, consider an arbitrary $m \in M$. As $\mathcal{T}$ is thin, the subfamily $\mathcal{T}_m$ of those members T of $\mathcal{T}$ with $T(m) \neq 0$ is finite. Thus, $E' := \{e_m\} \cup \bigcup_{T \in \mathcal{T}_m} E_T$ has, as a finite subset of E^*, a solution $b : \mathcal{T} \to F$. Since for every $T \in \mathcal{T}_m$, we have $E_T \subseteq E'$ it follows that $b(T) = f_T = a(T)$. As b solves e_m we see that a solves e_m, too. Thus a is a solution of E, and hence a representation of K in $\mathcal{T}$. $\square$

We need one more simple lemma:

Lemma 6. *Let M be a set, R be a ring, and let $\mathcal{N}$ be a family of elements of R^M. Then $\langle\langle \mathcal{N} \rangle\rangle \subseteq \overline{\langle \mathcal{N} \rangle}$.*

Proof. Let $K \in \langle\langle \mathcal{N} \rangle\rangle$, and consider an arbitrary finite subset M' of M. Denote the canonical projection of R^M to $R^{M'}$ by π, and observe that $\pi(K) \in \langle\langle \pi(\mathcal{N}) \rangle\rangle$. Since rings are closed under addition, and since all thin families in $R^{M'}$ are finite, it holds that $\langle\langle \pi(\mathcal{N}) \rangle\rangle = \langle \pi(\mathcal{N}) \rangle$. Choosing a representation of $\pi(K)$ in $\langle \pi(\mathcal{N}) \rangle$, and replacing each element of $\pi(\mathcal{N})$ in it by one of its preimages with respect to π yields an $N \in \langle\mathcal{N}\rangle$ so that $K(m) = N(m)$ for all $m \in M'$. This proves that $K \in \overline{\langle\mathcal{N}\rangle}$. $\square$

Lemma 6 combined with Lemmas 5 and 4 immediately implies the following:

Theorem 7. *Let M be a set, R be a ring, and let $\mathcal{T}$ be a thin family of elements of R^M. Then $\langle\mathcal{T}\rangle = \langle\langle\mathcal{T}\rangle\rangle$ if R is a field or a finite ring.*

As mentioned in the introduction, an immediate consequence of Theorem 7 is that the topological cycle space as well as the cut space of a locally finite graph is closed under taking thin sums. Both these spaces are generated by thin sets: the former by the fundamental cycles of a normal spanning tree, and the latter by the cuts separating a single vertex from the rest of the graph. For more details see [9], in particular Theorem 8.5.8.

Let us now argue that Theorem 7 gives, in a sense, a comprehensive answer to Question 3. In fact, we shall construct an example where R is neither a field nor finite, and where there exists a thin family $\mathcal{T} \subseteq R^M$ such that $\langle\mathcal{T}\rangle$ is not closed under taking thin sums.

For this, set $R := \mathbb{Z}$ and $M := \mathbb{N}$. Define $N \in \mathbb{Z}^{\mathbb{N}}$ by $N(i) = 1$ for every $i \in \mathbb{N}$. For $j = 1, 2, \ldots$, define $N_j \in \mathbb{Z}^{\mathbb{N}}$ by $N_j(j) = p_j$ and $N_j(i) = 0$ for every $i \neq j$, where p_j is the jth prime number. Let $\mathcal{T} = \{N, N_1, N_2, \ldots\}$, and note that $\mathcal{T}$ is thin. We will show that the function $K \in \mathbb{Z}^{\mathbb{N}}$ defined by $K(i) = i$ is in $\langle\langle\mathcal{T}\rangle\rangle$ but not in $\langle\mathcal{T}\rangle$.

Let us first prove that $K \notin \langle\mathcal{T}\rangle$. Suppose for contradiction, there is a representation $a : \mathcal{T} \to \mathbb{Z}$ of K in $\langle\mathcal{T}\rangle$. We distinguish two cases, depending on whether $n := a(N)$ is non-negative or not. If $n \geq 0$, then $n+1 = K(n+1) = \sum_{L\in\mathcal{T}} a(L)L(n+1) = a(N)N(n+1) + a(N_{n+1})N_{n+1}(n+1) = n + a(N_{n+1})p_{n+1}$, which implies that $1 = a(N_{n+1})p_{n+1}$, a contradiction. If, on the other hand, $n < 0$ then we have for $n' = -n$ that $n' = K(n') = n + a(N_{n'})p_{n'}$, i.e. $2n' = a(N_{n'})p_{n'}$. With $p_{n'} > n'$ we obtain $p_{n'} = 2$ and thus $n = -1$. This again leads to a contradiction, as it implies that $K(3) = 3 = -1 + a(N_3) \cdot 5$.

Having shown $K \notin \langle\mathcal{T}\rangle$, we now prove $K \in \langle\langle\mathcal{T}\rangle\rangle$. For this, we construct for every $i \in \mathbb{N}$ an $S_i \in \langle\mathcal{T}\rangle$ so that $S_i(i) = 1$ and $S_i(j) = 0$ for every $j < i$. Once we have done that we can represent K with these S_i: put $d(S_1) = 1$ and, inductively, set $d(S_i) = i - \sum_{j=1}^{i} d(S_j)S_j(i)$. Then $K = \sum_{i\in\mathbb{N}} d(S_i)S_i$.

Let us now find coefficients $a_0, \ldots, a_i \in \mathbb{Z}$ so that $S_i := a_0 N + \sum_{j=1}^{i} a_j N_j$ is as desired. It follows from the Chinese remainder theorem that the system of congruences

$$\begin{aligned} x &\equiv 0 \pmod{p_1} \\ &\vdots \\ x &\equiv 0 \pmod{p_{i-1}} \\ x &\equiv 1 \pmod{p_i} \end{aligned}$$

has a solution $a_0 \in \mathbb{Z}$. This allows us to choose $a_j \in \mathbb{Z}$ so that $a_0 + a_j p_j = 0$, for every $1 \leq j < i$, and $a_i \in \mathbb{Z}$ so that $a_0 + a_i p_i = 1$.

4 Thin sums and topological closure

In [18], Vella introduced for a family $\mathcal{N}$ of elements of R^M the following notation and spaces:

- the *weak span* $\mathcal{W}(\mathcal{N})$ is the set of all finite sums of elements of $\mathcal{N}$ with coefficients in R, i.e. $\mathcal{W}(\mathcal{N})$ is the R-module generated by $\mathcal{N}$;
- the *algebraic span* $\mathcal{A}(\mathcal{N})$ is the set of all thin sums of elements of $\mathcal{N}$ with coefficients in R, i.e. what we have called (and will continue to call) $\langle\mathcal{N}\rangle$; and
- the *strong span* $\mathcal{S}(\mathcal{N})$ is the intersection of all sets $\mathcal{M} \supseteq \mathcal{N}$ that are closed under taking thin sums, i.e. the smallest set $\mathcal{S}$ containing $\mathcal{N}$ with $\langle\mathcal{S}\rangle = \mathcal{S}$.

Let us add to this list a fourth space, namely $\overline{\mathcal{W}(\mathcal{N})}$, the topological closure of $\mathcal{W}(\mathcal{N})$ in the product space R^M. It is not hard to see that it contains the strong span of $\mathcal{N}$:

Lemma 8. *For any ring R, any set M and any family $\mathcal{N}$ of elements of R^M, $\overline{\mathcal{W}(\mathcal{N})}$ is closed under taking thin sums, i.e. $\left\langle\overline{\mathcal{W}(\mathcal{N})}\right\rangle = \overline{\mathcal{W}(\mathcal{N})}$.*

Proof. Consider a thin family $\mathcal{T}$ of elements of $\overline{\mathcal{W}(\mathcal{N})}$. We need to show that $S := \sum_{T\in\mathcal{T}} T \in \overline{\mathcal{W}(\mathcal{N})}$. For this, let M' be an arbitrary finite subset of M. As $\mathcal{T}$ is thin, the subfamily $\mathcal{T}'$ of those members T of $\mathcal{T}$ for which $T(m) \neq 0$ for some $m \in M'$ is finite. For each $T \in \mathcal{T}'$ there exists a finite subfamily $\mathcal{N}_T$ of $\mathcal{N}$ and coefficients $r_N^T \in R$ for $N \in \mathcal{N}_T$ so that $\sum_{N\in\mathcal{N}_T} r_N^T N(m) = T(m)$ for all $m \in M'$ since T lies in $\overline{\mathcal{W}(\mathcal{N})}$. Then, $S' := \sum_{T\in\mathcal{T}'}\sum_{N\in\mathcal{N}_T} r_N^T N$ is an element of $\mathcal{W}(\mathcal{N})$ and $S'(m) = S(m)$ for all $m \in M'$. As M' was arbitrary, this means that $S \in \overline{\mathcal{W}(\mathcal{N})}$. $\square$

We thus have the following inclusions:

$$\mathcal{W}(\mathcal{N}) \subseteq \langle\mathcal{N}\rangle \subseteq \mathcal{S}(\mathcal{N}) \subseteq \overline{\mathcal{W}(\mathcal{N})}.$$

Clearly, the first two inclusions can be proper. But can the third one also be proper?

Question 9. *Is $\mathcal{S}(\mathcal{N}) = \overline{\mathcal{W}(\mathcal{N})}$ for every family $\mathcal{N}$ of elements of R^M?*

In the special case when $\mathcal{N}$ is thin and R a field or a finite ring we obtain from the results of the previous section (Lemmas 4 and 5) that $\langle\mathcal{N}\rangle = \overline{\langle\mathcal{N}\rangle}$, which clearly implies $\mathcal{S}(\mathcal{N}) = \overline{\mathcal{W}(\mathcal{N})}$. This answers a question of Manfred Droste (personal communication).

For countable M, Question 9 has an affirmative answer, too. This can easily be seen using a telescoping sum argument:

Proposition 10. *If R is any ring and M a countable set then $\overline{\mathcal{W}(\mathcal{N})} = \mathcal{S}(\mathcal{N})$ for any family $\mathcal{N}$ of elements of R^M.*

Proof. We only need to prove that $\overline{\mathcal{W}(\mathcal{N})} \subseteq \mathcal{S}(\mathcal{N})$. Consider an arbitrary $K \in \overline{\mathcal{W}(\mathcal{N})}$, and let $m_1, m_2 \ldots$ be an enumeration of M. As K lies in $\overline{\mathcal{W}(\mathcal{N})}$ there is for every $i \in \mathbb{N}$ an $N_i \in \mathcal{W}(\mathcal{N})$ so that $N_i(m_j) = K(m_j)$ for all $j \leq i$.

Set $N_0 = \mathbf{0}$ and define $L_i = N_i - N_{i-1} \in \mathcal{W}(\mathcal{N})$ for every i. Then, $L_i(m_j) = 0$ for $j < i$ and, consequently, the L_i form a thin family. Furthermore:

$$\left(\sum_{j=1}^{\infty} L_j\right)(m_i) = \left(\sum_{j=1}^{i} L_j\right)(m_i) = N_i(m_i) = K(m_i).$$

As $L_j \in \mathcal{W}(\mathcal{N}) \subseteq \mathcal{S}(\mathcal{N})$ for every j and $\mathcal{S}(\mathcal{N})$ is closed under taking thin sums, we obtain that $K = \sum_{j=1}^{\infty} L_j$ lies in $\mathcal{S}(\mathcal{N})$. □

In the case of the topological cycle space $\mathcal{C}$ of a graph, the set M, the set of edges of the graph, is usually countable and so Proposition 10 is applicable (with $\mathcal{N} = \mathcal{C}$). Moreover, $\mathcal{C}$ is generated by a thin set (see [9, Theorem 8.5.8]), thus we obtain with Theorem 7 that $\mathcal{C} = \overline{\mathcal{C}}$. A technique that appears in a number of proofs, see e.g. [1, 2, 3, 14, 15, 16], makes implicit use of this fact. In those proofs, an infinite cycle or element of $\mathcal{C}$ with certain properties is sought. The standard way to construct the desired object is to approximate it by a sequence of finite cycles or elements of $\mathcal{C}$ and to consider the limit of this sequence. That this limit lies indeed in $\mathcal{C}$ is usually proved explicitly, but follows directly from our corollary that $\mathcal{C} = \overline{\mathcal{C}}$.

For uncountable M, Question 9 is not as easy to answer and indeed Proposition 10 becomes false. In the rest of the paper we will present a family $\mathcal{N}$ of elements of $\mathbb{Z}_2^{[0,1]}$ for which the inclusion $\mathcal{S}(\mathcal{N}) \subseteq \overline{\mathcal{W}(\mathcal{N})}$ is proper.

Again, we will view elements of $\mathbb{Z}_2^{[0,1]}$ as subsets of $[0,1]$. In particular, the elements of $\mathcal{N}$ will consist of disjoint unions of intervals of $[0,1]$. These intervals will be chosen from an ever finer subdivision of $[0,1]$. More precisely, we will construct $\mathcal{N}$ in ω steps each of which corresponds to a certain level of coarseness: we start in step 0 with the whole interval, so the first elements of $\mathcal{N}$ will be $[0,1]$ and the empty set. In the next step, we cut $[0,1]$ in half, and in the following step we subdivide each of these halves again into two halves, and so on. Thus, in step n, we have the intervals $[0, 1/2^n], \ldots, [(2^n-1)/2^n, 1]$ at our disposal, and an element N constructed in this step will be the union of some of these intervals. However, not every such union will be put in $\mathcal{N}$.

Before we start with the formal definition, let us make one amendment. When we add (perhaps many) intervals of the form $[\frac{i-1}{2^n}, \frac{i}{2^n}]$ it is not so easy to keep track of what happens with the points on the boundary of the intervals. While this is not a serious problem, it complicates the matter. To circumvent this, we will delete from our ground set all those points that can ever arise as a boundary of some interval. These are precisely the points $J := \{\frac{i}{2^n} : 0 \le i \le 2^n,\, i, n \in \mathbb{N}\}$, and the subsets of $\mathcal{N}$ will therefore be subsets of $[0,1] \setminus J$.

We begin with the definition of the "intervals": for $n \in \mathbb{N}$ and every $i \in \{1, 2, \ldots 2^n\}$ let $I_n^i = [\frac{i-1}{2^n}, \frac{i}{2^n}] \setminus J$, and define $\mathcal{I}_n := \{I_n^i : i \in \{1, 2, \ldots 2^n\}\}$. In step 0, we set $S_0 := [0,1] \setminus J$ and $\mathcal{N}_0 := \{\emptyset, S_0\}$. Then, in step $n+1$, assuming that we have already defined nested sets $\mathcal{N}_0 \subseteq \ldots \subseteq \mathcal{N}_n$ in the previous steps, we construct a new "seed" element by taking every second interval in $\mathcal{I}_{n+1}$:

$$S_{n+1} := \bigcup_{i=1}^{2^n} I_{n+1}^{2i}$$

By adding this seed to the existing elements we define the new ones:

$$\mathcal{N}_{n+1} := \mathcal{N}_n \cup \{N + S_{n+1} : N \in \mathcal{N}_n\}.$$

Once all these $\mathcal{N}_n$ are constructed, we put $\mathcal{N} := \bigcup_{n=0}^{\infty} \mathcal{N}_n$. See Figure 1 for the first few elements of $\mathcal{N}$.

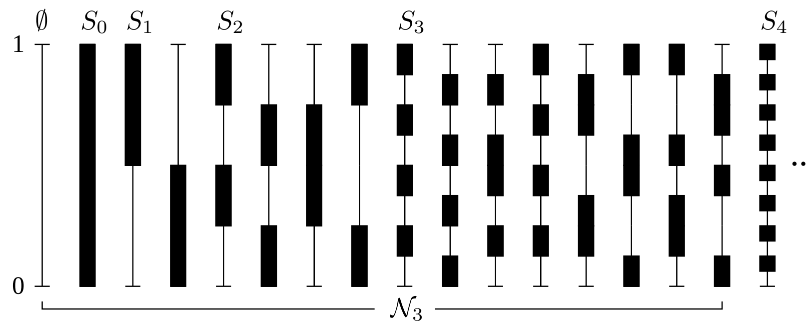

Figure 1: A schematic drawing of the first few elements of $\mathcal{N}$

We will accomplish our aim, to show that $\mathcal{S}(\mathcal{N}) \neq \overline{\mathcal{W}(\mathcal{N})}$, in three steps. In each of these we prove one of the following assertions:

(i) *$\mathcal{N}$ is closed under taking finite sums, i.e.* $\mathcal{N} = \mathcal{W}(\mathcal{N})$;

(ii) $\mathcal{W}(\mathcal{N}) = \mathcal{S}(\mathcal{N})$; and

(iii) $\mathcal{N} \neq \overline{\mathcal{N}}$.

Combining (i), (ii) and (iii) we immediately obtain $\mathcal{S}(\mathcal{N}) \neq \overline{\mathcal{W}(\mathcal{N})}$.

In order to establish (i), we will show inductively that each $\mathcal{N}_n$ is already closed under taking finite sums, i.e. that $\mathcal{W}(\mathcal{N}_n) = \mathcal{N}_n$. For this, consider $N, L \in \mathcal{N}_n$. If $N, L \in \mathcal{N}_{n-1}$ then we are done by induction. So we may assume that $N \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$, i.e. that there is an $N' \in \mathcal{N}_{n-1}$ with $N = S_n + N'$. Now if $L \in \mathcal{N}_{n-1}$ then $N' + L \in \mathcal{N}_{n-1}$ (by induction) and thus $N + L = S_n + (N' + L) \in \mathcal{N}_n$. If, on the other hand, $L = S_n + L'$ for some $L' \in \mathcal{N}_{n-1}$, then $N + L = S_n + N' + S_n + L' = N' + L' \in \mathcal{N}_{n-1}$, which completes the proof of (i).

For the proof of (ii), we will need some intermediate assertions. The first one states that

$$\textit{if } N \in \mathcal{N}_n \textit{ and } I \in \mathcal{I}_n \textit{ then either } I \subseteq N \textit{ or } I \cap N = \emptyset. \tag{9}$$

To prove this, we perform induction on n. Note that I is contained in an $I' \in \mathcal{I}_{n-1}$. Thus, if $N \in \mathcal{N}_{n-1}$ then the assertion holds by the induction hypothesis. If, however, $N \in \mathcal{N}_n \setminus \mathcal{N}_{n-1}$, then $N = S_n + N'$ for some $N' \in \mathcal{N}_{n-1}$. The assertion holds for N' (by induction) and for S_{n+1} (by construction), and therefore it is also true for $N = S_{n+1} + N'$.

Next, we prove that

$$\textit{every } N \in \mathcal{N}_{n+1} \setminus \mathcal{N}_n \textit{ meets every } I \in \mathcal{I}_n. \tag{10}$$

Indeed, it is easy to see that this holds for $n = 0$. Now, suppose that $N = S_{n+1} + N'$ where $N' \in \mathcal{N}_n$. By construction, $I \cap S_{n+1}$ is a non-empty proper subset of I. Since, by (9), either $I \subseteq N'$ or $I \cap N' = \emptyset$, it follows that $S_{n+1} + N'$ meets I.

Before we deduce (ii), we need one final assertion:

$$\begin{array}{l}\textit{if } \mathcal{L} \textit{ is an infinite subset of } \mathcal{N} \textit{ and if } I \in \mathcal{I}_n \textit{ for some } n \in \mathbb{N} \\ \textit{then there exists an } r \in I \textit{ lying in infinitely many } L \in \mathcal{L}.\end{array} \tag{11}$$

To prove this, let $N_1, N_2, \ldots$ be distinct elements of $\mathcal{L}$, and define $n_k \in \mathbb{N}$ by $N_k \in \mathcal{N}_{n_k} \setminus \mathcal{N}_{n_k-1}$. We may assume that the N_k are ordered so that $n_1 \leq n_2 \leq \ldots$. By going to a subsequence we may even assume that $n_{k+1} - n_k \geq 2$ for all k, and that $n_1 \geq n+1$ (note that $|\mathcal{N}_n| < \infty$ for all n). Thus there is, by (9) and (10), an $I_1 \in \mathcal{I}_{n_1}$ with $I_1 \subseteq N_1 \cap I$. Now $I_1 = I^j_{n_1}$ for some j, and I_1 is the union of two elements of $\mathcal{I}_{n_1+1}$: the "left half" $I_l := I^{2j-1}_{n_1+1}$ and the "right half" $I_l := I^{2j}_{n_1+1}$. Since $n_2 - n_1 \geq 2$ it follows from (10) that N_2 meets both of I_l and I_r, and therefore both of $I_l \cap N_2$ and $I_r \cap N_2$ contain, by (9), an element of $\mathcal{I}_{n_2}$ as a subset. We choose an $I_2 \in \mathcal{I}_{n_2}$ with $I_2 \subseteq I_l \cap N_2$. Continuing in this manner, we find nested sets $I \supseteq I_1 \supseteq I_2 \supseteq \ldots$ with $I_k \subseteq N_k$. Since $\mathbb{R}$ is complete and since the lengths of the intervals $\overline{I_k}$ converge to zero, there is precisely one point $r \in \mathbb{R}$ lying in all $\overline{I_k}$ (where $\overline{I_k}$ is the closure of I_k in the usual topology of $\mathbb{R}$). By choosing I_k to lie in the right half of I_{k-1} for odd k and in the left half for even k, we ensure that $r \notin J$. Thus, r lies in $\overline{I_k} \setminus J = I_k \subseteq N_k$ for every k. Since $I_1 \subseteq I$ this proves (11).

Now (ii) follows directly from (11), as the latter implies that no thin sum can have infinitely many non-zero summands.

Let us deduce an easy corollary of (11) that we will use in order to prove (iii):

$$\begin{array}{l}\textit{if } \mathcal{L} \textit{ is an infinite subset of } \mathcal{N} \textit{ and if } I \in \mathcal{I}_n \textit{ for some } n \in \mathbb{N} \textit{ then} \\ \textit{there exists an } s \in I \textit{ that is missed by infinitely many } L \in \mathcal{L}.\end{array} \tag{12}$$

Indeed, this follows immediately if we apply (11) to $\mathcal{L}' := \{([0,1] \setminus J) \setminus L : L \in \mathcal{L}\}$. (Note that $\mathcal{L}' \subseteq \mathcal{N}$ as $\mathcal{L}' = \{L + S_0 : L \in \mathcal{L}\}$.)

To prove (iii), take any infinite subset $\mathcal{L}_0$ of $\mathcal{N}$, and choose an $I_1 \in \mathcal{I}_1$. By subsequent application of (11) and (12) we find $r_1, s_1 \in I_1$ and an infinite subset $\mathcal{L}_1$ of $\mathcal{L}$ so that $r_1 \in L$ but $s_1 \notin L$ for all $L \in \mathcal{L}_1$. Pick any element of $\mathcal{L}_1$ and denote it by L_1. Next, we choose some $I_2 \in \mathcal{I}_2$ and find, again, $r_2, s_2 \in I_2$ and an infinite subset $\mathcal{L}_2$ of $\mathcal{L}_1$ so that r_2 lies in all the $L \in \mathcal{L}_2$ and s_2 in none. Pick any $L_2 \in \mathcal{L}_2$ and continue in this manner.

This process yields a sequence $L_1, L_2, \ldots$ of elements of $\mathcal{N}$. By Tychonoff's theorem, the space $\mathbb{Z}_2^{[0,1]}$ is compact. Hence, the sequence $L_1, L_2, \ldots$ has an accumulation point $X \in \mathbb{Z}_2^{[0,1]}$, and clearly $X \in \overline{\mathcal{N}}$.

We claim that $X \notin \mathcal{N}$. To see this, first note that since this is the case for almost all L_k, $r_i \in X$ but $s_i \notin X$ for any i. Now, suppose that $X \in \mathcal{N}_n$ for some n. As $I_n \in \mathcal{I}_n$, it follows from (9) that either $I_n \subseteq X$ or $I_n \cap X = \emptyset$. However, the former contradicts $s_n \notin X$ and the latter contradicts $r_n \in X$. This completes the proof of (iii).

References

[1] H. Bruhn, *The cycle space of a* 3*-connected locally finite graph is generated by its finite and infinite peripheral circuits*, J. Combin. Theory (Series B) **92** (2004), 235–256.

[2] H. Bruhn and R. Diestel, *Duality in infinite graphs*, Comb., Probab. Comput. **15** (2006), 75–90.

[3] H. Bruhn, R. Diestel, and M. Stein, *Cycle-cocycle partitions and faithful cycle covers for locally finite graphs*, J. Graph Theory **50** (2005), 150–161.

[4] H. Bruhn, S. Kosuch, and M. Win Myint, *Bicycles and left-right tours in locally finite graphs*, Preprint 2007.

[5] H. Bruhn and M. Stein, *MacLane's planarity criterion for locally finite graphs*, To appear in *J. Combin. Theory (Series B).*

[6] ———, *On end degrees and infinite circuits in locally finite graphs*, Submitted.

[7] K. Casteels and B. Richter, *The cycle spaces of an infinite graph*, Preprint 2006.

[8] R. Diestel, *The cycle space of an infinite graph*, Comb., Probab. Comput. **14** (2005), 59–79.

[9] ———, *Graph theory* (3rd edition), Springer-Verlag, 2005.

[10] R. Diestel and D. Kühn, *On infinite cycles I*, Combinatorica **24** (2004), 69–89.

[11] ———, *On infinite cycles II*, Combinatorica **24** (2004), 91–116.

[12] ———, *Topological paths, cycles and spanning trees in infinite graphs*, Europ. J. Combinatorics **25** (2004), 835–862.

[13] R. Diestel and P. Sprüssel, *The homology of locally finite graphs with ends*, Preprint 2007.

[14] A. Georgakopoulos, *Infinite hamilton cycles in squares of locally finite graphs*, Preprint 2006.

[15] A. Georgakopoulos, *Topological cycles and Euler tours in locally finite graphs*, Preprint 2007.

[16] A. Georgakopoulos and P. Sprüssel, *Geodesic topological cycles in locally finite graphs*, Preprint 2006.

[17] R. Göbel and J. Trlifaj, *Approximations and endomorphism algebras*, Walter de Gruyter, 2006.

[18] A. Vella, *A fundamentally topological perspective on graph theory*, Ph.D. thesis, University of Waterloo, 2005.

Henning Bruhn `<hbruhn at gmx dot net>`
Agelos Georgakopoulos `http://www.math.uni-hamburg.de/home/georgakopoulos/`
Mathematisches Seminar
Universität Hamburg
Bundesstraße 55
20146 Hamburg
Germany