Twins of rayless graphs

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Abstract

Two non-isomorphic graphs are twins if each is isomorphic to a subgraph of the other. We prove that a rayless graph has either infinitely many twins or none.

1   Introduction

Up to isomorphism, the subgraph relation $\subseteq$ is antisymmetric on finite graphs: If a finite graph G is (isomorphic to) a subgraph of H, i.e. $G \subseteq H$, and if also $H \subseteq G$, then G and H are isomorphic. For infinite graphs this need no longer be the case, see Figure 1. Two non-isomorphic graphs G and H are *weak twins* if G is isomorphic to a subgraph of H and vice versa, and *strong twins* if both these subgraph embeddings are induced. When G and H are trees the two notions coincide, and we just speak of *twins*.
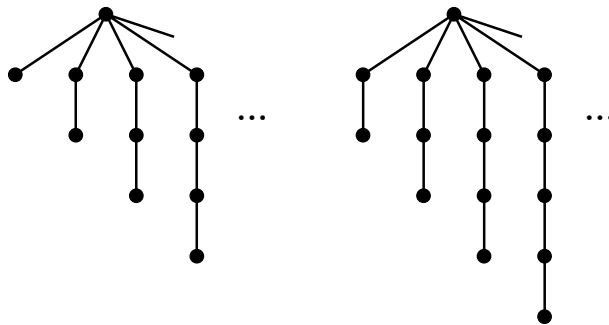

Figure 1: Each of the two graphs is a subgraph of the other.

The trees in Figure 1 are twins, and by deleting some of their leaves we can obtain infinitely many further trees that are twinned with them. On the other hand, no tree is a twin of the infinite star. Bonato and Tardif [3] conjectured that every tree is subject to this dichotomy: that it has either infinitely many trees as twins or none. They call this the *tree alternative conjecture*.

In this paper we prove the corresponding assertion for *rayless* graphs, graphs that contain no infinite path:

Theorem 1. *The following statements hold with both the weak and the strong notion of 'twin'.*

(i) *A rayless graph has either infinitely many twins or none.*

(ii) *A connected rayless graph has either infinitely many connected twins or none.*

We do not know of any counterexamples to the corresponding statements for arbitrary graphs, rayless or not.

Note that the 'strong twin' version of Theorem 1 does not directly imply the 'weak twin' version. Indeed, consider the complete bipartite graph $K_{2,\infty}$ with one partition class consisting of two and the other of (countable-)infinitely many vertices. By deleting any edge we obtain a weak twin of $K_{2,\infty}$. However, it is straightforward to check that $K_{2,\infty}$ has no strong twin.

We have stressed in the theorem that for a connected rayless graph we may restrict ourselves to twins that are also connected. This is indeed a stronger statement: For example, an infinite star has disconnected weak twins—add isolated vertices—but no connected ones. We do not know whether the same can occur for strong twins.

Twins were first studied in [2]. The tree alternative conjecture was formulated in [3], where it was proved in the special case of rayless trees. (Note that Theorem 1 reproves this case.) Most of the work there was spent on showing that the conjecture holds for *rooted* rayless trees, which motivated Tyomkyn [12] to verify the conjecture for arbitrary rooted trees. Moreover, Tyomkyn established the tree alternative conjecture for certain types of locally finite trees. (A graph is *locally finite* if all its vertices have finite degree.) A proof of the conjecture for arbitrary unrooted locally finite trees has remained elusive.

In [12], a slightly different approach is outlined as well. If a graph G has a twin, then mapping G to that twin and back embeds it as a proper subgraph in itself. Tyomkyn conjectures that, with the exception of the ray, every locally finite tree that is a proper subgraph of itself has infinitely many twins.

In this paper we consider only embeddings as subgraphs or induced subgraphs, leading to weak or strong twins. It seems natural, however, to ask a similar question for other relations on graphs, such as the minor relation or the immersion relation. Does a graph always have either infinitely many 'minor-twins' or none at all? Conceivably, the question of when a graph is a proper minor of itself, as is claimed for countable graphs by Seymour's *self-minor conjecture*, should play a role in this context. The self-minor conjecture is described in Chapter 12.5 in [5]; partial results are due to Oporowski [8] and Pott [10]. In related work, Oporowski [9] characterises the minor-twins of the infinite grid, and Matthiesen [7] studies a complementary question with respect to the topological minor relation, restricted to rooted locally finite trees.

In the next section we introduce a recursive technique for handling rayless graphs, which we will use in Section 3 to prove Theorem 1.

2 A rank function for rayless graphs

All our graphs are simple. For general graph theoretical concepts and notation we refer the reader to [5].

Our proof of Theorem 1 is based on a construction by Schmidt [11] (see also Halin [6] for an exposition in English) that assigns an ordinal rk(G), the *rank of G*, to all rayless graphs G as follows:

Definition 2. *Let* $\mathrm{rk}(G) = 0$ *if and only if G is a finite graph. Then recursively for ordinals $\alpha > 0$, let $\mathrm{rk}(G) = \alpha$ if and only if*

(i) *G has not been assigned a rank smaller than α; and*

(ii) *there is a finite set $S \subseteq V(G)$ such that every component of $G - S$ has rank smaller than α.*

It is easy to see that the graphs that receive a rank are precisely the rayless ones. The rank function makes the class of rayless graphs accessible to induction proofs. One of the first applications of the rank was the proof of the reconstruction conjecture restricted to rayless trees by Andreae and Schmidt [1]. Recently, the rank was used to verify the unfriendly partition conjecture for rayless graphs, see [4].

We shall need a few properties of the rank function that are either simple consequences of the definition or can be found in [6]. Let G be an infinite rayless graph, and let S be minimal among the sets as in (ii) of Definition 2. It is not hard to see that S is unique with this property. We call S the *kernel of G* and denote it by $K(G)$. Furthermore, it holds that:

- if H is a subgraph of G, then $\mathrm{rk}(H) \leq \mathrm{rk}(G)$; and

- if G is connected, then $K(G)$ is non-empty; and

- $\mathrm{rk}(G - X) = \mathrm{rk}(G)$ for any finite $X \subseteq V(G)$.

In particular, if C is a component of $G - K(G)$, then $G[C \cup K(G)]$ has smaller rank than G.

To illustrate the definition of the rank, let us note that an infinite star has rank 1, and its kernel consists of its centre. The same holds for the graphs in Figure 1. On the other hand, the disjoint union of infinitely many infinite stars (or in fact, of any graphs of rank 1) has rank 2 and an empty kernel.

3   The proofs

In this section we prove the 'strong twin' version of Theorem 1. All proofs will apply almost literally to the case of weak twins instead of strong twins. For that reason we will often drop the qualifiers 'strong' and 'weak'.

Let G, H be two rayless graphs and let $X \subseteq V(G)$ and $Y \subseteq V(H)$ be finite vertex subsets. We call a homomorphism $\phi : G \to H$ a *strong embedding of (G, X) in (H, Y)* if it is injective, $\phi(G)$ is an induced subgraph of H, and $\phi(X) \subseteq Y$. Alternatively, we shall say that $\phi : (G, X) \to (H, Y)$ is a strong embedding. Observe that ϕ preserves edges as well as non-edges. We call (G, X) and (H, Y) *isomorphic* if there is an isomorphism $\gamma : (G, X) \to (H, Y)$, i.e. if γ is a graph-isomorphism between G and H with $\gamma(X) = Y$. We say that (G, X) and (H, Y) are *strong twins* if they are not isomorphic and there exist strong embeddings $\phi : (G, X) \to (H, Y)$ and $\psi : (H, Y) \to (G, X)$; note that $\phi(X) = Y$ and $\psi(Y) = X$ in this case. For (G, X) and (H, Y) to be *weak twins* we only require ϕ and ψ to be injective homomorphisms with $\phi(X) = Y$ and $\psi(Y) = X$. Let us point out that rayless graphs G and H are (strong resp. weak) twins if and only if the tuples $(G, \emptyset)$ and $(H, \emptyset)$ are (strong resp. weak) twins.

As we have noted, a subgraph of a rayless graphs does not have larger rank than the graph itself. Moreover, if a subgraph G' of a rayless graph G has the same rank as G, then $K(G') \subseteq K(G)$ since $K(G) \cap V(G')$ is a set as in (ii) of Definition 2. We thus have:

Lemma 3. *Let G and H be rayless graphs, and let there be injective homomorphisms $\phi : G \to H$ and $\psi : H \to G$. Then $\phi(K(G)) = K(H)$ and $\psi(K(H)) = K(G)$.*

In particular, the lemma implies that if (G, X) and (H, Y) are twins, then $(G, X \cup K(G))$ and $(H, Y \cup K(H))$ are twins too.

Let G and H be rayless graphs, and let $X \subseteq V(G)$ and $Y \subseteq H$ be finite vertex sets. We write $\bar{X}$ as a shorthand for $X \cup K(G)$, and define $\bar{Y}$ analogously. Assume there are (strong) embeddings $\phi : (G, X) \to (H, Y)$ and $\psi : (H, Y) \to (G, X)$ and set $\iota := \psi \circ \phi$. Since, by Lemma 3, ι induces an automorphism on (the subgraph induced by) the finite set $\bar{X}$ there exists a k with $\iota^k \restriction \bar{X} = \mathrm{id}_{\bar{X}}$. By replacing ϕ with $\phi \circ \iota^{k-1}$, we may assume that

$$\phi : (G, X) \to (H, Y) \text{ and } \psi : (H, Y) \to (G, X) \text{ are embeddings} \tag{1}$$
$$\text{so that the restriction of } \iota = \psi \circ \phi \text{ to } \bar{X} \text{ coincides with } \mathrm{id}_{\bar{X}}.$$

Assume now that (G, X) and (H, Y) are isomorphic by virtue of an isomorphism γ. In that case, abusing symmetry and notation, let us write $(G, X) \simeq_\eta (H, Y)$, where η denotes the isomorphism $\bar{X} \to \bar{Y}$ induced by γ. Denote by $\mathcal{C}_G$ the set of all subgraphs $G[C \cup \bar{X}]$ of G, where C is a component of $G - \bar{X}$. For $A \in \mathcal{C}_G$ set

$$I_G(A) := \{D \in \mathcal{C}_G : (D, \bar{X}) \simeq_{\mathrm{id}} (A, \bar{X})\}.$$

Lemma 4. *Let G and H be rayless graphs, and let $X \subseteq V(G)$ and $Y \subseteq V(H)$ be finite. The following statements are equivalent.*

(i) *(G, X) and (H, Y) are isomorphic.*

(ii) *There is a bijection $\alpha : \mathcal{C}_G \to \mathcal{C}_H$ and an isomorphism $\eta : G[\bar{X}] \to G[\bar{Y}]$ with $\eta(X) = Y$ so that $(A, \bar{X}) \simeq_\eta (\alpha(A), \bar{Y})$ and $|I_G(A)| = |I_H(\alpha(A))|$ for all $A \in \mathcal{C}_G$.*

Moreover, if (i) and (ii) hold, then α, η, and the isomorphism $\phi : (G, X) \to (H, Y)$ can be chosen so that $\phi \restriction \bar{X} = \eta$ and $\phi(A) = \alpha(A)$ for every $A \in \mathcal{C}_G$.

Proof. First assume that (i) holds and let $\phi : (G, X) \to (H, Y)$ the isomorphism certifying this fact. Put $\eta := \phi \restriction \bar{X}$. Observe that, by Lemma 3, for every $A \in \mathcal{C}_G$ there is a $B \in \mathcal{C}_H$ with $\phi(A) = B$; set $\alpha(A) := B$. Clearly, α is a bijection and $(A, \bar{X}) \simeq_\eta (\alpha(A), \bar{Y})$. It remains to show that $|I_G(A)| = |I_H(\alpha(A))|$ for all $A \in \mathcal{C}_G$. Indeed, for every $C \in I_G(A)$ we have $\alpha(C) \in I_H(\alpha(A))$: Since $(A, \bar{X}) \simeq_{\mathrm{id}} (C, \bar{X})$, by virtue of an isomorphism γ say, $\phi \circ \gamma \circ \phi^{-1}$ is an isomorphism certifying $(\alpha(A), \bar{Y}) \simeq_{\mathrm{id}} (\alpha(C), \bar{Y})$. Hence we obtain $|I_H(\alpha(A))| \geq |I_G(A)|$ and analogously $|I_G(A)| \geq |I_H(\alpha(A))|$.

Now assume that (ii) holds. Then for every $A \in \mathcal{C}_G$ there is an isomorphism $\phi_A : A \to \alpha(A)$ that witnesses $(A, \bar{X}) \simeq_\eta (\alpha(A), \bar{Y})$. Now the function $\phi : G \to H$ defined by $\phi \restriction A := \phi_A$ for every A is an isomorphism of (G, X) and (H, Y) satisfying $\phi \restriction \bar{X} = \eta$ and $\phi(A) = \alpha(A)$ for every $A \in \mathcal{C}_G$. $\qquad\square$

We call the tuple (G, X) *connected* if $G - X$ is connected.

Lemma 5. *Let (G, X) and (H, Y) be strong twins, where G and H are rayless graphs, and $X \subseteq V(G)$ and $Y \subseteq V(H)$ finite. Then (G, X) has infinitely many strong twins. If both (G, X) and (H, Y) are connected, then (G, X) has infinitely many connected strong twins.*

Before we prove the lemma let us remark that it immediately implies the strong version of Theorem 1 if we set $X = Y = \emptyset$.

Proof of Lemma 5. We proceed by transfinite induction on the rank of G. For rank 0 the statement is trivially true as finite graphs do not have twins. We may thus assume that G has rank $\kappa > 0$ and that the lemma is true for rank smaller than κ.

Assume there exists a $C_0 \in \mathcal{C}_G$ so that $(C_0, \bar{X})$ has a connected twin. Then, as C_0 has rank smaller than κ, the inductive hypothesis provides us with infinitely many connected twins (C_i, X_i), $i > 0$, of $(C_0, \bar{X})$. By applying (1) to $(C_0, \bar{X})$ and (C_i, X_i) we may assume that the restrictions to $\bar{X}$ and X_i, respectively, of the mutual embeddings are inverse isomorphisms. Hence, by identifying X_i with $\bar{X}$ by this isomorphism we may assume that the twins have the form $(C_i, \bar{X})$ and that the corresponding embeddings induce the identity on $\bar{X}$. Denote by $\mathcal{T}$ the set of $C \in \mathcal{C}_G$ for which either $(C, \bar{X}) \simeq_{\mathrm{id}} (C_0, \bar{X})$, or for which $(C, \bar{X})$ is a twin of $(C_0, \bar{X})$ by virtue of mutual embeddings that each induce the identity on $\bar{X}$. For every $i \in \mathbb{N}$, define G_i to be the graph obtained from G by replacing every $C \in \mathcal{T}$ by a copy of C_i.

The construction ensures two properties. First, there are strong embeddings $(G, X) \to (G_i, X)$ and $(G_i, X) \to (G, X)$ for every i. So, if infinitely many of the (G_i, X) are non-isomorphic, we have found infinitely many twins of (G, X). Second, for $j \neq k$ it follows that $|I_{G_k}(C_j)| = 0 \neq |I_{G_j}(C_j)|$. Consequently, Lemma 4 implies

$$(G_j, \bar{X}) \not\simeq_{\mathrm{id}} (G_k, \bar{X}). \tag{2}$$

Assume that for distinct i, j, k the tuples (G_i, X), (G_j, X) and (G_k, X) are isomorphic. Thus, by Lemma 4 there are isomorphisms η between $\bar{X} \subseteq V(G_i)$ and $\bar{X} \subseteq V(G_j)$ and η' between $\bar{X} \subseteq V(G_i)$ and $\bar{X} \subseteq V(G_k)$ so that $(G_i, \bar{X}) \simeq_\eta (G_j, \bar{X})$ and $(G_i, \bar{X}) \simeq_{\eta'} (G_k, \bar{X})$. Now, if $\eta = \eta'$, then the resulting isomorphism between G_j and G_k would induce the identity on $\bar{X}$, which is impossible by (2). As there are only finitely many automorphisms of the finite set $\bar{X}$, we deduce that each (G_i, X) is isomorphic to only finitely many (G_j, X). Therefore we can easily find among the (G_i, X) infinitely many that are pairwise non-isomorphic.

Finally, we claim that if (G, X), i.e. $G - X$, is connected, then so is each (G_i, X), i.e. $G_i - X$. Indeed, by construction there is an embedding $(G, X) \to (G_i, X)$ that restricts to the identity on $\bar{X}$ and whose image meets all components of $G_i - \bar{X}$. As $G - X$ is connected, as well as each component of $G_i - \bar{X}$, we deduce that $G_i - X$ is connected.

Thus, we may assume from now on that

$$\text{for each } C \in \mathcal{C}_G, \ (C, \bar{X}) \text{ has no connected twin.} \tag{3}$$

By symmetry, the same holds for (H, Y).

Let $\phi : (G, X) \to (H, Y)$ and $\psi : (H, Y) \to (G, X)$ be strong embeddings, and recall that by (1) we may assume that $\iota := \psi \circ \phi$ induces the identity map on $\bar{X}$. By Lemma 4 and symmetry, we may assume that for $\eta := \phi \upharpoonright \bar{X}$ there are $A \in \mathcal{C}_G$ and $B \in \mathcal{C}_H$ with $(A, \bar{X}) \simeq_\eta (B, \bar{Y})$ so that $|I_G(A)| > |I_H(B)|$.

Observe that by Lemma 3

$$\text{for every } C \in \mathcal{C}_G \text{ there is a (unique) } D \in \mathcal{C}_G \text{ with } \iota(C) \subseteq D. \tag{4}$$

Furthermore, we point out that ι is a strong self-embedding of (G, X), and also of $(G, \bar{X})$.

We define a directed graph Γ on $\mathcal{C}_G$ as vertex set by declaring (C, D) to be an edge if $\iota(C) \subseteq D$ for $C, D \in \mathcal{C}_G$. We do allow Γ to have loops and parallel edges (which then, necessarily, are pointing in opposite directions). Note that by (4) every vertex in Γ has out-degree one. Define $\mathcal{A}$ to be the set of those $A' \in I_G(A)$ for which the unique out-neighbour does not lie in $I_G(A)$.

Suppose that distinct $A_1, A_2 \in I_G(A)$ are mapped by ϕ into the same $B' \in \mathcal{C}_H$. If A_1 (and then also A_2) is finite, then $|V(B')| > |V(A_i)|$ for $i = 1, 2$ since the injectivity of ϕ implies $\phi(A_1) \cap \phi(A_2) = \bar{Y}$. Consequently, we obtain $B' \notin I_H(B)$. Let now A_1 and A_2 be infinite. Unless $\mathrm{rk}(B') > \mathrm{rk}(A_1) = \mathrm{rk}(A_2)$ it follows that $\phi(K(A_i)) \subseteq K(B')$ for $i = 1, 2$. Since $A_1 - \bar{X}$ and $A_2 - \bar{X}$ are connected the kernels $K(A_i - \bar{X})$ are non-empty (but finite). Again from $\phi(A_1) \cap \phi(A_2) = \bar{Y}$ we obtain that $K(B')$ has larger cardinality than either of $K(A_1)$ and $K(A_2)$, which implies $B' \notin I_H(B)$. Therefore, we have in all cases that $B' \notin I_H(B)$. Since (3) and (4) necessitate that $\phi(A')$ is contained in an element of $I_H(B)$ for every $A' \in I_G(A) \setminus \mathcal{A}$ we deduce that $|\mathcal{A}| \geq |I_G(A)| - |I_H(B)|$. Thus, it holds that

$$\mathcal{A} \neq \emptyset, \text{ and if } I_G(A) \text{ is infinite, then we have } |\mathcal{A}| = |I_G(A)|. \tag{5}$$

By construction, the set $\mathcal{A}$ is independent in Γ. Moreover,

$$\begin{array}{l} \text{there is no directed path in } \Gamma \text{ starting in } \mathcal{A} \text{ and ending in } I_G(A), \\ \text{and there is no directed cycle containing any } A' \in \mathcal{A}. \end{array} \tag{6}$$

To prove (6), suppose that $C_1, \ldots, C_k$ is a directed path in Γ with $C_1 \in \mathcal{A}$ and $C_k \in I_G(A)$ (possibly even $C_k \in \mathcal{A}$). Since repeated application of ι maps every $(C_1, \bar{X})$ into any $(C_i, \bar{X})$ and likewise $(C_i, \bar{X})$ into $(C_k, \bar{X}) \simeq_{\mathrm{id}} (C_1, \bar{X})$, we deduce that $(C_i, \bar{X}) \simeq_{\mathrm{id}} (C_j, \bar{X})$ for $i, j \in \{1, \ldots, k\}$, as they cannot be twins by (3) (recall that $\iota \upharpoonright \bar{X} = \mathrm{id}_{\bar{X}}$ by (1)). However, $(C_1, \bar{X}) \simeq_{\mathrm{id}} (C_2, \bar{X})$ violates $C_1 \in \mathcal{A}$. The same arguments hold if $C_1, \ldots, C_k$ is a directed cycle that meets $\mathcal{A}$.

Define $\mathcal{A}^-$ to be the set of all $C \in \mathcal{C}_G$ from which there is a nontrivial directed path in Γ ending in $I_G(A)$ (in particular, $I_G(A) \setminus \mathcal{A} \subseteq \mathcal{A}^-$). Setting $\mathcal{A}^+ := \mathcal{C}_G \setminus (\mathcal{A} \cup \mathcal{A}^-)$ we see with (6) that $(\mathcal{A}^-, \mathcal{A}, \mathcal{A}^+)$ partitions $\mathcal{C}_G$. By definition, the out-neighbour of an $A' \in \mathcal{A}$ does not lie in $\mathcal{A}$, and by (6) the out-neighbour does not lie in $\mathcal{A}^-$ either. Hence, we have $\iota(A') \subseteq \bigcup \mathcal{A}^+$. On the other hand, the definition of $\mathcal{A}^-$ implies that the out-neighbour of every $A^+ \in \mathcal{A}^+$ is contained in $\mathcal{A}^+$. Thus it follows that $\iota(A^+) \subseteq \bigcup \mathcal{A}^+$. In summary, we obtain:

$$(\mathcal{A}^-, \mathcal{A}, \mathcal{A}^+) \text{ partitions } \mathcal{C}_G \text{ and } \iota\left(\bigcup \mathcal{A} \cup \bigcup \mathcal{A}^+\right) \subseteq \bigcup \mathcal{A}^+. \tag{7}$$

We claim that there exists a strong self-embedding $\gamma : (G, X) \to (G, X)$ that induces the identity on $\bigcup I_G(A) \setminus \mathcal{A}$ (in particular on $\bar{X}$) and satisfies

$$\gamma(G) \cap \bigcup \mathcal{A} = \bar{X}. \tag{8}$$

On $\bar{X}$ we define γ to be the identity. For every other vertex $v \in V(G)$ we consider the unique $C \in \mathcal{C}_G$ containing v. If $C \in \mathcal{A}^-$ we set $\gamma(v) := v$, and if $C \in \mathcal{A} \cup \mathcal{A}^+$ we put $\gamma(v) := \iota(v)$. Note that by (4) it holds that for every $C \in \mathcal{C}_G$ we have $\gamma \restriction C = \mathrm{id}_C$ or $\gamma \restriction C = \iota \restriction C$. It is immediate from (7) that (8) holds. Moreover, since the identity as well as ι are strong self-embeddings it follows from (7) that γ is one, too.

If $I_G(A)$ is infinite, then by (5) we change γ on each component in $I_G(A) \setminus \mathcal{A}$ so as to obtain a strong self-embedding φ whose image avoids $\bigcup I_G(A) \setminus \mathcal{A} - \bar{X}$. Then $\beta := \varphi^2$ is a strong self-embedding that induces the identity on $\bar{X}$ and satisfies

$$\beta(G) \cap \bigcup I_G(A) = \bar{X}. \tag{9}$$

Let us now construct infinitely many strong twins of (G, X). Assume first that $I_G(A)$ is a finite set. Add a disjoint copy $\tilde{A}$ of A to G and identify every vertex in $\bar{X}$ with its copy in $\tilde{A}$. The resulting graph G_1 is clearly a supergraph of G. But by (8) we can also embed (G_1, X) in (G, X): extend γ to an embedding of (G_1, X) in (G, X) by mapping $\tilde{A} - \bar{X}$ to $A' - \bar{X}$ for some $A' \in \mathcal{A}$. Here, we use that $\mathcal{A} \neq \emptyset$, by (5). Note that $|I_{G_1}(A)| = |I_G(A)| + 1$. Now we repeat this process, with G_1 in the role of G, so as to obtain G_2, and so on. Since $|I_{G_i}(A)| \neq |I_{G_j}(A)|$ for all $i \neq j$, we can deduce from Lemma 4, as in the proof of (3), that each (G_i, X) is isomorphic to only finitely many (G_j, X). Therefore we can find among the (G_i, X) infinitely many twins of (G, X).

So, consider the case when $I_G(A)$ contains infinitely many elements $A_1, A_2, \ldots$. Set $G_i := G - (\bigcup I_G(A) \setminus \{A_1, \ldots, A_i\} - \bar{X})$ for $i \in \mathbb{N}$. Since, by (9), β can be used to embed (G, X) in (G_i, X) we can again find infinitely many twins of (G, X)—note that $|I_{G_i}(A)|$ takes different (finite) values.

Finally, observe that in both cases, all the strong twins we constructed are connected if (G, X) is. $\square$

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