All graphs have tree-decompositions displaying their topological ends

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Abstract

We show that every connected graph has a spanning tree that displays all its topological ends. This proves a 1964 conjecture of Halin in corrected form, and settles a problem of Diestel from 1992.

1 Introduction

In 1931, Freudenthal introduced a notion of *ends* for second countable Hausdorff spaces [16], and in particular for locally finite graphs [17]. Independently, in 1964, Halin [19] introduced a notion of *ends* for graphs, taking his cue directly from Carathéodory's *Primenden* of simply connected regions of the complex plane [4]. For locally finite graphs these two notions of ends agree.

For graphs that are not locally finite, Freudenthal's topological definition still makes sense, and gave rise to the notion of *topological ends* of arbitrary graphs [13]. In general, this no longer agrees with Halin's notion of ends, although it does for trees.

Halin [19] conjectured that the end structure of every connected graph can be displayed by the ends of a suitable spanning tree of that graph. He proved this for countable graphs. Halin's conjecture was finally disproved in the 1990s by Seymour and Thomas [23], and independently by Thomassen [26].

In this paper we shall prove Halin's conjecture in amended form, based on the topological notion of ends rather than Halin's own graph-theoretical notion. We shall obtain it as a corollary of the following theorem, which proves a conjecture of Diestel [11] of 1992 (again, in amended form):

Theorem 1. *Every graph has a tree-decomposition $(T, \mathcal{V})$ of finite adhesion such that the ends of T define precisely the topological ends of G.*
See Section 2 for definitions.

The tree-decompositions constructed for the proof of Theorem 1 have several further applications. In [5] we use them to answer the question to what extent the ends of a graph - now in Halin's sense - have a tree-like structure at all. In [6], we apply Theorem 1 to show that the topological cycles of any graph together with its topological ends induce a matroid.

This paper is organised as follows. In Section 2 we explain the problems of Diestel and Halin in detail, after having given some basic definitions. In Section 3 we continue with examples related to these problems. Section 4 only contains material that is relevant for Section 5 in which we prove that every graph has a nested set of separations distinguishing the vertex ends efficiently. In Section 6, we use this theorem to prove Theorem 1. Then we deduce Halin's amended conjecture.

2 Definitions

Throughout, notation and terminology for graphs are that of [12] unless defined differently. And G always denotes a graph.

A *vertex end* in a graph G is an equivalence class of rays (one-way infinite paths), where two rays are equivalent if they cannot be separated in G by removing finitely many vertices. Put another way, this equivalence relation is the transitive closure of the relation relating two rays if they intersect infinitely often.

Let X be a locally connected Hausdorff space. Given a subset $Y \subseteq X$, we write $\overline{Y}$ for the closure of Y, and $F(Y) := \overline{Y} \cap \overline{X \setminus Y}$ for its frontier. In order to define the topological ends of X, we consider infinite sequences $U_1 \supseteq U_2 \supseteq ...$ of non-empty connected open subsets of X such that each $F(U_i)$ is compact and $\bigcap_{i \geq 1} \overline{U_i} = \emptyset$. We say that two such sequences $U_1 \supseteq U_2 \supseteq ...$ and $U_1' \supseteq U_2' \supseteq ...$ are *equivalent* if for every i there is some j with $U_i \supseteq U_j'$. This relation is transitive and symmetric [16, Satz 2]. The equivalence classes of those sequences are the *topological ends* of X [13, 16, 22].

For the simplical complex of a graph G, Diestel and Kühn described the topological ends combinatorically: a vertex *dominates* a vertex end ω if for some (equivalently: every) ray R belonging to ω there is an infinite fan of v-R-paths that are vertex-disjoint except at v. In [13], they proved that the topological ends are given by the undominated vertex ends. Hence in this paper, we take this as our definition of *topological end of G*.

We denote the complement of a set X by $X^{\complement}$. For an edge set X, we denote by $V(X)$, the set of vertices incident with edges from X. For a vertex set W, we denote by s_W, the set of those edges with at least one endvertex in W.

For us, a *separation* is just an edge set. A *vertex-separation* in a graph G is an ordered pair (A, B) of vertex sets such that there is no edge of G with one endvertex in $A \setminus B$ and the other in $B \setminus A$. A separation X *induces* the vertex-separation $(V(X), V(X^{\complement}))$. Thus in general there may be several separations inducing the same vertex-separation. The *boundary* $\partial(X)$ of a separation X is the set of those vertices adjacent with an edge from X and one from $X^{\complement}$. The *order* of X is the size of $\partial(X)$. A separation X is *componental* if there is a component C of $G - \partial(X)$ such that $s_C = X$. Two separations X and Y are *nested* if one of the following 4 inclusions is true: $X \subseteq Y$, $X^{\complement} \subseteq Y$, $Y \subseteq X$ or $Y \subseteq X^{\complement}$. If there is a vertex in $\partial(Y) \setminus V(X)$, then it is incident with an edge from $Y \setminus X$ and an edge from $Y^{\complement} \setminus X$. Thus if additionally, X and Y are nested, then either $X^{\complement} \subseteq Y$ or $Y \subseteq X^{\complement}$. We shall refer to the four sets $\partial(Y) \setminus V(X)$, $\partial(Y) \setminus V(X^{\complement})$, $\partial(X) \setminus V(Y)$ or $\partial(X) \setminus V(Y^{\complement})$ as the *links of X and Y*.

A vertex end ω *lives* in a separation X of finite order if $V(X)$ contains one (equivalently: every) ray belonging to ω. Similarly, we define when a vertex end lives in a component. A separation X of finite order *distinguishes* two vertex ends ω and μ if one of them lives in X and the other in $X^{\complement}$. It distinguishes them *efficiently* if X has minimal order amongst all separations distinguishing ω and μ.

A *tree-decomposition* of G consists of a tree T together with a family of subgraphs $(P_t | t \in V(T))$ of G such that every vertex and edge of G is in at least one of these subgraphs, and such that if v is a vertex of both P_t and P_w, then it is a vertex of each P_u, where u lies on the v-w-path in T. Moreover, each edge of G is contained in precisely one P_t. We call the subgraphs P_t, the *parts* of the tree-decomposition. Sometimes, the "Moreover"-part is not part of the definition of tree-decomposition. However, both these two definitions give the same concept of tree-decomposition since any tree-decomposition without this additionally property can easily be changed to one with this property by deleting edges from the parts appropriately. The *adhesion* of a tree-decomposition is finite if adjacent parts intersect only finitely. Given a directed edge tu of T, the *separation corresponding to tu* consists of those edges contained in parts P_w, where w is in the component of $T - t$ containing

u.

In [2, 21, 25], tree-decompositions of finite adhesion are used to study the structure of infinite graphs. In [11, Problem 4.3], Diestel wanted to know whether every graph G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion that somehow encodes the structure of the graph with its ends.

Let us be more precise: Given a vertex end ω, we take $O(\omega)$ to consist of those oriented edges tu of T such that ω lives in its corresponding separation. Note that $O(\omega)$ contains precisely one of tu and ut. Furthermore this orientation $O(\omega)$ of T points towards a node of T or to an end of T. We say that ω *lives* in the part for that node or that end, respectively.

A vertex end ω is *thin* if every set of vertex-disjoint rays belonging to ω is finite; otherwise ω is *thick*. Diestel asked whether every graph has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that different thick vertex ends live in different parts and such that the ends of T *define precisely* the thin vertex ends. Here the ends of T *define precisely* a set W of vertex ends of G if in every end of T there lives a unique vertex end and it is in W and conversely every vertex end in W lives in some end of T.

Unfortunately, that is not true: In Example 3.1, we construct a graph such that each of its tree-decompositions of finite adhesion has a part in which two (thick) vertex ends live. Moreover, in Example 3.5, we construct a graph that does not have a tree-decomposition of finite adhesion such that the ends of its decomposition tree define precisely the thin vertex ends.

Hence the remaining open question is whether there is a natural subclass of the vertex ends (similar to the class of thin vertex ends) such that every graph has a tree-decomposition of finite adhesion such that the ends of its decomposition tree define precisely the vertex ends in that subclass. Theorem 1 above answers this question affirmatively.

It is impossible to construct a tree-decomposition as in Theorem 1 with the additional property that for any two topological ends ω and μ, there is a separation corresponding to an edge of the tree that separates ω and μ efficiently, see Example 3.6.

A recent development in the theory of infinite graphs seeks to extend theorems about finite graphs and their cycles to infinite graphs and the topological circles formed with their ends, see for example [1, 3, 14, 15, 18, 24], and [10] for a survey. We expect that Theorem 1 has further applications in this direction aside from the one mentioned in the Introduction.

A rooted spanning tree T of a graph G is *end-faithful* for a set Ψ of vertex ends if each vertex end $\omega \in \Psi$ is uniquely represented by T in the sense that T contains a unique ray belonging to ω and starting at the root. For example, every normal spanning tree is end-faithful for all vertex ends. Halin conjectured that every connected graph has an end-faithful tree for all vertex ends. At the end of Section 6, we show that Theorem 1 implies the following nontrivial weakening of this disproved conjecture:

Corollary 2.1. *Every connected graph has an end-faithful spanning tree for the topological ends.*

One might ask whether it is possible to construct an end-faithful spanning tree for the topological ends with the additional property that it does not include any ray to any other vertex end. However, this is not possible in general. Indeed, Seymour and Thomas constructed a graph G with no topological end that does not have a rayless spanning tree [23].

3 Example section

Throughout this section, we denote by T_2 the infinite rooted binary tree, whose nodes are the finite 0-1-sequences and whose ends are the infinite ones. In particular, its root is denoted by the empty sequence ϕ.

Example 3.1. In this example, we construct a graph G such that all its tree-decompositions of finite adhesion have a part in which two vertex ends live. We obtain G from T_2 by adding a single vertex v_ω for each of the continuum many ends ω of T_2, which we join completely to the unique ray belonging to ω starting at the root. Note that the vertex ends of G are the ends of T_2. For a finite path P of T_2 starting at ϕ, we denote by $A(P)$, the set of those vertex ends of G whose corresponding 0-1-sequence begins with the finite 0-1-sequence which is the last vertex of P.

Lemma 3.2. *The set of vertex ends of G that live in a finite order separation Z of G is open and closed in the end-topology of T_2 restricted to the set of vertex ends* [1]*.*

Proof. The set of vertex ends of G that live in a finite order separation of G are finite unions of sets of the form $A(P)$, so they are open and closed in the end-topology of T_2. □

Suppose for a contradiction that there is a tree-decomposition $(T, P_t | t \in V(T))$ of G of finite adhesion such that in each of its parts lives at most one vertex end.

[1] A basis of this topology is given by the sets of vertex ends living in sets of the form $A(P)$.

Lemma 3.3. *For each $k \in \mathbb{N}$, there is a separation X_k corresponding to a directed edge $t_k u_k$ of T together with a finite path P_k of T of length k starting at ϕ satisfying the following.*

1. *uncountably many vertex ends of $A(P_k)$ live in X_k;*
2. *$X_{k+1} \subseteq X_k$;*
3. *$P_k \subseteq P_{k+1}$;*
4. *If $v_\omega \in \partial(X_k)$, then ω does not live in X_{k+1}.*

Proof. We start the construction with picking $P_0 = \phi$ and X_0 such that uncountably many vertex ends live in it. Assume that we already constructed for all $i \leq k$ separations X_i and P_i satisfying the above. Let Q_k and R_k be the two paths of T_2 starting at ϕ of length $k+1$ extending P_k. Then $A(P_k)$ is a disjoint union of $A(Q_k)$ and $A(R_k)$. For P_{k+1} we pick one of these two paths of length $k+1$ such that uncountably many vertex ends of $A(P_{k+1})$ live in X_k;

Let S_k be the component of $T - t_k$ containing u_k. Let F_k be the set of those directed edges of S_k directed away from u_k. Note that if some separation X corresponds to some $ab \in F_k$, then $X \subseteq X_k$. Actually, we will find $t_{k+1}u_{k+1}$ in F_k. We colour an edge of F_k *red* if uncountably many vertex ends of $A(P_{k+1})$ live in the separation corresponding to that edge. If $ab \in F_k$ is not red, then in its separation does not live any vertex-end of $A(P_{k+1})$ by Lemma 3.2.

Suppose for a contradiction that there is a constant c such that for each r, there are at most c red edges of F_k with distance r from $t_k u_k$ in T. Let W be the subforest of T consisting of the red edges. Note that W is a tree with at most c vertex ends. If no vertex end of $A(P_{k+1})$ lives in parts of nodes of W, then all vertex ends of $A(P_{k+1})$ that live in X_k live in ends of W. If a vertex end lives in an end τ of W, then the vertex dominating it must eventually be contained in the separators on the rooted ray to τ. Since W has only countably many ends, we get the desired contradiction.

So it remains to consider the case that a vertex end ω of $A(P_{k+1})$ lived in a part of a node t of W. Then ω is in the closure of the set of all vertex ends of $A(P_{k+1})$ that live in separations of red outedges of t by Lemma 3.2 applied to the inedge of t. Applying Lemma 3.2 to each outedge, we deduced that there must be infinitely many outedges in which vertex ends of $A(P_{k+1})$ live. So there are infinitely many red edges at a fixed distance from $t_k u_k$, which is a contradiction.

Hence there is some distance r such that there are at least $|\partial(X_k)| + 1$ red edges of F_k with distance r from $t_k u_k$ in T. Each vertex end ω with $v_\omega \in \partial(X_k)$ can live in at most one separation corresponding to one of these edges. Hence amongst these red edges we can pick $t_{k+1}u_{k+1}$ such that no such ω lives in its corresponding separation X_{k+1}. Clearly, X_{k+1} and P_{k+1} have the desired properties, completing the construction. □

Lemma 3.4. *Let X_k and P_k be as in Lemma 3.3. Then $P_k \subseteq V(X_k)$.*

Proof. By 1, uncountably many vertex ends of $A(P_k)$ live in X_k. Thus infinitely many of their corresponding vertices v_ω are in $V(X_k)$. Since only finitely many of these vertices can be in $\partial(X_k)$, one of these vertices has all its incident edges in X_k. Since P_k is in its neighbourhood, it must be that $P_k \subseteq V(X_k)$. □

Having proved Lemma 3.3 and Lemma 3.4, it remains to derive a contradiction from the existence of the X_k and P_k. By construction $R = \bigcup_{k\in\mathbb{N}} P_k$ is ray. Let μ be its vertex end. By Lemma 3.4, $R \subseteq V(X_k)$ so that μ lives in each X_k. Hence $v_\mu \in V(X_k)$ for all k. Let e be any edge of G incident with v_μ. As each edge of G is in precisely one part P_t, the edge e is eventually not in X_k. Hence v_μ is eventually in $\partial(X_k)$, contradicting 4 of Lemma 3.3. Hence there is no tree-decomposition $(T, P_t | t \in V(T))$ of G of finite adhesion such that in each of its parts lives at most one vertex end.

Example 3.5. In this example, we construct a graph G that does not have a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that the thin vertex ends of G define precisely the ends of T. Let Γ be the set of those ends of T_2 whose 0-1-sequences are eventually constant and let $\omega_1, \omega_2, \dots$ be an enumeration of Γ. We represent each end ω of T_2 by the unique ray $R(\omega)$ starting at the root and belonging to ω.

For $n \in \mathbb{N}^*$, let H_n be the graph obtained by T_2 by deleting each ray $R(\omega_i)$ for each $i \leq n$. We obtain G from T_2 by adding for each natural number n the graph H_n where we join each vertex of T_2 with each of its clones in the graphs H_n. Note that a vertex in $R(\omega_n)$ has at most n clones.

It is clear from this construction that T_2 is a subtree of G whose ends are those of G. For every vertex end ω not in Γ, there are infinitely many vertex-disjoint rays in G belonging to ω, one in each H_n. For $\omega_n \in \Gamma$ and $v \in R(\omega_n)$, let $S_n(v)$ be the set of v and its clones. Each ray in G belonging to ω intersects the separators $S_n(v)$ eventually. Thus as $|S_n(v)| \leq n$, there are at most n vertex-disjoint rays belonging to ω_n. Hence the thin vertex ends of G are precisely those in Γ.

Suppose G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that the thin vertex ends live in different ends of T. It remains to show that there is a vertex end of T in which no vertex end of Γ lives. For that, we shall recursively construct a sequence of separations $(A_n | n \in \mathbb{N}^*)$ that correspond to edges of T satisfying the following.

1. A_{n+1} is a proper subset of A_n;
2. infinitely many vertex ends of Γ live in A_n but none of $\{\omega_1, \dots, \omega_n\}$.

We start the construction by picking an edge of T arbitrarily; one of the two separations corresponding to that edge satisfies 2 and we pick such a separation for A_1. Now assume that we already constructed $A_1, \dots, A_n$ satisfying 1 and 2. By assumption, there are two distinct vertex ends α and β in Γ that live in A_n. If possible, we pick $\beta = \omega_{n+1}$. Since α and β live in different ends of T, there must be some separation A_{n+1} corresponding to an edge of T such that α lives in A_{n+1} but β does not.

We claim that A_{n+1} is a proper subset of A_n. Indeed, A_{n+1} and A_n are nested and as α lives in both of them, either $A_n \subseteq A_{n+1}$ or $A_{n+1} \subseteq A_n$. Since β witnesses that the first cannot happen, it must be that A_{n+1} is a proper subset of A_n.

Having seen that A_{n+1} satisfies 1, note that it also satisfies 2 since by construction one vertex end of Γ lives in A_{n+1}, which entails that infinitely many vertex ends of Γ live in A_{n+1} because for each finite separator S of G, each infinite component of $G - S$ contains infinitely many vertex ends from Γ.

Having constructed the sequence of separations $(A_n | n \in \mathbb{N}^*)$ as above, let e_n be the edge of T to which A_n corresponds. The set of the edges e_n lies on a ray of T but no vertex end in Γ lives in the end of that ray by 2, completing this example.

Example 3.6. In this example, we construct a graph G such that for any tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion that distinguishes the topological ends, there are two topological ends such that no separation corresponding to an edge of T distinguishes them efficiently.

Given two graphs G and H, by $G \times H$, we denote the graph with vertex set $V(G) \times V(H)$ where we join two vertices (g, h) and (g', h') by an edge if both $g = g'$ and $hh' \in E(G)$ or both $h = h'$ and $gg' \in E(G)$. Given a set of natural numbers X, by $\overline{X}$ we denote the graph with vertex set X where two vertices are adjacent if they have distance 1.

We start the construction with the graph $W = \overline{\mathbb{N}^*} \times \overline{\{1, 2, 3, 4, 5\}}$. Then for each $k \geq 2$, we glue on the vertex set $R_k = \{1, ..., k\} \times \{4\} + (k, 5) + (k -$

1, 5) the graph $H_k = \overline{\mathbb{N}^*} \times W[R_k]$ by identifying $(l, i) \in R_k$ with $(1, l, i)$.[2] Let ω_k be the vertex end whose subrays are eventually in H_k. Note that ω_k is undominated.

Similarly, we glue the graphs $H'_k = \overline{\mathbb{N}^*} \times W[R'_k]$ on the vertex sets $R'_k = \{1, ..., k\} \times \{2\} + (k, 1) + (k-1, 1)$. By μ_k we denote the vertex end whose subrays are eventually in H'_k.

For $k < m$, the separator $S_k = (\{1, ..., k\} \times \{4\}) + (k, 5)$ separates ω_k from μ_m and every other separator separating ω_k from ω_m has strictly larger order. Note that $G - S_k$ has precisely two components, one containing $(1, 1)$ and the other containing $(1, 5)$. Thus every separation X with $\partial(X) = S_k$ has the property that precisely one of $(1, 1)$ and $(1, 5)$ is in $V(X)$.

Now let $(T, P_t | t \in V(T))$ be a tree-decomposition of finite adhesion that distinguishes the set of topological ends. Let P_t be a part containing $(1, 1)$ and P_u be a part containing $(1, 5)$. If X is a separation corresponding to an edge e of T and precisely one of $(1, 1)$ and $(1, 5)$ is in $V(X)$, then e lies on the finite t-u-path in T. Thus there are only finitely many such X so that there is some $k \in \mathbb{N}^*$ such that S_k is not the separator of any X corresponding to an edge of T. Thus there are two topological ends that are not distinguished efficiently by $(T, P_t | t \in V(T))$.

4 Separations and profiles

In this section, we define profiles and prove some intermediate lemmas that we will apply in Section 5.

4.1 Profiles

Profiles [7] are slightly more general objects than tangles which are a central concept in Graph Minor Theory. Readers familiar with tangles will not miss a lot if they just think of tangles instead of profiles. In fact, they can even skip the definition of robustness of a profile below as tangles are always robust.

For two separations X and Y, we denote by $L(X, Y)$ the intersection of $V(X) \cap V(Y)$ and $V(X^\complement) \cup V(Y^\complement)$. Note that $\partial(X \cap Y) \subseteq L(X, Y)$ and there may be vertices in $L(X, Y)$ that only have neighbours in $X \setminus Y$ and $Y \setminus X$ so that they are not in $\partial(X \cap Y)$.

Remark 4.1. $|L(X, Y)| + |L(X^\complement, Y^\complement)| = |\partial(X)| + |\partial(Y)|$. □

[2] Here $W[R_k]$ denotes the induced subgraph of W with vertex set R_k.

Definition 4.2. A profile[3] P of order $k+1$ is a set of separations of order at most k that does not contain any singletons and that satisfies the following.

(P0) for each X with $\partial(X) \leq k$, either $X \in P$ or $X^{\complement} \in P$;

(P1) no two $X, Y \in P$ are disjoint;

(P2) if $X, Y \in P$ and $|L(X, Y)| \leq k$, then $X \cap Y \in P$;

(P3) if $X \in P$, then there is a componental separation $Y \subseteq X$ with $Y \in P$.

Note that (P1) implies that $\emptyset \notin P$. Under the presence of (P0) the axiom (P1) is equivalent to the following: if $X \in P$ and $X \subseteq Y$ with $\partial(Y) \leq k$, then $Y \in P$. So far profiles have only been defined for finite graph [7], and for them the definition given here is equivalent to one in [7]. Indeed, for finite graphs, there is an easy induction argument which proves (P3) from the other axioms. In infinite graphs, we get a different notion of profile if we do not require (P3) - for example if we leave out (P3), there is a profiles of order 3 on the infinite star.

If we replace '$L(X, Y)$' by '$\partial(X, Y)$', then this will define *tangles*; indeed, under the presence of (P1) it can be shown that the modified (P2) is equivalent to the axiom that no three small sides cover G. Thus every tangle of order $k+1$ induces a profile of order $k+1$, where a separation X of order at most k is in the induced profile if and only if the tangle says that it is the big side (formally, this means that X is not in the tangle). However, there are profiles of order $k+1$ that do not come from tangles, see [8, Section 6].

A separation X *distinguishes* two profiles P and Q if $X \in P$ and $X^{\complement} \in Q$ or vice versa: $X \in Q$ and $X^{\complement} \in P$. It distinguishes them *efficiently* if X has minimal order amongst all separations distinguishing P and Q. Given $r \in \mathbb{N} \cup \{\infty\}$ and $k \in \mathbb{N}$, a profile P of order $k+1$ is *r-robust* if there does not exist a separation X of order at most r together with a separation Y of order $\ell \leq k$ such that $L(X, Y) < \ell$ and $L(X^{\complement}, Y) < \ell$ and $Y \in P$ but both $Y \setminus X$ and $Y \setminus X^{\complement}$ are not in P. Note that every profile of order $k+1$ is r-robust for every $r \leq k$.

The notion of a profile is closely related to the well-known notion of a haven, defined next. Two subgraph of an ambient graph *touch* if they share a vertex or there is an edge of the ambient graph connecting a vertex from the first subgraph with a vertex from the second one. A vertex *touches* if the subgraph just consisting of that vertex touches. A *haven* of order $k+1$

[3] In [7], profiles were introduced using vertex-separations. However, it is straightforward to check that the definition given here gives the same concept of profiles.

consists of a choice of a component of $G - S$ for each separator S of size at most k such that any two of these chosen components touch. Note that if a component C is a component of both $G - S$ and $G - T$ for separators of order at most k, then it is in the haven for S if and only if it is in haven for T. Hence we can just say that a component is in a haven without specifying a particular separator.

Given a profile P of order $k+1$, for each separator S of order at most k, there is a unique component C of $G - S$ such that $s_C \in P$ by (P1) and (P3). By (P1), the collection of these components is a haven of order $k+1$. We say that this *haven* is induced by P. A haven of order $k+1$ is *good* if for any two separators S and T of size at most k and the components C and D of $G-S$ and $G-T$ that are in the haven, the set $C \cap D$ is also in the haven as soon as there are at most k vertices in $S \cup T$ that touch both C and D.

Remark 4.3. *A haven is good if and only if it is induced by a profile.* □

In [5], we further explain the connections between vertex ends, havens and profiles.

4.2 Torsos

An $\mathcal{N}$*-block* is a maximal set of vertices no two of which are separated by a separation in $\mathcal{N}$. A separation $X \in \mathcal{N}$ *distinguishes* two $\mathcal{N}$-blocks B and D if there are vertices in $B \setminus \partial(X)$ and $D \setminus \partial(X)$. Note that if B and D are different $\mathcal{N}$-blocks, then there is some $X \in \mathcal{N}$ distinguishing them.

Until the end of this subsection, let us fix a nested set $\mathcal{N}$ of separations and an $\mathcal{N}$-block B. We obtain the *torso* $G_T[B]$ of B from $G[B]$ by adding those edges xy such that there is some $X \in \mathcal{N}$ with $x, y \in \partial(X)$. This definition is compatible with the usual definition of torso [12] in the context of tree-decompositions: if $\mathcal{N}$ is the set of separations corresponding to the edges of a tree-decomposition, then the vertex set of every maximal part is an $\mathcal{N}$-block and its torso is just the torso of that part.

Lemma 4.4. *Let C be a component of $G - B$ whose neighbourhood $N(C)$ is finite. Then there is some $X \in \mathcal{N}$ such that $N(C) \subseteq \partial(X)$.*

In particular, $N(C)$ is complete in $G_T[B]$.

Proof. Let $U \subseteq N(C)$ be maximal such that there is some $X \in \mathcal{N}$ separating a vertex of C from B with $U \subseteq \partial(X)$. Suppose for a contradiction there is some $y \in N(C) \setminus U$. Pick $X \in \mathcal{N}$ with $U \subseteq \partial(X)$. Then $\partial(X)$ contains a vertex of C. Pick such an X such that the distance from y to $\partial(X) \cap C$ is minimal. Let $z \in \partial(X) \cap C$ with minimal distance to y and let $Z \in \mathcal{N}$ be a separation separating z from B. Without loss of generality we may assume that $B \subseteq V(X)$ and $B \subseteq V(Z)$. Since z is in the link $\partial(X) \setminus V(Z)$ and X and Z are nested, the link $\partial(X) \setminus V(Z^{\complement})$ is empty. Thus $U \subseteq \partial(Z)$. By the minimality of the distance, it cannot be that $X^{\complement} \subseteq Z^{\complement}$. So $X \subseteq Z^{\complement}$ as this is the only left possibility for X and Z to be nested. Hence $B \subseteq \partial(Z) \cap \partial(X)$. Hence $y \in U$, which is the desired contradiction. Thus $U = N(C)$. □

Given a separation Y of G that is nested with $\mathcal{N}$, the separation Y_B *induced by* Y in the torso $G_T[B]$ is obtained from $Y \cap E(G[B])$ by adding those edges $xy \in E(G_T[B])$ such that there is some $X \in \mathcal{N}$ with $x, y \in \partial(X)$ and $V(X) \subseteq V(Y)$ or $V(X^{\complement}) \subseteq V(Y)$.

Remark 4.5. $\partial(Y_B) \subseteq \partial(Y)$. □

The vertex-separation (C, D) of G induced by Y *induces* the separation $(C \cap B, D \cap B)$ of $G_T[B]$. In general $(C \cap B, D \cap B)$ differs from the vertex-separation induced by Y_B.

Remark 4.6. *Let H be a haven of order $k+1$. Assume that for every vertex set $S \subseteq B$ of at most k vertices the unique component C_S of $G - S$ in H intersects B. Let H_B be the haven induced by H: for each $S \subseteq B$ of at most k vertices, H_B picks the unique component C^S of $G_T[B] - S$ that includes $C_S \cap B$. Then H_B is a haven of order $k+1$. Moreover, if H is good, then so is H_B.*

Proof. If C_S and D_S touch, then so do C^S and D^S by Lemma 4.4. Thus H_B is a haven of order $k+1$. The 'Moreover'-part is clear. □

Let P be a profile of order $k+1$ and H be its induced good haven, then under the circumstances of Remark 4.6 we define the profile P_B *induced* by P on $G_T[B]$ to be the profile induced by H_B. Note that P_B has order $k+1$.

Remark 4.7. *If P is r-robust, then so is P_B.* □

Lemma 4.8. *Let $r \in \mathbb{N} \cup \{\infty\}$, and $k \leq r$ be finite. Let $\mathcal{N}$ be a nested set of separations of order at most k. Let P and Q be two r-robust profiles distinguished efficiently by a separation Y of order $l \geq k+1$ that is nested with $\mathcal{N}$. Then there is a unique $\mathcal{N}$-block B containing $\partial(Y)$.*

Moreover, P_B and Q_B are well-defined and r-robust profiles of order at least $l+1$, which are distinguished efficiently by Y_B.

Proof. Since Y is nested with any $Z \in N$, no Z can separate two vertices in $\partial(Y)$ because then both links $\partial(Y) \setminus V(Z)$ and $\partial(Y) \setminus V(Z^{\complement})$ would be nonempty. Let B be the set of those vertices that are not separated by any $Z \in \mathcal{N}$ from $\partial(Y)$. Clearly, B is the unique $\mathcal{N}$-block containing $\partial(Y)$.

Let H be the haven induced by P. Let $S \subseteq B$ be so that there is a component C of $G - S$ that is in H. Suppose for a contradiction that C does not intersect B. Then by Lemma 4.4, the neighbourhood $N(C)$ of C is complete in $G_T[B]$ and $|N(C)| \leq k$.

Since $(V(Y) \cap B, V(Y^{\complement}) \cap B)$ is a vertex-separation of $G_T[B]$ either $N(C) \subseteq V(Y) \cap B$ or $N(C) \subseteq V(Y^{\complement}) \cap B$. By symmetry, we may assume that $Y \in P$. Then the second cannot happen since the component of $G - \partial(Y)$ that is in H touches C. Hence s_C distinguishes P and Q, contradicting the efficiency of Y. Thus H_B is well-defined and a good haven of order $l+1$ by Remark 4.6. Thus P_B is an r-robust profile of order at least $l+1$. The same is true for Q_B whose corresponding havens we denote by J and J_B.

If P_B and Q_B are distinguished by a separation X of order less than l, then H_B and J_B will pick different components of $G_T[B] - \partial(X)$. Then in turn H and J will pick different components of $G - \partial(X)$, which is impossible by the efficiency of Y. Thus by Remark 4.5 it remains to show that Y_B distinguishes P_B and Q_B.

Let U and W be the components of $G_T[B] - \partial(Y)$ picked by H_B and J_B, respectively. Since $s_U \subseteq Y_B$ and $s_W \subseteq Y_B^{\complement}$, the separation Y_B distinguishes P_B and Q_B by (P1). □

Given a set $\mathcal{P}$ of r-robust profiles of order at least $l+1$, in the circumstances of Lemma 4.8, we let $\mathcal{P}_B$ be the set of those $P \in \mathcal{P}$ distinguished efficiently from some other $Q \in \mathcal{P}$ by a separation Y nested with $\mathcal{N}$ with $|\partial(Y)| \geq k+1$ and $\partial(Y) \subseteq B$. By $\mathcal{P}(B)$ we denote the set of *induced profiles* P_B for $P \in \mathcal{P}_B$.

4.3 Extending separations of the torsos

We define an operation $Y \mapsto \hat{Y}$ that extends each separation Y of the torso $G_T[B]$ to a separation $\hat{Y}$ of G in such a way that $\hat{Y}$ is nested with every separation of $\mathcal{N}$.

For each $X \in \mathcal{N}$ at least one of $V(X)$ and $V(X^{\complement})$ includes B. We pick $X[B] \in \{X, X^{\complement}\}$ such that $B \subseteq V(X[B])$. Let $\mathcal{M} = \{X[B]^{\complement} \mid X \in \mathcal{N}\}$. We shall ensure that $X \subseteq \hat{Y}$ or $X \subseteq \hat{Y}^{\complement}$ for every $X \in \mathcal{M}$, which implies that $\hat{Y}$ is nested with every separation in $\mathcal{N}$.

Let (C, D) be the vertex-separation of the torso $G_T[B]$ induced by Y. An edge e of G is *forced at step* 1 (by Y) if one of its incident vertices is in $C \setminus D$. A separation $X \in \mathcal{M}$ is *forced at step* $2n+2$ if there is an edge $e \in X$ that is forced at step $2n+1$ and X is not forced at some step $2j+2$ with $j < n$. An edge e of G is *forced at step* $2n+1$ for $n > 0$ if there is some $X \in \mathcal{M}$ containing e that is forced at step $2n$ and e is not forced at some step $2j+1$ with $j < n$.

The separation $\hat{Y}$ consists of those edges that are forced at some step.

Remark 4.9. *If $Y \subseteq Z$, then $\hat{Y} \subseteq \hat{Z}$.* □

Remark 4.10. *$X \subseteq \hat{Y}$ or $X \subseteq \hat{Y}^{\complement}$ for every $X \in \mathcal{M}$.*

In particular, $\hat{Y}$ is nested with every separation of $\mathcal{N}$.

Proof. If X intersects $\hat{Y}$, then $X \subseteq \hat{Y}$ by construction. □

There are easy examples of nested separations Y and Z of the torso $G_T[B]$ such that $\hat{Y}$ and $\hat{Z}$ are not nested. These examples motivate the definition of $\tilde{\mathcal{L}}$ below.

Given a nested set $\mathcal{L}$ of separations of $G_T[B]$, the *extension* $\tilde{\mathcal{L}}$ of $\mathcal{L}$ (depending on a well-order $(Y_\alpha \mid \alpha \in \beta)$ of $\mathcal{L}$) is the set $\{\tilde{Y} \mid Y \in \mathcal{L}\}$, where $\tilde{Y}$ is defined as follows: For the smallest element Y_0 of the well-order, we just let $\tilde{Y}_0 = \hat{Y}_0$ and $\widetilde{Y_0^{\complement}} = (\hat{Y}_0)^{\complement}$.

Assume that we already defined $\tilde{Y}_\alpha$ and $\widetilde{Y_\alpha^{\complement}}$ for all $\alpha < \gamma$. Let $Z_\alpha \in \{Y_\alpha, Y_\alpha^{\complement}\}$ be such that $Z_\alpha \subseteq Y_\gamma$ or $Y_\gamma \subseteq Z_\alpha$. We let $\tilde{Y}_\gamma$ consist of those edges that are first forced by Y_γ or second contained in some $\tilde{Z}_\alpha$ with $Z_\alpha \subseteq Y_\gamma$ or third both contained in every $\tilde{Z}_\alpha$ with $Y_\gamma \subseteq Z_\alpha$ and not forced by $Y_\gamma^{\complement}$. We define $\widetilde{Y_\gamma^{\complement}}$ similarly with '$Y_\gamma^{\complement}$' in place of 'Y_γ' and '$Z_\alpha^{\complement}$' in place of 'Z_α'.

Lemma 4.18 below says that no edge is forced by both Y and $Y^{\complement}$. Using that and Remark 4.9, a transfinite induction over $(Y_\alpha \mid \alpha \in \beta)$ gives the following:

Remark 4.11.
1. *If $Z_\alpha \subseteq Y_\gamma$, then $\tilde{Z}_\alpha \subseteq \tilde{Y}_\gamma$;*
2. *If $Y_\gamma \subseteq Z_\alpha$, then $\tilde{Y}_\gamma \subseteq \tilde{Z}_\alpha$;*
3. *$\widetilde{Y_\gamma^{\complement}} = (\tilde{Y}_\gamma)^{\complement}$;*
4. *$\tilde{Y}_\gamma$ contains all edges forced by Y_γ;*
5. *$\widetilde{Y_\gamma^{\complement}}$ contains all edges forced by $Y_\gamma^{\complement}$;*

□

Lemma 4.12. *Let $\mathcal{N}$ be a nested set of separations and let B and D be distinct $\mathcal{N}$-block. Let $\mathcal{L}_B$ and $\mathcal{L}_D$ be nested sets of separations of $G_T[B]$ and $G_T[D]$, respectively. Then $\tilde{\mathcal{L}}_B$ is a set of nested separations. If $X \in \mathcal{L}_B$ and $Y \in \mathcal{L}_D$, then $\tilde{X}$ and $\tilde{Y}$ are nested. Moreover, they are nested with every separation in $\mathcal{N}$.*

Proof. $\tilde{\mathcal{L}}_B$ is nested by 1 and 2 of Remark 4.11. It is easily proved by transfinite induction over the underlying well-order of $\mathcal{L}_B$ that for every $Z \in \mathcal{N}$ either $Z[B]^{\complement} \subseteq \tilde{X}$ or $\tilde{X} \subseteq Z[B]$. This implies the 'Moreover'-part.

There is some $Z \in \mathcal{N}$ distinguishing B and D. By exchanging the roles of B and D if necessary, we may assume that $Z[B] = Z$ and $Z[D] = Z^{\complement}$. Thus $\tilde{X} \subseteq Z$ or $\tilde{X}^{\complement} \subseteq Z$. And $\tilde{Y} \subseteq Z^{\complement}$ or $\tilde{Y}^{\complement} \subseteq Z^{\complement}$. Hence one of $\tilde{X}$ or $\tilde{X}^{\complement}$ is included in Z which in turn is included in one of $\tilde{Y}$ or $\tilde{Y}^{\complement}$. Thus $\tilde{X}$ and $\tilde{Y}$ are nested. □

Remark 4.13. *Let Y be a separation in a nested set $\mathcal{L}$ of $G_T[B]$. Then $\partial(\tilde{Y}) \subseteq \partial(Y)$.*

Proof. Let (C, D) be the vertex-separation induced by Y. If v is a vertex of B not in $C \cap D$, then all its incident edges are either all forced by Y at step 1 or else all forced by $Y^{\complement}$ at step 1, yielding that v cannot be in $\partial(\tilde{Y})$. If v is not in B then it is easily proved by induction on a well-order of $\mathcal{L}$ that all its incident edges are in $\tilde{Y}$ or else all of them are in $\tilde{Y}^{\complement}$. □

Remark 4.14. *Let B, P_B and Q_B as in Lemma 4.8. Let $\mathcal{L}$ be a nested set of separations in $G_T[B]$. If $X \in \mathcal{L}$ distinguishes P_B and Q_B in $G_T[B]$, then $\tilde{X}$ distinguishes P and Q.*

Proof. By construction there are different components F and K of $G - \partial(X)$ such that $s_F \in P$ and $s_K \in Q$. Clearly, every edge in s_F is forced by X, and every edge in s_K is forced by $X^{\complement}$. Thus $s_F \subseteq \tilde{X}$ and $s_K \subseteq \widetilde{X^{\complement}} = (\tilde{X})^{\complement}$. Hence $\tilde{X}$ distinguishes P and Q. □

Now we prepare to prove Lemma 4.18 below:

Remark 4.15. *Let $X \in \mathcal{M}$ that contains some edge e forced by Y. Then each endvertex v of e in $C \setminus D$ is in the boundary $\partial(X)$ of X.*

Proof. By assumption $v \in V(X^{\complement})$ and thus $v \in \partial(X)$. □

Remark 4.16. *Assume there is at least one edge forced by Y. Then no $X \in \mathcal{M}$ contains all edges of G which are forced by Y at steps 1.*

Proof. If X is not forced by Y at step 2, then this is clear. Otherwise there is a vertex $v \in \partial(X)$ that is in $C \setminus D$ by Remark 4.15. Thus there is an edge e incident with v contained in $X^{\complement}$. □

Remark 4.17. *1. No edge is forced by both Y and $Y^{\complement}$ at step 1.*

2. No $X \in \mathcal{M}$ contains edges forced by Y at step 1 and edges forced by $Y^{\complement}$ at step 1.

Proof. 1 follows from the fact that (C, D) is a vertex-separation of the torso $G_T[B]$. To see 2, we have to additionally apply Remark 4.15 and the corresponding fact for $Y^{\complement}$. □

Lemma 4.18. *No edge is forced by both Y and $Y^{\complement}$.*

Proof. In this proof, we run step m for forcing by $Y^{\complement}$ in between step m and step $m+1$ for forcing by Y. Suppose for a contradiction, there is some step m such that just after step m there is an edge e that is forced by both Y and $Y^{\complement}$ or there is some $X \in \mathcal{M}$ containing edges forced by Y and edges forced by $Y^{\complement}$. Let k be minimal amongst all such m. Thus k must be odd. By 1 and 2 of Remark 4.17, $k \geq 3$.

Case 1: there is some $X \in \mathcal{M}$ containing an edge e_C forced by Y and an edge e_D forced by $Y^{\complement}$ just after step k. Then precisely one of e_C and e_D was forced at step k, say e_D (the case with e_C will be analogue). Let $Z \in \mathcal{M}$ be a separation forcing e_D, which exists as $k \geq 3$.

We shall show that X and Z are not nested by showing that all the four intersections $X \cap Z$, $X \cap Z^{\complement}$, $X^{\complement} \cap Z$ and $X^{\complement} \cap Z^{\complement}$ are nonempty: First $e_D \in X \cap Z$. Let f an edge forcing Z for $Y^{\complement}$. By minimality of k, first $f \in X^{\complement} \cap Z$. Second, the separation Z does not contain any edge forced by Y just before step k. Thus $e_C \in X \cap Z^{\complement}$. Furthermore, there is some edge forced by Y in $X^{\complement} \cap Z^{\complement}$ by Remark 4.16. Thus X and Z are not nested, which gives the desired contradiction in this case.

Case 2: there is some edge e that is forced by both Y and $Y^{\complement}$ just after step k. We shall only consider the case that e was first forced by Y and then by $Y^{\complement}$ (the other case will be analogue). As $k \geq 3$, there is a separation $Z \in \mathcal{M}$ forcing e for $Y^{\complement}$. Let f be an edge forcing Z for $Y^{\complement}$. If e is forced by Y at step 1, then at the step before k the separation Z will contain edges forced by Y and edges forced by $Y^{\complement}$, which is impossible by minimality of k. Thus there is a separation $X \in \mathcal{M}$ forcing e for Y. Let g be an edge forcing

X for Y. By minimality of k, we have $g \in X \cap Z^{\complement}$ and $f \in X^{\complement} \cap Z$. Similar as in the last case we deduce that X and Z are not nested, which gives the desired contradiction. □

4.4 Miscellaneous

Lemma 4.19. *Let X and Y be two separations such that there is a component C of $G - \partial(X)$ with $s_C = X$ and C does not intersect $\partial(Y)$. Then X and Y are nested.*

Proof. By the definition of nestedness, it suffices to show that $X \subseteq Y$ or $X \subseteq Y^{\complement}$. For that, by symmetry, it suffices to show that if there is some edge $e_1 \in X \cap Y$, then any other edge e_2 of X must also be in Y. For that note that e_1 has an endvertex v in C and that there is a path P included in C from v to some endvertex of e_2. As no vertex of P is in $\partial(Y)$ and $e_1 \in Y$ it must be that $e_2 \in Y$, as desired. □

Lemma 4.20. *Let X, Y and Z be separations such that first X and Y are not nested and second $X \cap Y$ and Z are not nested. Then Z is not nested with X or Y.*

Proof. Recall that if A and Z are nested, then one of $A \subseteq Z$, $A \subseteq Z^{\complement}$, $A^{\complement} \subseteq Z$ or $A^{\complement} \subseteq Z^{\complement}$ is true. If one of $A \subseteq Z$ or $A \subseteq Z^{\complement}$ is false for $A = X \cap Y$, then it is also false for both $A = X$ and $A = Y$. If one of $A^{\complement} \subseteq Z$ or $A^{\complement} \subseteq Z^{\complement}$ is false for $A = X \cap Y$, then it is false for at least one of $A = X$ or $A = Y$. Suppose for a contradiction that $X \cap Y$ is not nested with Z but X and Y are. By exchanging the roles of X and Y if necessary, we may assume by the above that $X^{\complement} \subseteq Z$ and $Y^{\complement} \subseteq Z^{\complement}$. Then $X^{\complement} \subseteq Y$, contradicting the assumption that X and Y are not nested. □

A separation X is *tight* if $\partial(X) = \partial(s_C)$ for every component C of $G - \partial(X)$.

Lemma 4.21. *Let X be a separation of order k. Let Y be a tight separation such that $G - \partial(Y)$ has at least $k+1$ components. Then one of the links $\partial(Y) \setminus V(X)$ or $\partial(Y) \setminus V(X^{\complement})$ is empty.*

Proof. Suppose not for a contradiction, then there are $v \in \partial(Y) \setminus V(X)$ and $w \in \partial(Y) \setminus V(X^{\complement})$. Then v and w are in the neighbourhood of every component C of $G - \partial(Y)$. Thus there are $k+1$ internally disjoint paths from v to w, contradiction that fact that $\partial(X)$ separates v from w. □

Given two vertices v and w, a separator S separates v and w *minimally* if each component of $G - S$ containing v or w has the whole of S in its neighbourhood.

Lemma 4.22 ([20, Statement 2.4]). *Given vertices v and w and $k \in \mathbb{N}$, there are only finitely many distinct separators of size at most k separating v from w minimally.*

5 Distinguishing the profiles

The aim in this section is to construct a nested set of separations of finite order that distinguishes any two vertex ends efficiently, which is needed in the proof of Theorem 1. A related result is proved in [9]. Actually, we shall prove the stronger statement that for each $r \in \mathbb{N} \cup \{\infty\}$ there is a nested set $\mathcal{N}$ of separations that distinguishes any two r-robust profiles efficiently.

Overview of the proof

We shall construct the set $\mathcal{N}$ as an ascending union of sets $\mathcal{N}_k$ one for each $k \in \mathbb{N}$, where $\mathcal{N}_k$ is a nested set of separations of order at most k distinguishing efficiently any two r-robust profiles of order $k+1$. Any two r-robust profiles of order $k+2$ that are not distinguished by $\mathcal{N}_k$ will live in the same $\mathcal{N}_k$-block. We obtain $\mathcal{N}_{k+1}$ from $\mathcal{N}_k$ by adding for each $\mathcal{N}_k$-block a nested set $\tilde{\mathcal{N}}_{k+1}(B)$ that distinguishes efficiently any two r-robust profiles of order $k+2$ living in B. Working in the torsos $G_T[B]$ will ensure that the sets $\tilde{\mathcal{N}}_{k+1}(B)$ for different blocks B will be nested with each other.

Summing up, we are left with the task of finding in these torso graphs $G_T[B]$ a nested set distinguishing efficiently all r-robust profiles of order $k+2$. Theorem 5.2 deals with this problem if $G_T[B]$ is “nice enough”. In order to make all torso graphs nice enough, we add in an additional step in which we enlarge $\mathcal{N}_k$ a little bit so that for the larger nested set the new torso graphs are the old ones with the junk cut off. Lemma 5.1 will be the main lemma we use to enlarge $\mathcal{N}_k$.

Finishing the overview, we first state Lemma 5.1 and Theorem 5.2 and introduce the necessary definitions for that.

For any r-robust profile P and $k \in \mathbb{N}$, the restriction P_k of P to the set of separations of order at most k is an r-robust profile, whose order is the minimum of $k+1$ and the order of P. An *r-profile set* is a set of r-robust profiles such that if $P \in \mathcal{P}$ then for each $k \in \mathbb{N}$ the restriction P_k is in $\mathcal{P}$. Until the end of Subsection 5.2, let us a fix a graph G together numbers $k, r \in \mathbb{N} \cup \{\infty\}$ with $k \leq r$ and an r-profile set $\mathcal{P}$.

A set $\mathcal{N}$ of nested sets is *extendable* (for $\mathcal{P}$) if for any two distinct profiles in $\mathcal{P}$ of the same order, there is some separation X nested with $\mathcal{N}$ that distinguishes these two profiles efficiently.

By $R(k, r, \mathcal{P}, G)$ we denote the set of those separations whose order is finite and at most k that distinguish efficiently two profiles in $\mathcal{P}$ in the graph G. It may happen for some $X \in R(k, r, \mathcal{P}, G)$ that $G - \partial(X)$ has a component C such that $\partial(s_C)$ is a proper subset of $\partial(X)$. By $S(k, r, \mathcal{P}, G)$, we denote the set of all separations s_C for such components C of $G - \partial(X)$ for some $X \in R(k, r, \mathcal{P}, G)$. If it is clear from the context what G is, we shall just write $R(k, r, \mathcal{P})$ or $S(k, r, \mathcal{P})$, or even just $R(k, r)$ or $S(k, r)$.

Lemma 5.1. *If $R(k-1, r) = \emptyset$, then $S(k, r)$ is a nested extendable set of separations.*

A separation X *strongly disqualifies* a set Y if $|\partial(Y)|$ is strictly larger than both $|L(X, Y)|$ and $|L(X^{\complement}, Y)|$. A set X *disqualifies* a set Y if it strongly disqualifies Y or $Y^{\complement}$. Note that every $X \in R(k, r)$ is tight if and only if $S(k, r) = \emptyset$.

Theorem 5.2. *Let $k \in \mathbb{N}$ and $r \in \mathbb{N} \cup \{\infty\}$ with $k \leq r$. Assume that $S(k, r) = \emptyset$ and $R(k, r) = \emptyset$. Any set $\mathcal{N}$ of nested tight separations of order at most k that are not disqualified by any $X \in R(r, r)$ is extendable.*

In particular, any maximal such set distinguishes any two profiles of order $k+1$ in $\mathcal{P}$.

5.1 Proof of Lemma 5.1.

Lemma 5.3. *If X distinguishes two r-robust profiles P_1 and P_2 efficiently, then X is not disqualified by any separation Y with $\partial(Y) \leq r$.*

Proof. We may assume that $X \in P_1$ and $X^{\complement} \in P_2$. Suppose for a contradiction that Y strongly disqualifies X. Then $|L(X, Y)| < |\partial(X)|$ and $|L(X, Y^{\complement})| < |\partial(X)|$. As neither $X \cap Y$ nor $X \cap Y^{\complement}$ is in P_2, these two sets cannot be in P_1 either since X distinguishes P_1 and P_2 efficiently. This contradicts the assumption that P_1 is r-robust. Similarly, one shows that Y cannot strongly disqualify $X^{\complement}$, and thus Y does not disqualify X. $\square$

Lemma 5.4. *Let X and Y be two separations distinguishing profiles in $\mathcal{P}$ efficiently with $k = |\partial(X)| \leq |\partial(Y)|$. Let C be a component of $G - \partial(X)$ such that $\partial(s_C)$ is a proper subset of $\partial(X)$.*

If $R(k-1, r) = \emptyset$, then C does not intersect $\partial(Y)$.

Proof. Let P and P' be two profiles in $\mathcal{P}$ distinguished efficiently by X, where $X \in P$.

Sublemma 5.5. *$G - \partial(X)$ has two components D and K different from C such that $s_D \in P$ and $s_K \in P'$.*

Proof. s_C can be in at most one of P and P'. By the efficiency of X it actually cannot be in precisely one of them. Thus s_C is in none of them. Hence the components D and K of $G - \partial(X)$ such that $s_D \in P$ and $s_K \in P'$, which exist by (P3), are different from C. $\square$

Let Q and Q' be two profiles in $\mathcal{P}$ distinguished efficiently by Y, where $Y \in Q$. Since $|\partial(X)| \leq |\partial(Y)|$, we have $X \in Q$ or $X^{\complement} \in Q$. By exchanging the roles of X and $X^{\complement}$ if necessary, we may assume that $X \in Q$. By Sublemma 5.5, we may assume that $s_C \subseteq X$ by replacing X by $X \cup s_C$ if necessary.

Sublemma 5.6. *Either $|L(X,Y)| \leq |\partial(Y)|$ and $X \cap Y \in Q$ or else $|L(X,Y^{\complement})| \leq |\partial(Y)|$ and $X \cap Y^{\complement} \in Q'$.*

Proof. **Case 1: $X^{\complement} \in Q'$.**

If $|L(X^{\complement},Y^{\complement})| < |\partial(X)|$, then $X^{\complement} \cap Y^{\complement} \in Q'$ by (P2) so that $X^{\complement} \cap Y^{\complement}$ will distinguish Q and Q', which is impossible by the efficiency of Y. Thus $|L(X,Y)| \leq |\partial(Y)|$ by Remark 4.1, yielding that $X \cap Y \in Q$ by (P2), as desired.

Case 2: $X \in Q'$.

By Lemma 5.3, Y does not strongly disqualify $X^{\complement}$. Thus either $|L(Y^{\complement}, X^{\complement})| \geq |\partial(X)|$ or $|L(Y, X^{\complement})| \geq |\partial(X)|$. In the first case, $|L(Y^{\complement}, X)| \leq |\partial(Y)|$ by Remark 4.1. Then $Y^{\complement} \cap X \in Q'$ by (P2). Similarly in the second case, $|L(Y,X)| \leq |\partial(Y)|$. Then $Y \cap X \in Q$ by (P2), as desired. $\square$

Sublemma 5.7. *One of C and D does not meet $\partial(Y)$.*

Proof. First we consider the case that $|L(X,Y)| \leq |\partial(Y)|$ and $X \cap Y \in Q$. By (P3), there is a component F of $G - \partial(Y \cap X)$ such that $s_F \in Q$. By the efficiency of Y, it must be that $\partial(s_F) = \partial(Y \cap X)$ as s_F distinguishes Q and Q'. Thus the union F' of F and the link $\partial(Y) \setminus V(X^{\complement})$ is connected.

Suppose for a contradiction that both C and D meet $\partial(Y)$, then they both meet $\partial(Y)$ in vertices of the link $\partial(Y) \setminus V(X^{\complement})$. Since C and D are components, they both must contain F', and hence are equal, which is the desired contradiction. Thus at most one of C and D can meet $\partial(Y)$.

By Sublemma 5.6 it remains to consider the case where $|L(X,Y^{\complement})| \leq |\partial(Y)|$ and $X \cap Y^{\complement} \in Q'$, which is dealt with analogous to the above case. $\square$

Recall that $\partial(s_C) \subseteq \partial(s_D)$. By Sublemma 5.7, one of the links $\partial(s_C) \setminus V(Y)$ and $\partial(s_C) \setminus V(Y^{\complement})$ must be empty since otherwise there would a path joining these two links and avoiding $\partial(Y)$, which is impossible. By symmetry, we may assume that $\partial(s_C) \setminus V(Y)$ is empty. Thus $\partial(Y \setminus s_C) \subseteq \partial(Y)$. Since $R(k-1, r) = \emptyset$, and $s_C \notin P$, it must be that $s_C \notin Q$. Thus $Y \setminus s_C \in Q$ by (P2) so that $Y \setminus s_C$ distinguishes Q and Q'. By the efficiency of Y, it must be that $\partial(Y \setminus s_C) = \partial(Y)$. Hence $\partial(Y) \cap C$ is empty, as desired. □

Proof of Lemma 5.1. Let $X \in R(k, r)$ and $Y \in R(r, r)$ of order at least k. Let C be a component of $G - \partial(X)$ and D be a component of $G - \partial(Y)$. In order to see that $S(k, r)$ is a nested, it suffices to show that for any such C and D that the separations s_C and s_D are nested. This is true by Lemma 5.4 and Lemma 4.19. In order to see that $S(k, r)$ is an extendable, it suffices to show that for any such C and Y that the separations s_C and Y are nested. This is true by Lemma 5.4 and Lemma 4.19, as well. □

5.2 Proof of Theorem 5.2.

Before we prove Theorem 5.2, we need some intermediate lemmas. Throughout this subsection, we assume that $S(k, r)$ is empty. Let U be the set of those tight separations of order at most k that are not disqualified by any $X \in R(r, r)$. Note $R(k, r) \subseteq U$.

Lemma 5.8. *For any componental separation $X \in R(r, r)$, there are only finitely many $Y \in U$ not nested with X.*

Proof. First, we show that X is nested with every $Y \in U$ such that the link $\partial(X) \setminus V(Y)$ is empty. By Lemma 4.19, it suffices to show that $\partial(Y) \setminus V(X^{\complement})$ is empty. As X does not strongly disqualify $Y^{\complement}$, one of the links $\partial(Y) \setminus V(X)$ and $\partial(Y) \setminus V(X^{\complement})$ is empty. Hence we may assume that $\partial(Y) \setminus V(X)$ is empty. If Y is not nested with X, there must be a component of C of $G - \partial(Y)$ all of whose neighbours are in $\partial(X) \cap \partial(Y)$. As Y is tight, it must be that $\partial(Y) = \partial(X) \cap \partial(Y)$ so that $\partial(Y) \setminus V(X^{\complement})$ is empty. Hence X and Y are nested by Lemma 4.19.

Similarly one shows that X is nested with every $Y \in U$ such that the link $\partial(X) \setminus V(Y)$ is empty.

It remains to show that there are only finitely many $Y \in U$ not nested with X such that both links $\partial(X) \setminus V(Y)$ and $\partial(X) \setminus V(Y^{\complement})$ are nonempty. By Lemma 4.22, there are only finitely many triples (v, w, T) where $v, w \in \partial(X)$ and T is a separator of size at most k separating v and w minimally. Since each $\partial(Y)$ for some Y as above is such a separator T, it suffices to show that there are only finitely many $Z \in U$ with $\partial(Z) = \partial(Y)$. This is true as $G - \partial(Y)$ has at most $\partial(X) + 1$ components by Lemma 4.21. □

Lemma 5.9. *Let $\mathcal{N}$ be a nested subset of U. For any two distinct profiles P and Q in $\mathcal{P}$ of the same order that are not distinguished by any separation of order less than k, there is some separation $X \in R(k,r) \subseteq U$ that is nested with $\mathcal{N}$ and distinguishes P and Q efficiently.*

Proof. First, we show that there is some $X \in U$ distinguishing P and Q efficiently that is nested with all but finitely many separations of $\mathcal{N}$. Since $S(k,r)$ is empty, $R(k,r)$ is a subset of U. Thus U contains some separation A distinguishing P and Q efficiently. By (P3), we can pick such an A that is componental. By Lemma 5.8, A is nested with all but finitely many separations of $\mathcal{N}$. Hence we can pick X distinguishing P and Q efficiently such that it is not nested with a minimal number of $Y \in \mathcal{N}$.

Suppose for a contradiction that there is some $Y \in \mathcal{N}$ that is not nested with X. We may assume that Y does not distinguish P and Q since otherwise Y would distinguish P and Q efficiently. Thus either both $Y \in P$ and $Y \in Q$ or both $Y^{\complement} \in P$ and $Y^{\complement} \in Q$. Since $Y^{\complement}$ is nested with $\mathcal{N}$, we may by symmetry assume that $Y \in P$ and $Y \in Q$.

Since X does not strongly disqualify $Y^{\complement}$ by the definition of U, either $|L(X, Y^{\complement})| \geq |\partial(Y)|$ or $|L(X^{\complement}, Y^{\complement})| \geq |\partial(Y)|$. By symmetry, we may assume that $|L(X, Y^{\complement})| \geq |\partial(Y)|$. By exchanging the roles of P and Q if necessary, we may assume that $X \in P$ and $X^{\complement} \in Q$. By Remark 4.1, $|L(X^{\complement}, Y)| \leq |\partial(X)|$. Note that $X^{\complement} \cap Y \notin P$ as $X^{\complement} \notin P$ by (P1) but $X^{\complement} \cap Y \in Q$ by (P2). Thus $X^{\complement} \cap Y$ distinguishes P and Q efficiently. Any separation in $\mathcal{N}$ not nested with $X^{\complement} \cap Y$ is by Lemma 4.20 not nested with X. As Y is nested with $X^{\complement} \cap Y$, the separation $X^{\complement} \cap Y$ violates the minimality of X. Hence X is nested with $\mathcal{N}$, completing the proof. □

Proof of Theorem 5.2. By Lemma 5.9 any nested subset of U is extendable. □

5.3 Proof of the main result of this section.

In this subsection, we proof the following.

Theorem 5.10. *For any graph G and any $r \in \mathbb{N} \cup \{\infty\}$, there is a nested set of separation $\mathcal{N}$ that distinguishes efficiently any two r-robust profiles of the same order.*

First we need an intermediate lemma, for which we fix some notation. Let us fix some $r \in \mathbb{N} \cup \{\infty\}$, some finite $k \leq r$ and an r-profile set $\mathcal{P}$. Let $\mathcal{N}$ be a nested set of separations of order at most k that is extendable for $\mathcal{P}$ and that distinguishes efficiently any two profiles of $\mathcal{P}$ that can be distinguished by a separation of order at most k. For each $\mathcal{N}$-block B, let $\mathcal{P}(B)$ be defined as after Lemma 4.8. And let $\mathcal{N}_B$ be a set of nested separations of $G_T[B]$ that is extendable for $\mathcal{P}(B)$. We abbreviate $\mathcal{M} = \mathcal{N} \cup \bigcup \tilde{\mathcal{N}_B}$, where the union ranges over all $\mathcal{N}$-blocks B.

Lemma 5.11. *The set $\mathcal{M}$ is nested and extendable for $\mathcal{P}$.*

Proof. $\mathcal{M}$ is nested by Lemma 4.12.

It remains to show for every $l \geq k+1$ and any two profiles P and Q in $\mathcal{P}$ that are distinguished efficiently by a separation of order l that there is a separation nested with $\mathcal{M}$ that distinguishes P and Q efficiently. We may assume that P and Q both have order $l+1$ as $\mathcal{P}$ is an r-profile set. By Lemma 4.8 and since $\mathcal{N}$ is extendable, there is a unique $\mathcal{N}$-block B such that some separation Y of order l of $G_T[B]$ distinguishes P_B and Q_B.

As $\mathcal{N}_B$ is extendable, there is a separation Z of $G_T[B]$ nested with $\mathcal{N}_B$ that distinguishes P_B and Q_B efficiently. By Lemma 4.12, $\tilde{Z}$ is nested with $\mathcal{M}$, and it distinguishes P and Q by Remark 4.14 and it does so efficiently by Remark 4.13. □

Proof of Theorem 5.10. We shall construct the nested set $\mathcal{N}$ of Theorem 5.10 as a nested union of sets $\mathcal{N}_k$ one for each $k \in \mathbb{N} \cup \{-1\}$, where $\mathcal{N}_k$ is a nested extendable set of separations of order at most k that distinguishes any two r-robust profiles efficiently that are distinguished by a separation of order at most k. We start the construction with $\mathcal{N}_{-1} = \emptyset$. Assume that we already constructed $\mathcal{N}_k$ with the above properties. For an $\mathcal{N}_k$-block B, we define $\mathcal{P}(B)$ as indicated after Lemma 4.8.

Sublemma 5.12. *The set $R(k, r, \mathcal{P}(B), G_T[B])$ is empty.*

Proof. Suppose for a contradiction, two profiles P_B and Q_B in $\mathcal{P}(B)$ can be distinguished by a separation X of order at most k. Then $\tilde{X}$ has order at most $|\partial(X)|$ by Remark 4.13 and by Remark 4.14 it distinguishes the profiles P and Q which induce P_B and Q_B. So P and Q are distinguished by $\mathcal{N}_k$ by the induction hypothesis. This contradicts the assumption that P and Q are both in $\mathcal{P}(B)$. □

By Sublemma 5.12, we can apply Lemma 5.1 to $G_T[B]$ and $\mathcal{P}(B)$, yielding that the set $S(k+1, r, \mathcal{P}(B), G_T[B])$ is a nested extendable set of separations. For each $S(k+1, r, \mathcal{P}(B), G_T[B])$-block B', we define $\mathcal{P}(B')$ as indicated after Lemma 4.8.

Sublemma 5.13. *The set $S(k+1, r, \mathcal{P}(B'), G_T[B'])$ is empty.*

Proof. Suppose for a contradiction, there is some $X \in S(k+1, r, \mathcal{P}(B'), G_T[B'])$. Then there is some $Y \in R(k+1, r, \mathcal{P}(B'), G_T[B'])$ so that there is a component C of $G_T[B'] - \partial(Y)$ with $s_C = X$. By Remark 4.13, Remark 4.14 and the definition of $\mathcal{P}(B')$, the separation $\tilde{Y}$ distinguishes efficiently two profiles in $\mathcal{P}(B)$ so that $\tilde{Y} \in R(k+1, r, \mathcal{P}(B), G_T[B])$. By Remark 4.13, $\tilde{Y}$ has order precisely $k+1$ since $\tilde{\tilde{Y}}$ has order $k+1$ because it distinguishes two profiles that are not distinguished by $\mathcal{N}_k$. Hence $\tilde{X} \in S(k+1, r, \mathcal{P}(B), G_T[B])$ by Remark 4.13. Thus X is the empty, which is the desired contradiction. □

By Zorn's Lemma we pick a maximal set $\mathcal{N}(B')$ of nested tight separations of order at most k in $G_T[B']$ that are not disqualified by any $X \in R(r, r, \mathcal{P}(B'), G_T[B'])$. By Theorem 5.2 the set $\mathcal{N}(B')$ is extendable and distinguishes any two r-robust profiles of order $k+2$ in $\mathcal{P}(B')$.

Let $\mathcal{N}_{k+1}(B)$ be the union of the sets $\tilde{\mathcal{N}}(B')$ together with $S(k+1, r, \mathcal{P}(B), G_T[B])$ where the union ranges over all $S(k+1, r, \mathcal{P}(B), G_T[B])$-blocks B'. By Lemma 5.11, $\mathcal{N}_{k+1}(B)$ is a nested and extendable set of separation of order at most $k+1$ in $G_T[B]$. Let $\mathcal{N}_{k+1}$ be the union of the sets $\tilde{\mathcal{N}}_{k+1}(B)$ together with $\mathcal{N}_k$, where the union ranges over all $\mathcal{N}_k$-blocks B. Applying Lemma 5.11 again, we get that $\mathcal{N}_{k+1}$ is a nested and extendable set of separation of order at most $k+1$ in G.

Sublemma 5.14. *$\mathcal{N}_{k+1}$ distinguishes efficiently any two r-robust profiles P and Q of G that are distinguished by a separation of order at most $k+1$.*

Proof. We may assume that P and Q both have order $k+2$. Let A distinguish P and Q efficiently. If A has order at most k, by the induction hypothesis, there is a separation $\hat{A}$ in $\mathcal{N}_k$ distinguishing P and Q efficiently. So $\hat{A}$ is in $\mathcal{N}_{k+1}$ by construction.

Otherwise there is a separation X distinguishing P and Q efficiently that is nested with $\mathcal{N}_k$ as $\mathcal{N}_k$ is extendable. By Lemma 4.8, there is an $\mathcal{N}_k$-blocks B such that P_B and Q_B are r-robust profiles in $G_T[B]$ of order $k+2$ in $\mathcal{P}(B)$, which are distinguished efficiently by X_B. Using the fact that $\mathcal{N}_{k+1}(B)$ is extendable and then applying Lemma 4.8 again, we find an $S(k+1, r, \mathcal{P}(B))$-block B' such that P_B and Q_B induce different r-robust profiles of order $k+2$ in $G_T[B']$, which are distinguished efficiently by some separation Z of order at most $k+1$. By construction, we find such a Z in

$\mathcal{N}(B')$. Applying Remark 4.13 twice yields that the order of $\tilde{\tilde{Z}}$ is at most $k+1$. Thus $\tilde{\tilde{Z}}$ distinguishes P and Q efficiently by Remark 4.14. As $\tilde{\tilde{Z}}$ is in $\mathcal{N}_{k+1}$, this completes the proof. $\square$

Finally, the nested union $\mathcal{N}$ of the sets $\mathcal{N}_k$ is a nested set of separations that distinguishes efficiently any two r-robust profiles of the same order, as desired. $\square$

For a vertex end ω, let P_ω^k be the set of those separations of order at most k, in which ω lives. It is straightforward to show that P_ω^k is an ∞-robust profile of order $k+1$. Hence Theorem 5.10 has the following consequence.

Corollary 5.15. *For any graph G, there is a nested set $\mathcal{N}$ of separations that distinguishes any two vertex ends efficiently.* $\square$

6 A tree-decomposition distinguishing the topological ends

In this section, we prove Theorem 1 already mentioned in the Introduction. A key lemma in the proof of Theorem 1 is the following.

Lemma 6.1. *Let G be a graph with a finite nonempty set W of vertices. Then G has a star decomposition $(S, Q_s | s \in V(S))$ of finite adhesion such that each topological end lives in some Q_s where s is a leaf.*

Moreover, only the central part Q_c contains vertices of W, and for each leaf s, there lives an topological end in Q_s, and $Q_s \setminus Q_c$ is connected.

Proof that Lemma 6.1 implies Theorem 1. We shall recursively construct a sequence $\mathcal{T}^n = (T^n, P_t^n | t \in V(T^n))$ of tree-decomposition of G of finite adhesion as follows. We starting by picking a vertex v of G arbitrarily and we obtain $\mathcal{T}^1$ by applying Lemma 6.1 with $W = \{v\}$. Assume that we already constructed $\mathcal{T}^n$. For each leaf s of $\mathcal{T}^n$, we denote by W_s the set of those vertices in Q_s also contained in some other part of $\mathcal{T}^n$. Note that W_s is contained in the part adjacent to Q_s and thus is finite. By Lemma 6.1, we obtain a star decomposition $\mathcal{T}_s$ of $G[Q_s]$ such that no $w \in W_s$ is contained in a leaf part of $\mathcal{T}_s$ and such that each topological end living in Q_s lives in a leaf of $\mathcal{T}_s$. We obtain $\mathcal{T}^{n+1}$ from $\mathcal{T}^n$ by replacing each leaf part Q_s by $\mathcal{T}_s$, which is well-defined as the set W_s is contained in a unique part of $\mathcal{T}_s$.

By r, we denote the center of $\mathcal{T}_1$. For each $j < m < n$, the balls of radius j around r in T^m and T^n are the same. Thus we take T to be the tree whose nodes are those that are eventually a node of T^n. For each $t \in V(T)$, the parts P_t^n are the same for n larger than the distance between t and r, and we take P_t to be the limit of the P_t^n.

It is easily proved by induction that each vertex in W_s for s a leaf of T^n has distance at least $n-1$ from v in G. Thus for each $j<n$ the ball of radius j around v in G is included in the union over all parts P_t^n where t is in the ball of radius j around r in $\mathcal{T}_n$. Hence $(T, P_t | t \in V(T))$ is a tree-decomposition, and it has finite adhesion by construction.

It remains to show that the ends of T define precisely the topological ends of G, which is done in the following four sublemmas.

Sublemma 6.2. *Each topological end ω of G lives in an end of T.*

Proof. There is a unique leaf s of T^n such that ω lives in P_s^n. Let s_n be the predecessor of s in T^n. Then ω lives in the end of T to which $s_1 s_2 \dots$ belongs. □

Sublemma 6.3. *In each end τ of T, there lives a vertex end of G.*

Proof. For a directed paths P, we shall denote by $\overleftarrow{P}$ the directed path with the inverse ordering of that of P.

Let $s_1 s_2 ...$ be the ray in T starting at r that belongs to τ. By construction, the sets W_{s_i} are disjoint and finite. For each $w \in W_{s_i}$, we pick a path P_w from w to v. Since $W_{s_{i-1}}$ separates w from v, there is a first $w' \in W_{s_{i-1}}$ appearing on P_w. Now we apply the Infinity Lemma in the form of [12, Section 8] on the graph whose vertex set is the disjoint union of the sets W_{s_i}, and we put in all the edges ww'. Thus this graph has a ray $w_1 w_2 ...$ where $w_i \in W_{s_i}$. Then $K = v\overleftarrow{P_{w_1}} w_1 \overleftarrow{P_{w_2}} w_2 \overleftarrow{P_{w_3}} ...$ is an infinite walk with the property that the distance between v and a vertex k on K is at least n if k appears after $\overleftarrow{P_{w_n}}$. In particular, K traverses each vertex only finitely many times. Thus K is a connected locally finite graph, and thus contains a ray R. Since R meets each of the sets W_{s_i}, the end to which R belongs lives in τ, as desired. □

Sublemma 6.4. *No two distinct vertex ends ω_1 and ω_2 of G live in the same end τ of T.*

Proof. Suppose for a contradiction, there are such ω_1, ω_2 and τ. Let U be a finite separator separating ω_1 from ω_2 and let n be the maximal distances between v and a vertex in U. Then there is a leaf s of T^{n+1} such that τ lives in Q_s. Let C_i be the component of $G-U$ in which ω_i lives. Since W_s separates U from $Q_s \setminus W_s$, it must be that the connected set $Q_s \setminus W_s$ is contained in a component of $G-U$. As ω_i lives in $Q_s \setminus W_s$ by assumption, it must be that $Q_s \setminus W_s \subseteq C_i$. Hence C_1 and C_2 intersect, which is the desired contradiction. $\square$

Sublemma 6.5. *No vertex u dominates a vertex end ω living in some end of T.*

Proof. Suppose for a contradiction u does. Let n be the distance between u and v in G. Then there is a leaf s of T^{n+1} such that ω lives in Q_s. Thus the finite set W_s separates u from ω, contradicting the assumption that u dominates ω. $\square$

Sublemma 6.2, Sublemma 6.3, Sublemma 6.4 and Sublemma 6.5 imply that the ends of T define precisely the topological ends of G, as desired. $\square$

Remark 6.6. *Let $(T, \leq)$ be the tree order on T as in the proof of Theorem 1 where the root r is the smallest element. We remark that we constructed $(T, \leq)$ such that $(T, P_t | t \in V(T))$ has the following additional property: For each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.*

Moreover, we construct $(T, P_t | t \in V(T))$ such that if st and tu are edges of T with $s \leq t \leq u$, then $V(P_s) \cap V(P_t)$ and $V(P_t) \cap V(P_u)$ are disjoint.

In order to prove Lemma 6.1, we need the following.

Lemma 6.7. *Let G be a connected graph and $W \subseteq V(G)$ finite and nonempty. Then there is a set $\mathcal{X}$ of disjoint edge sets X of finite boundary such that every vertex end not dominated by some $w \in W$ lives in some $X \in \mathcal{X}$ and no edge e in any $X \in \mathcal{X}$ is incident with a vertex of W.*

Proof that Lemma 6.7 implies Lemma 6.1. We may assume that G in Lemma 6.1 is connected. Let $C = V(E \setminus \bigcup \mathcal{X}) \cup \bigcup_{X \in \mathcal{X}} \partial(X)$. For $X \in \mathcal{X}$ let $\mathcal{Q}_X$ consist of sets of the form $\partial(X) \cup Q$, where Q is a component of $G - \partial(X)$ with $Q \subseteq V(X)$. Let $\mathcal{Q}$ be the union over $\mathcal{X}$ of the sets $\mathcal{Q}_X$. Let $\mathcal{R}$ be the set of those H in $\mathcal{Q}$ such that some topological end lives in $V(H)$. Note that each topological end lives in some $R \in \mathcal{R}$ and that W does not intersect any such R. We obtain C' from C by adding the vertex sets of all $H \in \mathcal{Q} \setminus \mathcal{R}$. We consider $S = \mathcal{R} \cup \{C'\}$ as a star with center C'. It is straightforward to verify that $(S, s | s \in V(S))$ is a star decomposition with the desired properties. $\square$

The rest of this section is devoted to the proof of Lemma 6.7. We shall need the following lemma.

Lemma 6.8. *Let G be a connected graph and $W \subseteq V(G)$ finite. There is a nested set N of nonempty separations of finite order such that every vertex end not dominated by some $w \in W$ lives in some $X \in N$ and no edge e in some $X \in N$ is incident with a vertex of W.*

Moreover, if $X, Y \in N$ are distinct with $X \subseteq Y$, then the order of Y is strictly larger than the order of X.

Proof. We obtain G_W from G by first deleting W and then adding a copy of K_ω, the complete graph on countably many vertices, which we join completely to the neighbourhood of W. Applying Corollary 5.15 to G_W, we obtain a nested set N' of separations of finite order such that any two vertex ends of G_W are distinguished efficiently by a separation in N'. Let τ be the vertex end to which the rays of the newly added copy of K_ω belong. Let N'' consist of those separations in N' that distinguish τ efficiently from some other vertex end. As the separations in N'' distinguish efficiency, no $X \in N''$ contains an edge incident with a vertex of the newly added copy of K_ω.

Given $k \in \mathbb{N}$, a *k-sequence* $(X_\alpha | \alpha \in \gamma)$ (for N'') is an ordinal indexed sequence of elements of N'' of order at most k such that if $\alpha < \beta$, then $X_\alpha \subseteq X_\beta$. We obtain N''' from N'' by adding $\bigcup_{\alpha\in\gamma} X_\alpha$ for all k-sequences (X_α) for all k. Clearly, $N'' \subseteq N'''$ and N''' is nested. Given $k \in \mathbb{N}$, the set N_k consists of those $X \in N'''$ of order at most k, and N'_k consists of the inclusion-wise maximal elements of N_k.

We let $N = \bigcup_{k\in\mathbb{N}} N'_k$. By construction, each $X \in N$ contains no edge incident with a vertex of the newly added copy of K_ω, and thus it can be considered as an edge set of G, whose boundary is the same as the boundary in G_W. We claim that N has all the properties stated in Lemma 6.8: By construction, each $X \in N$ is nonempty. Since $N \subseteq N'''$, the set N is nested. The "Moreover"-part is clear by construction. Thus it remains to show that each vertex end ω of G not dominated by some vertex in W lives in some element of N.

Let R be a ray belonging to ω. Since ω is not dominated by any vertex in W, for each $w \in W$ there is a finite vertex set S_w separating a subray R_w of R from w. Then $S = \bigcup_{w\in W} S_w \setminus W$ separates $R' = \bigcap_{w\in W} R_w$ from W in G but also in G_W. Let ω' be the vertex end of G_W to which R' belongs. Note that S witnesses that $\omega' \neq \tau$. Thus there is some $X \in N'''$ in which ω' lives. Let k be the order of X. By Zorn's lemma, N''' contains an inclusion-wise maximal element X' of order at most k including X. By construction X' is in N'_k and includes a subray of R'. Thus ω lives in X', which completes the proof. □

Next we show how Lemma 6.8 implies Lemma 6.7. A good candidate for $\mathcal{X}$ in Lemma 6.7 might be the inclusion-wise maximal elements of N. However, there might be an infinite strictly increasing sequence of members in N, whose orders are also strictly increasing, so that we cannot expect that the union over the members of this sequence has finite order, and hence cannot be in N. Thus we have to make a more sophisticated choice for $\mathcal{X}$ than just taking the maximal members of N.

Lemma 6.8 implies Lemma 6.7. Let N be as in Lemma 6.8. Let $X \in N$ be such that there is another $Y \in N$ with $X \subseteq Y$, then the order of Y is strictly larger than the order of X. We denote the set of such Y of minimal order by $D(X)$. Let H be the digraph with vertex set N where we put in the directed edge XY if $Y \in D(X)$. A *connected component of H*, is a connected component of the underlying graph of H.

Sublemma 6.9. *Let $X', Y' \in N$. Then $X' \subseteq Y'$ if and only if there is a directed path from X' to Y'. Moreover, if $X, Y \in N$ are not joined by a directed path, then they are disjoint.*

Proof. Clearly, if there is a directed path from X' to Y', then $X' \subseteq Y'$. Conversely, let $X', Y' \in N$ with $X' \subseteq Y'$. Let (X_n) be a sequence of distinct separations in N such that $X' \subseteq X_1 \subseteq ... \subseteq X_n \subseteq Y'$. By Lemma 6.8, $n \leq |\partial(Y')| - |\partial(X')| + 1$. Thus there is a maximal such chain (Z_n), which satisfies $Z_1 = X'$ and $Z_n = Y'$ and $X_{i+1} \in D(X_i)$ for all i between 1 and $n-1$. Hence $Z_1 ... Z_n$ is a path from X' to Y'.

To see that "Moreover"-part, let $X, Y \in N$. As G is connected, there is an edge e incident with some vertex in W. Since e is not in $X \cup Y$ and X and Y are nested, X and Y must be disjoint if they are not joined by a directed path. □

Sublemma 6.10. *Each vertex v of H has out-degree at most 1*

Proof. Suppose for a contradiction v has out-degree at least 2. Then there are distinct $X, Y \in D(v)$ so that neither $X \subseteq Y$ nor $Y \subseteq X$. Thus X and Y are disjoint by Sublemma 6.9. Since $v \subseteq X \cap Y$, this is the desired contradiction. □

Sublemma 6.11. *Any undirected path P joining two vertices v and w contains a vertex u such that vPu and wPu are directed paths which are directed towards u.*

Proof. It suffices to show that P contains at most one vertex of out-degree 0 on P. If it contained two such vertices then between them would be a vertex of out-degree 2, which is impossible by Sublemma 6.10. $\square$

We define $\mathcal{X}$ as the union of sets $\mathcal{X}_C$, one for each component C of H. The $\mathcal{X}_C$ are defined as follows: If C has a vertex v_C of out-degree 0, then by Sublemma 6.11 C cannot contain a second such vertex and for any other vertex v in C, there is a directed path from v to v_C directed towards v_C. Hence v_C includes any other $v \in V(C)$. We let $\mathcal{X}_C = \{v_C\}$.

Otherwise, C includes a ray $X_1X_2\dots$ as C cannot contain a directed cycle by Sublemma 6.9. In this case, we take $\mathcal{X}_C$ to be the set consisting of the $Y_i = X_i \setminus X_{i-1}$ for each $i \in \mathbb{N}$, where $Y_1 = X_1$. Note that the order of Y_i is bounded by the sum of the orders of X_i and X_{i-1}, and thus finite.

Since no $Y \in N$ contains an edge incident with some $w \in W$, the same is true for any $Y \in \mathcal{X}$. Any two distinct $X, Y \in \mathcal{X}$ are disjoint: If X and Y are in the same $\mathcal{X}_C$, this is clear by construction. Otherwise it follows from the definition of Y_i and Sublemma 6.9. Thus it remains to prove the following:

Sublemma 6.12. *Each vertex end ω not dominated by some vertex of W lives in some $X \in \mathcal{X}$.*

Proof. By Lemma 6.8, there is some $Z \in N$ in which ω lives. Let C be the component of H containing Z. If $\mathcal{X}_C = \{v_C\}$, then $Z \subseteq v_C$. Otherwise let the X_i and the Y_i be as in the construction of $\mathcal{X}_C$. If $Z = X_j$ for some j. Then we pick j minimal such that ω lives in X_j. Since ω does not live in X_{j-1}, it must live in Y_j, as desired.

Thus we may assume that Z is not equal to any X_j. Let P be a path joining Z and $X_1 = Y_1$. By Sublemma 6.11, P contains a vertex u such that ZPu and X_1Pu are directed paths which are directed towards u. If $u = X_1$, then $Z \subseteq Y_1$, and we are done. Otherwise X_1Pu is a subpath of the ray $X_1X_2\dots$ since the out-degree is at most 1 so that $u = X_j$ for some j.

We pick P such that the j with $u = X_j$ is minimal and have the aim to prove that then $Z \subseteq Y_j$. Since $Z \subseteq X_j$, it remains to show that Z and X_{j-1} are disjoint. Suppose for a contradiction, there is a directed path Q joining Z and X_{j-1}. If Q is directed towards Z, then $Z = X_m$ for some m, contrary to our assumption. Thus Q is directed towards X_{j-1}. But then $ZQX_{j-1}PX_1$ has a smaller j-value, which contradicts the minimality of P. Hence there cannot be such a Q, and thus Z and X_{j-1} are disjoint by Sublemma 6.9. Having shown that $Z \subseteq Y_j$, we finish the proof by concluding that then ω also lives in Y_j. $\square$

□

Finally we deduce Corollary 2.1.

Proof that Theorem 1 implies Corollary 2.1. By Theorem 1, G has a tree-decomposition $(T, P_t | t \in V(T))$ of finite adhesion such that the ends of T define precisely the topological ends of T, and we choose this tree-decomposition as in Remark 6.6. In particular, we can pick a root r of T such that for each edge tu with $t \leq u$, the vertex set $\bigcup_{w \geq u} V(P_w) \setminus V(P_t)$ is connected.

Thus for each such edge tu, there is a finite connected subgraph S_u of $G[\bigcup_{w \geq u} V(P_w)]$ that contains $V(P_t) \cap V(P_u)$. Let Q_t be a maximal subforest of the union of the S_u, where the union ranges over all upper neighbours u of t. We recursively build a maximal subset U of $V(T)$ such that if $a, b \in U$, then Q_a and Q_b are vertex-disjoint. In this construction, we first add the nodes of T with smaller distance from the root. This ensures by the "Moreover"-part of Remark 6.6 that U contains infinitely many nodes of each ray of T.

Let S' be the union of those Q_t with $t \in U$. We obtain S by extending S' to a spanning tree of G, and rooting it at some $v \in V(S)$ arbitrarily. By the Star-Comb-Lemma [12, Section 8], each spanning tree of G contains for each topological end ω a ray belonging to ω.

Thus it remains to show that S does not contain two disjoint rays R_1 and R_2 that both belong to the same topological end ω of G. Suppose there are such R_1, R_2 and ω. Let $t_1 t_2 \ldots$ be the ray of T in which ω lives. Let n be so large that both R_1 and R_2 meet P_{t_n}. Then for each $m \geq n$, the set S_{t_m} contains a path joining R_1 and R_2. Thus the set $Q_{t_{m-1}}$ contains such a path. Since $Q_{t_{m-1}} \subseteq S$ for infinitely many m, the tree S contains a cycle, which is the desired contradiction. □

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